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ON THE NUMERICAL METHODS BASED ON INTEGRAL EQUATIONS FOR INVERSE PARABOLIC PROBLEMS

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ABSTRACT. The paper presents some existing and new results for the numerical solution of some inverse parabolic problems. We focus our attention on the using of the integral equation method by the numerical solution.

1 Introduction

The inverse problems for a parabolic equation can be divided into the following principal groups [15,23]: 1) the problems of the estimation of the heat flux history along a boundary part of a domain from a known temperature measurements on the rest of the boundary and at interior locations; 2) the problems of determining the initial condition if the temperature distributions inside a domain are known at some time; 3) the problems recovering the diffusion coefficient from boundary measurements of the solution of a parabolic equation [22,24]; 4) the problems determining a boundary part for the bounded domain from a knowledge of the rest of the boundary, the heat and the heat flux on it.

In this paper we consider the inverse problems from some of the enumerated groups. Primarily we are interested in the aspects of the numerical solution of these problems with using of the integral equation method. In Section 2 we describe the numerical solution of inverse boundary value problems for heat equation. These problems are actual in non-destructive testing of materials. In this case, one tries to investigate the interior structure of a body using only some given information on the boundary. Among the strategies followed is the thermal imaging technique, where inclusions or interior cracks are detected by controlling the heat flux on the boundary body and monitoring the boundary temperature response over an appropriate time interval [1,2,3,4,5,6]. This inverse problem is solved by using Newton or Landweber method and boundary integral equation method [7,10, 11,12]. Section 3 contains the case of the identification of the heat conductivity. Here are used regularized Newton method for the inverse problem and Rothe method with integral equation for direct problems. In Section 4 we discuss some kind of parametric identification problems. Here it is identified the thermal diffusion coefficient of a body by exposing to a temperature field and then by measurement of the temperature in some points outside of body. The materials of this section is based on the papers [14,17,18,20].

2 Boundary reconstruction

Let D_1 and D_2 be two simply connected bounded domains in $\mathbb{R}^2$ such that $\bar{D}_1 \subset D_2$. We assume that the boundaries of D_1 and D_2 are of class C^2 and denote them by Γ_1 and Γ_2, respectively. Further denote $D := D_2 \setminus \bar{D}_1$. Let $T > 0$ and φ a given function on $\partial D \times (0,T)$. Define φ_1 and φ_2 as restrictions of φ on $\Gamma_1 \times (0,T)$ and $\Gamma_2 \times (0,T)$,

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respectively. Let $T_1 > T_0$, $[T_0, T_1] \subseteq (0, T)$, $\Sigma \subseteq \Gamma_2$ by nonempty open subset and ν is the outward unit normal to ∂D.
A function $u(x, t)$ satisfies the heat equation

$$\frac{\partial u}{\partial t} = \Delta u \quad \text{in } D \times (0, T) \tag{2.1}$$

and the homogeneous initial condition

$$u(\cdot, 0) = 0 \quad \text{in } D. \tag{2.2}$$

We consider the following inverse problems:

- Inverse Dirichlet boundary value problem Ia. Under the assumption that

$$u = 0 \quad \text{on } \Gamma_1 \times (0, T) \quad \text{and} \quad u = \varphi_2 \quad \text{on} \Gamma_2 \times (0, T), \tag{2.3}$$

  to determine the interior boundary Γ_1 from a knowledge of the heat flux $\partial u / \partial \nu$ on $\Sigma \times [T_0, T_1]$.
- Inverse Dirichlet boundary value problem Ib. Under the assumption that $u = 0$ on $\Gamma_1 \times (0, T)$ and $\partial u / \partial \nu = \varphi_2$ on $\Gamma_2 \times (0, T)$ to determine the interior boundary Γ_1 from a knowledge of the temperature u on $\Sigma \times [T_0, T_1]$.
- Inverse Neumann boundary value problem IIa. Under the assumption that $\partial u / \partial \nu = 0$ on $\Gamma_1 \times (0, T)$ and $u = \varphi_2$ on $\Gamma_2 \times (0, T)$, to determine the interior boundary Γ_1 from a knowledge of the heat flux $\partial u / \partial \nu$ on $\Sigma \times [T_0, T_1]$.
- Inverse Neumann boundary value problem IIb. Under the assumption that $\partial u / \partial \nu = 0$ on $\Gamma_1 \times (0, T)$ and $\partial u / \partial \nu = \varphi_2$ on $\Gamma_2 \times (0, T)$ to determine the interior boundary Γ_1 from a knowledge of the temperature u on $\Sigma \times [T_0, T_1]$.

The formulated inverse parabolic problems can be rewritten as some operator equations with nonlinear operators that map the curve Γ_1 onto the heat flux or temperature on the curve Γ_2. For example we have in cases Ia and IIa

$$A_d(\Gamma_1) = \left.\frac{\partial u}{\partial \nu}\right|_{\Sigma \times [T_0, T_1]} \quad \text{and} \quad A_n(\Gamma_1) = u|_{\Sigma \times [T_0, T_1]}.$$

These equations are ill-posed, since the construction of the solution to the heat equation from the Cauchy data is ill-posed linear problem, and they are nonlinear, since the solution to the initial boundary value problem depends nonlineary on the boundary curves.

Let us assume that Γ_1 is starlike, i.e.

$$x(s) = r(s)(\cos s, \sin s), \quad 0 \le s \le 2\pi$$

with a some positive function r. Clearly r is to be found. Then we transform the operator equations into the parametric form

$$A_d(r) = \gamma_d(s, t), \quad A_n(r) = \gamma_n(s, t), \quad (s, t) \in \Sigma^*, \tag{2.4}$$

where γ_d and γ_n are given data and $\Sigma^* := [\sigma_0, \sigma_1] \times [T_0, T_1]$.

2.1 Newton method

We describe shortly the algorithm for numerical solution of first nonlinear equation that is based on Newton method. We assume that the curve $\tilde{\Gamma}_1$, with the parametric representation $z(s)$ is an approximation for the curve Γ_1 and let $h(s)$ be the unknown correction such that $\tilde{z}(s) = z(s) + h(s)$ is a new approximation. We look for h in the form

$$h(s) = q(s)(\cos s, \sin s),$$

where q is an unknown function . After the linearization in (2.4) we get the following approximating linear equation with respect to h:

$$A_d(r) + A_d'(r; h) = \gamma_d(s,t), \quad (s,t) \in \Sigma^*. \tag{2.5}$$

We approximate q in the form

$$q(s) = \sum_{j=1}^{K} a_j q_j(s)$$

with basis functions q_j. The collocation method for (2.5) with respect to the collocation points $(\tilde{s}_k, \tilde{t}_i) \in \Sigma^*$, $k = 1, \dots, M_{\text{inv}}$, $i = 1, \dots, N_{\text{inv}}$, yields the system of linear equations

$$\sum_{j=1}^{K} a_j A_d'(r; h_j)(\tilde{s}_k, \tilde{t}_i) = \gamma_d(\tilde{s}_k, \tilde{t}_i) - A_d(r)(\tilde{s}_k, \tilde{t}_i), \tag{2.6}$$

where $h_j(s) := (q_j(s)\cos s, q_j(s)\sin s)$ and $M_{\text{inv}} N_{\text{inv}} > K$.

The following theorem about the domain derivative of operator A_d is proved (see [10]).

Theorem 2.1. *Let $\tilde{D}_1$ be a bounded domain with the boundary $\tilde{\Gamma}_1$ and $\tilde{D} := D_2 \setminus \tilde{D}_1$. Let $\varphi_2 \in L^2(\Gamma_2 \times [0,T])$, $h \in C^2(\tilde{\Gamma}_1; \mathbb{R}^2)$ and u be a weak solution of the initial boundary value problem (2.1)– (2.3) in $\tilde{D} \times (0,T)$. Then the domain derivative $A_d'(r;h)$ exists and is given by*

$$A_d'(r;h) = \left.\frac{\partial u'}{\partial \nu}\right|_{\Sigma},$$

where u' solves the heat equation

$$\frac{\partial u'}{\partial t} = \Delta u' \quad \text{in } \tilde{D} \times (0,T)$$

with the homogeneous initial condition and the boundary condition

$$u' = -h \cdot \nu \frac{\partial u}{\partial \nu} \quad \text{on } \tilde{\Gamma}_1 \times (0,T) \quad \text{and} \quad u' = 0 \quad \text{on } \Gamma_2 \times (0,T).$$

Here ν is the outward unit normal on $\tilde{\Gamma}_1$.

For the case of inverse Neumann boundary value problems see [11]. Due to the linear equation (2.5) being an ill-posed equation, we use Tikhonov regularization to stabilize our problem. Hence, we replace (2.6) by the following least-squares problem to minimize the penalized residual

$$T := \alpha \sum_{k=1}^{K} w_k a_k^2 + \sum_{i=1}^{M_{\text{inv}}} \sum_{j=1}^{N_{\text{inv}}} \left| \sum_{k=1}^{K} a_k A_d'(r;h_k)(\tilde{s}_i, \tilde{t}_j) - \gamma_d(\tilde{s}_i, \tilde{t}_j) + A_d(r)(\tilde{s}_i, \tilde{t}_j) \right|^2$$

with some regularization parameter $\alpha > 0$ and some positive weights $w_1, \dots, w_K$. Minimizing of T with respect to $a_1, \dots, a_K$ is equivalent to solving the following linear system

$$\alpha w_p a_p + \sum_{k=1}^{K} a_k \sum_{i=1}^{M_{\text{inv}}} \sum_{j=1}^{N_{\text{inv}}} A_d'(r;h_k)(\tilde{s}_i, \tilde{t}_j) A_d'(r;h_p)(\tilde{s}_i, \tilde{t}_j)$$
$$= \sum_{i=1}^{M_{\text{inv}}} \sum_{j=1}^{N_{\text{inv}}} \{\gamma_d(\tilde{s}_i, \tilde{t}_j) - A_d(r)(\tilde{s}_i, \tilde{t}_j)\} A_d'(r;h_p)(\tilde{s}_i, \tilde{t}_j), \quad p = 1, \dots, K. \tag{2.7}$$

We choose the weights w_p as in the Levenberg-Marquardt algorithm

$$w_p = \sum_{i=1}^{M_{\rm inv}} \sum_{j=1}^{N_{\rm inv}} A_d'(r; q_p)(\tilde{s}_i, \tilde{t}_j) A_d'(r; q_p)(\tilde{s}_i, \tilde{t}_j), \quad p = 1, \dots, K.$$

Finally, we summarize the description of one step of the Newton method as follows

1. With the radial function r_n given, solve the direct problem (2.1)– (2.3) by the boundary integral equation method and compute $\frac{\partial u}{\partial \nu}$ on Γ_2.
2. Compute numerical solutions for the sequence of the direct initial boundary value problems (2.1)– (2.3) with the corresponding boundary conditions via boundary integral equation method.
3. Solve the system of linear equations (2.7).
4. Compute the correction h and find new approximation $r_{i+1} = r_i + q$ for the boundary Γ_1.

As a stopping rule for the number of iterations we use the condition

$$\frac{||q||_{L^2}}{||r_i||_{L^2}} < \delta,$$

where δ is a given precision.

In [7] we apply the described method for the reconstruction of some bounded inclusion in the semi-infinite region. In contrast with previous case we have used the Green's function approach for direct initial boundary value problems and these are reduced to boundary integral equations with integrals over infinite axis. Then the quadrature method with combination of trigonometric and sinc approximation [25] is used. Note that numerical experiments indicate the feasibility of our method in all cases.

2.2 Landweber method

The main ingredient for an efficient implementation of Landweber iteration is the adjoint operator of $A_d'(\Gamma_1, a)$. In [12] it was proved the following result.

Theorem 2.2. *Let $\psi \in H^{-\frac{1}{2},-\frac{1}{4}}(\Gamma_2 \times (0,T))$. Then the corresponding adjoint operator $A_d'(\Gamma_1, a)^* : H^{-\frac{1}{2},-\frac{1}{4}}(\Gamma_2 \times (0,T)) \to C^2(\Gamma_1; \mathbb{R}^2)$ is given by*

$$A_d'(\Gamma_1, a)^* \psi = \nu \int_0^T \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} \bigg|_{\Gamma_2 \times (0,T)} dt, \tag{2.8}$$

where v solves the initial boundary value problem

$$\frac{\partial v}{\partial t} = -\Delta v \quad in D \times (0,T), \tag{2.9}$$

$$v(\cdot, T) = 0 \quad in \quad D, \tag{2.10}$$

$$v = 0 \quad on\, \Gamma_1 \times (0,T) \quad and \quad v = \psi \quad on\, \Gamma_2 \times (0,T). \tag{2.11}$$

Here we used the anisotropic Sobolev spaces

$$H^{p,s}(\Gamma_2 \times (0,T)) := L^2((0,T); H^p(\Gamma_2)) \cap H^s((0,T); L^2(\Gamma_2)).$$

We assume that the boundary curves have parametric representations

$$\Gamma_1 := \{x_1(s) = r(s)(\cos s, \sin s), 0 \le s \le 2\pi\}$$

and

$$\Gamma_2 := \{x_2(s) = (x_{21}(s), x_{22}(s)), 0 \le s \le 2\pi\}.$$

Here $r > 0$ is the unknown radial function and we assume $r, x_{21}, x_{22} \in C^2[0, 2\pi]$.

Now we have in this case the following representation for the adjoint operator $A'_d(r)^*$ according to (2.8)

$$(A'_d(r)^*)\psi(s,t) = r(s)\int_0^T \frac{\partial u}{\partial \nu}(x_2(s),t)\frac{\partial v}{\partial \nu}(x_2(s),t)dt. \tag{2.12}$$

Denote by $r^\dagger$ the exact solution of (2.4) and assume that the input data γ_d are given by noisy data γ_d^δ with the known noise level δ, i.e. $||\gamma_d^\delta - \gamma_d|| \le \delta$. Due to [4] we define the iterative Landweber method as follows

$$r_{n+1}^\delta = r_n^\delta + \mu A'_d(r_n^\delta)^*(\gamma_d^\delta - A_d(r_n^\delta)), \tag{2.13}$$

where $\mu > 0$ is an appropriate scaling parameter, that has to be chosen such that $||A'_d(r)|| \le 1/\mu$ for all r in a neighborhood of $r^\dagger$. This method can become a regularization property if it stopped "at the right time", i.e. only for a suitable stopping index k_*

We assume that the condition

$$||A_d(r) - A_d(\tilde r) - A'_d(r)(r-\tilde r)|| \le \eta ||A_d(r) - A_d(\tilde r)|| \tag{2.14}$$

holds for all r, $\tilde r$ in a neighborhood of $r^\dagger$ and some $\eta < \frac{1}{2}$. In [4] it is proved that in this case the Landweber iteration together with the discrepancy principle as stopping rule

$$||\gamma_d^\delta - A_d(r_{k_*}^\delta)|| \le \tau\delta < ||\gamma_d^\delta - A_d(r_n^\delta)||, \tag{2.15}$$

for $n = 1, \dots, k_*$ and with fixed $\tau > 2\frac{1+\eta}{1-2\eta}$ is a regularization method.

Now we summarize one step of the Landweber iteration of the following parts.

1. With the radial function r_n^δ given, solve the direct problem (2.1)– (2.3), evaluate $A_d(r_n^\delta)$ in term of the heat flux on Γ_2 and compute the heat flux on Γ_1 entering the first term in the integral of (2.12) for the adjoint $A'_d(r_n^\delta)^*$.
2. Solve the adjoint direct problem (2.9)– (2.11) with the boundary function $\psi = \gamma_d^\delta - A_d(r_n^\delta)$ on $\Gamma_2 \times (0,T)$ and compute the heat flux on Γ_1 entering the second term in the integral of (2.12) for the adjoint $A'_d(r_n^\delta)^*$.
3. Compute $A'_d(r_n^\delta)^*\psi$ as (2.12) by using the quadrature formula, for example trapeciodal rule.
4. Find new approximation for the radial function by (2.13). If for new approximation a stopping criterion (2.15) (or other suitable condition) is satisfied, then terminate, otherwise go back to 1.

2.3 Boundary integral equation method for direct problem

We use the indirect variant of boundary integral equation method and seek the solution in the form of a single-layer potential

$$u(x,t) = \frac{1}{4\pi}\sum_{i=1}^{2}\int_0^t \int_{\Gamma_i} q_i(y,\tau)G(x-y,t-\tau)ds(y)\,d\tau, \quad (x,t) \in D \times (0,T]. \tag{2.16}$$

Here q_i are unknown densities on $\Gamma_i \times (0,T]$, $i = 1,2$ and G is the fundamental solution of the heat equation in $\mathbb{R}^2$. Then the problem (2.1)– (2.3) can be reduced to the system of integral equation of the first kind

$$\begin{cases} \dfrac{1}{4\pi}\displaystyle\sum_{i=1}^{2}\int_0^t\int_{\Gamma_i} q_i(y,\tau)G(x-y,t-\tau)\,ds(y)d\tau = 0, \\ \qquad\qquad (x,t)\in\Gamma_1\times(0,T], \\ \dfrac{1}{4\pi}\displaystyle\sum_{i=1}^{2}\int_0^t\int_{\Gamma_i} q_i(y,\tau)G(x-y,t-\tau)\,ds(y)d\tau = \varphi_2(x,t), \\ \qquad\qquad (x,t)\in\Gamma_2\times(0,T]. \end{cases} \tag{2.17}$$

This system of integral equations is well posed in the corresponding anisotropic Sobolev spaces (see [13]).

We assume that the boundary curves are given through parametric representations

$$\Gamma_k = \{x_k(s) = (x_{k,1}(s), x_{k,2}(s)) : 0 \le s \le 2\pi\}, \quad k = 1,2, \tag{2.18}$$

where $x_k : \mathbb{R} \to \mathbb{R}^2$, $x_k \in C^2[0,2\pi]$ with $|x_k'(s)| > 0$ for all s. After the transformation of (2.17) into the parametric form and by application of the collocation method with respect to time variable with piecewise constant basis functions we reduce (2.17) to the sequence of Fredholm integral equations of first kind

$$\frac{1}{4\pi}\sum_{k=1}^{2}\int_0^{2\pi} \mu_{k,n}(\sigma)K_{\ell k}^{(0)}(s,\sigma)\,d\sigma = F_{\ell,n}(s), \quad s\in[0,2\pi],\ \ell=1,2,\ n=1,\dots,N, \tag{2.19}$$

with the right-hand sides

$$F_{\ell,n}(s) = f_{\ell,n}(s) - \frac{1}{4\pi}\sum_{m=1}^{n-1}\sum_{k=1}^{2}\int_0^{2\pi} \mu_{k,m}(\sigma)K_{\ell k}^{(n-m)}(s,\sigma)\,d\sigma, \tag{2.20}$$

where we have denoted $\mu_{k,n}(s) \approx q(x_k(s),t_n)|x_k'(s)|$, $f_{1,n}(s,t_n) := 0$, $f_{2,n}(s,t_n) := \varphi_2(x_2(s),t_n)$ and where the kernels are given by

$$K_{\ell k}^{(n-m)}(s,\sigma) := \int_{t_{m-1}}^{t_m} G(x_\ell(s)-x_k(\sigma), t_n-\tau)\,d\tau. \tag{2.21}$$

Here $t_n = nT/N$, $n = 0,\dots,N$ are the collocation points in time. The exact integration in (2.21) yields to explicit representation of all kernels [10]. The kernels $K_{\ell\ell}^{(0)}$ have logarithmic singularities and they can be written in the form

$$K_{\ell\ell}^{(0)}(s,\sigma) = -\ln\left(\frac{4}{e}\sin^2\frac{s-\sigma}{2}\right) + K_{\ell\ell}^{(0,1)}(s,\sigma), \quad s\ne\sigma$$

and can also be shown that the functions $K_{\ell\ell}^{(0,1)}$ and $K_{\ell\ell}^{(p)}$, $\ell = 1,2$, $p = 1,\dots,N$, are continuous.

For the numerical solution of the sequence of integral equations (2.19) we propose the discrete collocation method based on trigonometric interpolation. This method was suggested and analyzed for one integral equation of the type (2.19) in [8] in a Hölder space setting and in [21] in a Sobolev space setting.

For this method we choose an equidistant mesh by setting $s_j := j\pi/M$, $j = 0,\dots,2M-1$, and use the trapeciodal quadrature and the following quadrature rule

$$\frac{1}{2\pi}\int_0^{2\pi} g(\sigma)\ln\left(\frac{4}{e}\sin^2\frac{s_j-\sigma}{2}\right)d\sigma \approx \sum_{k=0}^{2M-1} R_{|j-k|}\,g(s_k) \tag{2.22}$$

with the weights

$$R_j := -\frac{1}{2M}\left\{1 + 2\sum_{m=1}^{M-1}\frac{1}{m}\cos m s_j + \frac{(-1)^j}{M}\right\}.$$

Now we approximate the integrals in (2.19) using the quadrature rules and then discretize the resulting approximate equations by trigonometric collocation using the quadrature points as collocation points. As a result we obtain the sequence of linear systems

$$\sum_{j=0}^{2M-1}\sum_{k=1}^{2}\tilde{\mu}_{k,n;j}a_{\ell k;ij} = \tilde{F}_{\ell,n;i}, \quad i = 0,\dots,2M-1,\ \ell = 1,2,\ n = 1,\dots,N, \tag{2.23}$$

with the matrix elements

$$a_{\ell k;ij} = \begin{cases} -R_{|i-j|} + \dfrac{1}{2M}\,K_{\ell\ell}^{(0,1)}(s_i,s_j), & \ell = k,\\[2mm] \dfrac{1}{2M}\,K_{\ell k}^{(0)}(s_i,s_j), & \ell \neq k,\end{cases}$$

and with the right-hand sides $\tilde{F}_{\ell,n;i}$ corresponding to (2.20). As to be expected, the numerical approximations show a linear convergence with respect to the time discretization and exponential convergence with respect to the discretization of the boundary integral equation provided that the boundaries and the boundary data are analytic.

For evaluation of $A(\Gamma_1)$, the boundary condition of the derivative $A'(\Gamma_1;a)$ and the adjoint operator $A'(\Gamma_1;a)^*$ we need approximations for the normal derivative of the single-layer potential (2.16) on Γ_1 and Γ_2. From the jump-relations for the normal derivative we have

$$\begin{cases} \dfrac{\partial u}{\partial\nu}(x,t) = -\dfrac{1}{2}\,q_1(x,t) + \\ +\dfrac{1}{4\pi}\displaystyle\sum_{i=1}^{2}\int_0^t\int_{\Gamma_i} q_i(y,\tau)\,\frac{\partial}{\partial\nu(x)}\,G(x-y,t-\tau)\,ds(y)d\tau,\ x\in\Gamma_1,\\ \dfrac{\partial u}{\partial\nu}(x,t) = \dfrac{1}{2}\,q_2(x,t) + \\ +\dfrac{1}{4\pi}\displaystyle\sum_{i=1}^{2}\int_0^t\int_{\Gamma_i} q_i(y,\tau)\,\frac{\partial}{\partial\nu(x)}\,G(x-y,t-\tau)\,ds(y)d\tau,\ x\in\Gamma_2 \end{cases} \tag{2.24}$$

for $t \in (0,T]$. Next we apply to (2.24) the collocation for the semidiscretization in time and then use the trigonometrical quadrature rules.

3 Identification of the heat conductivity

These problems arise by non-contact detection and evaluation of defects in materials. The goal is to receive the information about the interior or another inaccessible part of material object boundary after exterior measurements. These methods are used in various industrial applications (for example the corrosion and cracks detection in aircraft) and in the medical field (infrared thermography). Very often thermal imaging technique is used which consists of applying a heat flux to a part of the boundary of the object and observing of resulting boundary temperature. From these Cauchy data one attempts to identify the shape of some unknown inaccessible part of the boundary.

The inverse problem consists in the identification of the geometry (location and boundary) of discontinuities in a material body from the boundary measurements. These discontinuities are characterized by thermal diffusion coefficient.

The problem can be formulated as follows. Let D and D_1 be two simply connected bounded domains in $\mathbb{R}^2$ such that $\bar{D}_1 \subset D$ and denote the boundaries of domains by Γ_2

and Γ_1, respectively. We assume that f is a given function on $\partial D \times (0,T)$, where $T > 0$. Let $T_1 > T_0$, $[T_0, T_1] \subseteq (0,T)$, $\Sigma \subseteq \Gamma_2$ is nonempty open subset and ν_2 is the outward unit normal to Γ_2.

Then consider the solution $u(x,t)$ of the second initial boundary value problem

$$\frac{\partial u}{\partial t} - \operatorname{div}(a \operatorname{grad} u) = 0 \quad \text{in } D \times [0,T], \tag{3.1}$$

$$u = 0 \qquad \text{on } D \times \{0\}, \tag{3.2}$$

$$u = f \qquad \text{on } \Gamma_2 \times (0,T), \tag{3.3}$$

where thermal diffusion coefficient a has the form

$$a(x) = \begin{cases} k, & x \in D_1, \\ 1, & x \in D_2 = D \setminus \bar{D}_1 \end{cases}$$

with known constant $k > 0$ and $k \neq 1$.

The presence of discontinuos coefficients requires us to specify in which sense solutions of (3.1)- (3.3) exist. If u is sufficiently smooth then it satisfies the parabolic differential equation (3.1) in $D_1 \times (0,T)$ and $D_2 \times (0,T)$ with boundary conditions

$$[u]_\pm = 0 \quad \text{and} \quad \left[a\frac{\partial u}{\partial \nu_1}\right]_\pm = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{3.4}$$

where ν_1 denotes the unit outward directed normal vector at Γ_1 and the notation $[\cdot]_\pm$ abbreviates the difference of the traces of a function at Γ_1 approaching the boundary from D_2 and from D_1 respectively.

The inverse problem is to determine the region D_1 from the known temperature f on $\Gamma_2 \times (0,T)$ and heat flux $\frac{\partial u}{\partial \nu_2}$ on $\Sigma \times [T_0, T_1]$.

3.1 Newton method for the inverse problem

The solution of the direct initial boundary value problem (3.1)- (3.4) defines a non-linear operator

$$A : \Gamma_1 \to \left.\frac{\partial u}{\partial \nu_2}\right|_{\Sigma \times [T_0, T_1]}$$

which maps the curve Γ_1 onto the heat flux on the curve Γ_2. In this sense the solution of our inverse problem consists in the solution of the nonlinear equation

$$A(\Gamma_1) = g,$$

where $g(x,t) = \frac{\partial u}{\partial \nu_2}(x,t), (x,t) \in \Sigma \times [T_0, T_1]$.

We use the regularized Newton method, described in section 2.1, to find an approximation of Γ_1 by given some noisy data g^δ, a bound δ on the error $||g^\delta - g||$ and an initial guess on Γ_1.

To compute the domain derivative we use the theorem suggested by Hettlich and Rundell (see [16]).

Let $\Omega \subseteq \mathbb{R}^N, N = 2,3$, be a bounded domain with smooth boundary in the class C^2. In the interval $(0,T) \subseteq \mathbb{R}$ we consider the initial boundary value problem

$$\partial_t u - div(a\nabla u) + cu = f \quad \text{in} \quad Q = \Omega \times [0,T], \tag{3.5}$$

$$u = 0 \quad \text{on} \quad \Omega \times 0, \quad \text{and} \quad u = 0 \quad \text{on} \quad \partial\Omega \times (0,T),$$

$$[u]_\pm = 0 \quad \text{and} \quad \left[a\frac{\partial u}{\partial \nu}\right]_\pm = 0 \quad \text{on} \quad \partial D \times (0,T),$$

and initial condition

$$u(x,0) = \delta(x - x^*) \quad \text{in} \quad \mathbb{R}^2. \tag{4.2}$$

Here $a(x)$ is the thermal diffusion coefficient and we assume that $a(x) = 1 + f(x)$ with $\operatorname{supp} f(x) \in \bar{D}$, δ is the Dirac function and $T > 0$ is a given constant.
The inverse problem consists in the following: from the given final temperature measurement $u(x^*, T)$ it is necessary to find the function f.
The coefficient a in (4.1) is not continuously and for the riddance of the singularity in (4.1) the fundamental solution of heat equation

$$G(x,t;y) := \frac{1}{4\pi t} \exp -\frac{|x-y|}{4t}$$

is subtracted from u. Then the difference $v(x,t) := u(x,t) - G(x,t;x^*)$ satisfies the following initial value problem

$$\frac{\partial v}{\partial t}(x,t) - \operatorname{div}((1+f(x)) \operatorname{grad} v(x,t)) = \operatorname{div}(f(x) \operatorname{grad} G(x,t;x^*)) \quad \text{in} \quad \mathbb{R}^2 \times (0,T), \tag{4.3}$$

$$v(x,0) = 0 \quad \text{in} \quad \mathbb{R}^2. \tag{4.4}$$

The formulated inverse problem is equivalent to solving the nonlinear operator equation

$$F(f) = g, \tag{4.5}$$

where $F : H^s(D) \to L^2(D^*)$ is the parameter-to-data mapping and g is the given data. For $s > 0$ the operator F is compact and therefore the equation (4.5) is ill-posed and the regularization is needed.

4.1 Linearized version and its Tikhonov regularization

In [14] it is shown that for "ε-small" f the nonlinear term $\operatorname{div}(f \operatorname{grad} v)$ is small relative to the term $\operatorname{div}(f \operatorname{grad} G)$. Therefore for f with $||f||_{\infty,D} \le \varepsilon$ we get the linearized Cauchy problem related to (4.3), (4.4)

$$\frac{\partial v}{\partial t}(x,t) - \Delta v = \operatorname{div}(f(x) \operatorname{grad} G(x,t;x^*)) \quad \text{in} \quad \mathbb{R}^2 \times (0,T), \tag{4.6}$$

$$v(x,0) = 0 \quad \text{in} \quad \mathbb{R}^2. \tag{4.7}$$

If f is sufficiently smooth then according to the integral representation of the solution to the Cauchy problem we have the integral approach for the solution of (4.6), (4.7)

$$v(x,t;x^*) = -\int_0^t \int_{\mathbb{R}^2} f(y)(\operatorname{grad}_y G(x,t-\tau;y), \operatorname{grad}_y G(y,\tau,x^*))dyd\tau.$$

Thus the linearized inverse problem can be written as integral equation

$$(Af)(x) = g(x), \quad x \in D^*, \tag{4.8}$$

where

$$(Af)(x) = -\frac{1}{64\pi^2} \int_0^T \int_D f(y) \frac{|x-y|^2}{(\tau(T-\tau))^2} \exp\left(-\frac{|x-y|^2 T}{4\tau(T-\tau)}\right) dyd\tau$$

and g is the given measurement in $D^* \times \{T\}$.
In [14] it is proved uniqueness of the solution $f \in L^2(D)$ for the integral equation (4.8). Due to the classical Tikhonov regularization the integral equation of the first kind (4.8) is replaced by the following integral equation of the second kind

$$(\alpha I + A^* A) f_\alpha = A^* g, \tag{4.9}$$

$$u_n^1 = u_n^2 \quad \text{and} \quad k\frac{\partial u_n^1}{\partial \nu_1} = \frac{\partial u_n^2}{\partial \nu_1} \quad \text{on} \quad \Gamma_1, \tag{3.9}$$

$$u_{-1}^i = 0 \quad \text{in} \quad D_i, \quad i = 1, 2, \tag{3.10}$$

where

$$f_n = f(\cdot, t_n), \quad \kappa_i^2 = \frac{1}{\tau}\begin{cases} k, & i = 1, \\ 1, & i = 2, \end{cases} \quad i = 1, 2, n = 0, ..., N_t - 1.$$

The numerical solution of the sequence of the boundary value problems (3.7)- (3.10) is based on an indirect variant of the boundary integral equation approach (see [19]). Note here, that in [9] the fundamental solution to the sequence of equations (3.7) is to be found. Therefore, we seek the solution in such integral form:

$$\begin{cases} u_n^1(x) = \displaystyle\sum_{m=0}^{n} \int_{\Gamma_1} q_m^3(y)\Phi_{n-m}(\kappa_1, x, y)dS(y), \\ u_n^2(x) = \displaystyle\sum_{i=1}^{2}\sum_{m=0}^{n} \int_{\Gamma_i} q_m^i(y)\frac{\partial}{\partial \nu_i(y)}\Phi_{n-m}(\kappa_2, x, y)dS(y), \end{cases}$$

where q_n^i, $i = 1, 2, 3$, $n = \overline{0, N_t}$ are unknown densities,
$\Phi_n(\kappa_j, x, y) = K_0(\kappa_j r)v_n(\kappa_j, r) + K_1(\kappa_j r)\omega_n(\kappa_j, r)$, $r = |x - y|$, $x \neq y$ - fundamental solution of the recurrence sequence of elliptic equations, K_0, K_1 - modified Bessel functions, v_n, ω_n - polynomials.

Then the problem (3.7)- (3.10) can be reduced to the sequence of system of integral equations of the second kind, which can be rewritten as such operator equation

$$\frac{1}{2}\vec{q}_n(x) + (A^0\vec{q}_n(x)) = \vec{\psi}_n(x) - \frac{1}{2}\sum_{m=0}^{n-1}\vec{q}_m(x) - \sum_{m=0}^{n-1}(A^{n-m}\vec{q}_m)(x), \tag{3.11}$$

where

$$(A_{ij}^m\mu)(x) = (-1)^{i+1}\int_{\Gamma_j}\mu(y)\frac{\partial}{\partial \nu_j(y)}\Phi_m(\kappa_2, x, y)dS(y), \quad x \in \Gamma_i, \quad i, j = 1, 2,$$

$$(A_{13}^m\mu)(x) = -\int_{\Gamma_1}\mu(y)\Phi_m(\kappa_1, x, y)dS(y), \quad x \in \Gamma_1, \quad (A_{23}^m\mu)(x) = 0, \quad x \in \Gamma_2,$$

$$(A_{3j}^m\mu)(x) = -\frac{1}{k}\frac{\partial}{\partial \nu_1(x)}\int_{\Gamma_j}\mu(y)\frac{\partial}{\partial \nu_j(y)}\Phi_m(\kappa_2, x, y)dS(y), \quad x \in \Gamma_1, \quad j = 1, 2,$$

$$(A_{33}^m\mu)(x) = \frac{\partial}{\partial \nu_1(x)}\int_{\Gamma_1}\mu(y)\Phi_m(\kappa_1, x, y)dS(y), \quad x \in \Gamma_1,$$

$$\vec{q}_n = (q_n^1, q_n^2, q_n^3)^T, \quad \vec{\psi}_n = (-f_n, 0, 0)^T.$$

Note that the integral operator A_{31}^m has a hypersigular kernel and operators A_{13}^m, A_{11}^m, A_{22}^m and A_{33}^m have logarithmic singularity. Full discretization of (3.11) is made by Nyström method based on trigonometric quadrature rules (for details see [19]).

4 Parametric identification

In this case we need to identify the thermal diffusion coefficient of a body by exposing to a temperature field and then by measurement of the temperature in some points outside of body.

Let D and $D^* \in \mathbb{R}^2$ be bounded simple connected domains that are separated. The point sources are placed at the position $x^* \in D^*$. The temperature u formally satisfies the heat equation

$$\frac{\partial u}{\partial t}(x, t) - \operatorname{div}(a(x)\operatorname{grad} u(x, t)) = 0 \quad \text{in} \quad \mathbb{R}^2 \times (0, T) \tag{4.1}$$

where ν denotes the unit outward directed normal vector at ∂D, the coefficients do not depend on time and $a = a_0 + \chi_D a_1 \geq \gamma > 0$, $c = c_0 + \chi_D c_1$, $f = f_0 + \chi_D f_1$ with smooth functions $a_j, c_j, f_j \in C^1(\bar{\Omega}), j = 0, 1$, χ_D denotes the characteristic function of D.

Multiplying the differential equation (3.5) by a test function $v \in C^\infty(Q)$ which vanishes on $\partial\Omega \times (0,T)$, Greens theorem yields

$$L(u,v) = \int_0^T \int_\Omega f v dx dt \tag{3.6}$$

with the bilinear form $L(u,v) = \int_0^T \int_\Omega (a\nabla u \cdot \nabla v + \partial_t u v + c u v) dx dt$.

The variational problem (3.6) is considered in the usual anisotropic Sobolev spaces $H_0^{r,s}(Q) = \{L^2((0,T), H_0^r(\Omega)) \cap H^s((0,T), L^2(\Omega))\}$ for $r, s \geq 0$. For the case of homogeneous Dirichlet boundary values the subspaces are defined.

$$\check{H}_0^{r,s}(Q) = \{u = U|_Q : U \in H_0^{r,s}(\Omega \times \mathbb{R}), U(t,\cdot) = 0, \quad \text{for} \quad t < 0\},$$

$$\hat{H}_0^{r,s}(Q) = \{u = U|_Q : U \in H_0^{r,s}(\Omega \times \mathbb{R}), U(t,\cdot) = 0, \quad \text{for} \quad t > T\}.$$

Theorem 3.1. *Let $f = f_0 + \chi_D f_1$, $f_0, f_1 \in C^1(\Omega)$ and $u \in \check{H}_0^{1,\frac{1}{2}}(Q))$ denote the solution of (3.6) for all $v \in \hat{H}_0^{1,\frac{1}{2}}(Q)$. Analogously u_h is defined replacing D by the perturbed domain D_h. Then u on $\partial\Omega$ is differentiable at ∂D in the sense that there exists $\omega \in \check{H}_0^{1,\frac{1}{2}}(Q))$ linearly depending on h such that*

$$\lim_{h\to 0} \frac{1}{||h||_{C^1}} ||\tilde{u}_h - u - \omega|| = 0.$$

Furthermore $\omega = u' + h \cdot \nabla u$, where the domain derivative $u'|_D \in \check{H}_0^{1,\frac{1}{2}}(D \times (0,T))$ and $u'|_{\Omega\setminus\bar{D}} \in \check{H}_0^{1,\frac{1}{2}}(\Omega \setminus \bar{D} \times (0,T))$ is defined by the solution of the initial boundary value problem

$$\partial_t u' - div(a\nabla u') + cu' = 0 \quad in \quad Q$$

with

$$[u']_\pm = -h_\nu \left[\frac{\partial u}{\partial \nu}\right]_\pm \quad on \quad \partial D \times (0,T),$$

$$\left[a\frac{\partial u'}{\partial \nu}\right]_\pm = Div(h_\nu [a]_\pm \nabla_\tau u) - ([\partial_t u]_\pm + [c]_\pm u - [f]_\pm) h_\nu \quad on \quad \partial D \times (0,T),$$

$$u' = 0 \quad on \quad \partial\Omega \times (0,T),$$

$$u' = 0 \quad on \quad \Omega \times 0.$$

Here, $\nabla_\tau u$ is the tangential (surface) gradient of a scalar valued function u, $DivV$ - the tangential divergence of a tangential field V.

3.2 Rothe's method and the integral equation method for the direct problem

Rothe's method for parabolic initial boundary value problems consists of a time discretization by a finite difference approximation. We choose $N_t \in \mathbb{N}$ and with the stepsize $\tau = T/N_t$ consider the grid points $t_n = (n+1)\tau, n = 0, 1, ..., N_t - 1$. Then we replace the initial boundary value problem for the heat equation (3.1)- (3.4) by the sequence of the N_t Dirichlet boundary value problems:

$$\Delta u_n^i - \kappa_i^2 u_n^i = -\kappa_i^2 u_{n-1}^i, \quad \text{in} \quad D_i, \quad i = 1, 2, \tag{3.7}$$

$$u_n^2 = f_n \quad \text{on} \quad \Gamma_2, \tag{3.8}$$

where α is a regularization parameter and $A^* : L^2(D) \to L^2(D)$ is an adjoint operator for A.
The numerical solution of this integral equation (4.9) can be realized for example by the finite element method.

4.2 Newton-type methods

An alternative to Tikhonov regularization of the linearized problem are regularized iterative methods. In this case we linearize the equation (4.5) in each step around the current f_n and find new approximation f_{n+1} as a solution of the linear equation

$$F'(f_n)(f_{n+1} - f_n) = g - F(f_n). \tag{4.10}$$

In general the linearized equation (4.10) will not have the continuous inverse and the linear regularization method for its solution is to be used. As we see we need to calculate the Frechet-derivative of the parameter-to-data mapping F.

Theorem 4.1 [18,20]. *The operator $F : H^s(D) \to L^2(D^*)$ with $s > 1$ has the Frechet-derivative $F'(f)$ given as*

$$F'(f)h = \tilde{w}(x, T; x^*),$$

where $\tilde{w}$ is the solution of the initial value problem

$$\frac{\partial \tilde{w}}{\partial t}(x, t; x^*) - div((1 + f(x))\, grad\, \tilde{w}(x, t; x^*)) \tag{4.11}$$

$$= div(h(x)\, grad\, G(x, t; x^*)) + div(h(x)\, grad\, v(x, t; x^*)) \quad in \quad \mathbb{R}^2 \times (0, T),$$

$$\tilde{w}(x, 0; x^*) = 0 \quad in \quad \mathbb{R}^2. \tag{4.12}$$

Now we can solve the linear equations (4.10) by collocation method with the future regularization of the received linear system (see Sec. 2). On the other hand the nonlinear equation can be solved by iterative regularized method. Some aspects of these procedures are discussed in [20]. Note that solutions of the initial value problems (4.3)– (4.4) and (4.11)– (4.12) have integral representations according to the classical potential theory for the heat equation. For example

$$\tilde{w}(x, T; x^*) = -\int_0^T \int_D h(y)(\mathrm{grad}_y\, u(y, \tau; x), \mathrm{grad}_y\, u(y, T - \tau; x^*))dyd\tau.$$

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