Журнал обчисл. прикл. матем.
Вип. 1, 2003, стор. 17–27

J. Comput. Appl. Math.
No. 88, 2003, pp. 17–27

ON NUMERICAL APPROACH TO SOLVE SOME THREE-DIMENSIONAL BOUNDARY VALUE PROBLEMS IN POTENTIAL THEORY BASED ON INTEGRAL EQUATION METHOD

UDC 518:517.948

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ABSTRACT. In the article we consider the calculation of electronic field formed by a number of charged electrodes which have to be represented as unclosed surfaces. The effective algorithms for solving this problem can be obtained by integral equation method. We discuss two ways to solve the obtained integral equation. To approach the problem we combine Schwartz algorithm with Green function method. The described numerically analytical approach to solve two-dimensional integral equation of the first kind with weak singularity in the kernels is illustrated by numerical experiments.

Introduction

At the planning of electron-ray devices it is necessary to define an electrostatic field being formed by the charged electrode set, which is named as optoelectronic system. Making a suitable mathematical model each electrode has to be reasonably represented as unclosed surface with a given potential value. The effective algorithms for solving this problem can be obtained based on integral equation method (IEM), as since its application associates with the finding of unknown values only on the domain boundary. Nevertheless traditional using of this method is hampered over the electrode shape complicated sufficiently. Some difficulties emerge when electrode shape corresponds to unbounded surface as well. This and similar cases, when accepted surfaces are presented, demand to combine IEM with Green function means and to make domain decomposition.

Formulation of the problem

Assume that some closed separated Liapunov surfaces Σ_k $(k = \overline{1,n})$ are placed in Euclidean space $\mathbb{R}^3$. Let Ω_k^+ be a domain bounded by Σ_k. Let us note that

$$\Sigma = \bigcup_{k=1}^{n} \Sigma_k, \quad \Omega_k^- = \mathbb{R}^3 \backslash \overline{\Omega_k^+} \quad \text{where} \quad \overline{\Omega_k^+} = \Omega_k^+ \bigcup \Sigma_k, \quad \text{and}$$

$$\Omega^+ = \bigcup_{k=1}^{n} \Omega_k^+, \quad \Omega^- = \mathbb{R}^3 \backslash \overline{\Omega^+}, \quad \text{where} \quad \overline{\Omega^+} = \Omega^+ \bigcup \Sigma.$$

Suppose $S = \bigcup_{i=1}^{m} S_i$ be an aggregate of piecewise smooth open separated surfaces in unbounded domain Ω^-, and also $\overline{S} = \bigcup_{i=1}^{m} \overline{S}_i$, where $\overline{S}_i = S_i \bigcup \partial S_i$, ∂S_i is a piecewise smooth boundary of S_i $(i = \overline{1,m})$.

Let us calculate the electrostatic field, being formed by the charges distributed on surfaces Σ and S. It is necessary to define a potential u in domain $\Omega_s^- = \Omega^- \backslash \overline{S}$ under boundary conditions of the first kind. The generalized treatment of the sought solution

Key words and phrases. Dirichlet problem, two-dimensional integral equations, numerical methods.

has to be taken into account in the problem due to the suitable mathematical means being used during the integral equation obtaining and investigating [1].

Thus, we define the function $u \in H^1(\Omega_s^-, \Delta)$, which holds on

$$\Delta u = 0 \quad \text{in} \quad \Omega_s^-, \quad \gamma_0^- u = g_0, \quad \delta^{\pm} u = g_{\pm}, \quad \lim_{|P| \to \infty} u(P) = 0, \quad P \in \Omega_s^-, \qquad (1)$$

where $\gamma_0^- : H^1(\Omega_s^-) \to H^{1/2}(\Sigma)$, $\quad \delta^{\pm} : H^1(\Omega_s^-) \to H^{1/2}(S)$ are the trace maps [2], and

$$H^1(\Omega_s^-) = \left\{ u \;\middle|\; u,\ |\nabla u| \in L_2(\Omega_s^-) \right\},$$

$$H^1(\Omega_s^-, \Delta) = \left\{ u \;\middle|\; u \in H^1(\Omega_s^-),\ \Delta u \in L_2(\Omega_s^-) \right\}.$$

To approach problem (1) we use Schwartz method [3, 4], supposing that it is easy to solve Dirichlet problem in domain Ω_k^- for arbitrary $k \in \{1, 2, ..., n\}$ under conditions on Σ_k and $S^k = \bigcup\limits_{i_k \in M_k} S_{i_k}$, where $M_k \subset M' = \{1, 2, ..., m\}$, at that $\bigcup_{k=1}^{n} M_k = M'$. Then, iterative procedure on l-stage ($l = 1, 2, ...$) leads to determination of the function $u_{lk} \in H^1(\Omega_k^- \backslash \overline{S}^k, \Delta)$, which holds on the following for arbitrary k

$$\Delta u_{lk} = 0 \quad \text{in} \quad \Omega_s^- \backslash \overline{S}^k,$$

$$\gamma_0^- u_{lk} = \doteq \sum_{j=1, j \neq k}^{n} \gamma_0^- u_{l-1,j} \quad \text{on} \quad \Sigma_k, \quad \delta^{\pm} u_{lk} = - \sum_{j=1, j \neq k}^{n} \delta^{\pm} u_{l-1,j} \quad \text{on} \quad S^k, \qquad (2)$$

$$\lim_{|P| \to \infty} u_{lk}(P) = 0, \quad P \in \Omega_k^-,$$

where u_{0k} are the solutions of the problem resembling (1) under conditions g_0 and $g_{\pm}$ given onto Σ_k and S^k respectively. It is easy to verify that function

$$u = \sum_{i=0}^{\infty} \sum_{j=1}^{n} u_{ij} \quad \text{in} \quad \Omega_s^-$$

is a solution of problem (1) and convergence order of this series depends on the shape of domain Ω_s^-.

Thus, boundary value Dirichlet problem (2) for Laplace equation has to be solved by iterations in all domains $\Omega_s^- \backslash \overline{S}^k$. Therefore, without loss of generality it can be assumed that only surface Σ is given. Having supposed the existence of Green function in Ω^-, the solution of problem (1) can be represented as $u = u_1 + u_2$ [5, 6], where u_1 and u_2 satisfy the following

$$\Delta u_1 = 0 \quad \text{in} \quad \Omega^-, \quad \gamma_0^- u_1 = g_0, \quad \lim_{|P| \to \infty} u_1(P) = 0, \quad P \in \Omega^-, \qquad (3)$$

$$\Delta u_2 = 0 \quad \text{in} \quad \Omega_s^-, \gamma_0^- u_2 = 0, \quad \delta^{\pm} u = d_{\pm} = g_{\pm} - u_1, \quad \lim_{|P| \to \infty} u_2(P) = 0, \quad P \in \Omega_s^-. \qquad (4)$$

As is well known [7], the solution of problem (3) can be produced as

$$u_1(P) = - \iint_{\Sigma} \frac{\partial G(P, M)}{\partial n_M} g_0(M)\, d\Sigma_M, \quad P \in \Omega^-,$$

where $G(P, M)$ is a Green function of Laplace operator. That is why to define u it is necessary to solve problem (4). As it was proved [1, 5], the latter is equivalent to the

integral equation

$$\iint\limits_S G(P,M)\tau(M)\,dS_M = g(P), \quad P \in S, \tag{5}$$

where $\tau \in \left(H^{1/2}(S)\right)'$ is a sought density of surface charge, $g = \frac{1}{2}\left(d_- + d_+\right) + u_0$, at that

$$u_0(P) = \iint\limits_S \frac{\partial G(P,M)}{\partial n_M}\left[d_-(M) - d_+(M)\right] dS_M, \quad (d_- - d_+) \in H_{00}^{1/2}(S).$$

Let us note that

$$H_{00}^{1/2}(S) = \left\{ u \;\middle|\; u \in H^{1/2}(S),\ \omega^{-1/2} \cdot u \in L_2(S) \right\},$$

where ω is a function smooth enough and vanishing near ∂S as dist $(P, \partial S)$. The solution of problem (4) can be represented as

$$u_2(P) = \iint\limits_S G(P,M)\tau(M)\,dS_M - u_0(P), \quad P \in \Omega_s^-.$$

Equation (5) has a unique solution under $g \in H^{1/2}(S)$.

Moreover, surface Σ can be given as open and even unbounded in space. Only the existence of Green function is an essential condition for us. The electrostatic treatment of problem (1) means that $g_+ = g_- = g \equiv$ const and $d = d_+ = d_-$. Thus, $g = d$ in integral equation (5).

The ways to solve two-dimensional integral equations

At the establishment of equation's kind the shape of S does not really matter. The fact is that treatment of S as $\bigcup_{i=1}^m S_i$ makes no difficulties at the numerical analysis of integral equation (5). Thus, S is supposed to be single surface smooth enough with piecewise smooth contour ∂S. The solution of problem (1) is often independent of one argument, in particular, in the axially symmetric case. According to these constraints, equation (5) can be produced in the canonical form as

$$K\tau = (L+N)\tau \equiv \int\limits_a^b \left\{ \ln\frac{1}{|x-t|} + N(x,t) \right\} \tau(t)\,dt = g(x), \quad x \in (a,b), \quad |b-a| \neq 4. \tag{6}$$

where $g(x)$ and $N(x,t)$ are given functions smooth enough. The solvability of equation (6) can be analyzed in various functional spaces. Thus, if its solution belongs to Hölder-type space, which corresponds to the nature of phenomenon and the spaces $H^{1/2}(S)$ and $\left(H^{1/2}(S)\right)'$, having been used at the numerical analysis above, then it can be proved [8] that operator L is inverted continuously and operator N is compact. This means an existence of the left regulator L^{-1}. So, equation (6) can be replaced by the equivalent integral equation of the second kind. The latter is easy to solve by one of the known methods.

Let us consider integral equation (5) in the essential space case. Let S be represented parametrically

$$S = \left\{ x_i(\alpha,\beta) \in C^1(\mathcal{D}^2);\ i = 1,2,3;\ (\alpha,\beta) \in \mathcal{D}^2,\ \mathcal{D} = [-1,1] \right\}.$$

Then arbitrary point $M \in S$ has coordinates $\{x_i(\alpha,\beta)\}_{i=1}^3$ and some settled point $P \in S$ has coordinates $\{x_i(\alpha_0,\beta_0)\}_{i=1}^3$, where $(\alpha_0,\beta_0) \in \mathcal{D}^2$. Taking into account the known

properties of Green function it is easy to verify that kernel of integral equation (5) is a function of the distance between P and M. Square of the latter can be presented as

$$\mathrm{dist}^{\,2}(P,M)=\sum_{i=1}^{3}\left[x_i(\alpha,\beta)-x_i(\alpha_0,\beta_0)\right]^2.$$

According to the type of surface S, equation (5) can be represented as

$$\hat{K}\tau=(\hat{L}+\hat{N})\tau\equiv\iint\limits_{\mathcal{D}^2}\left[L(\alpha,\beta;\alpha_0,\beta)+N(\alpha,\beta;\alpha_0,\beta_0)\right]\tau(\alpha,\beta)\,d\alpha\,d\beta=g(\alpha_0,\beta_0),\quad(7)$$

where

$$L(\alpha,\beta;\alpha_0,\beta_0)=\frac{D(\alpha,\beta;\alpha_0,\beta_0)}{Q(\alpha,\beta;\alpha_0,\beta_0)},$$

at that functions D, N and g are smooth enough, and

$$Q(\alpha,\beta;\alpha_0,\beta_0)=\left[A(\alpha-\alpha_0)^2+2B(\alpha-\alpha_0)(\beta-\beta_0)+C(\beta-\beta_0)^2\right]^{1/2}.$$

Thus, equation (7) is a two-dimensional integral equation with the weak singularity in the kernel. There are some difficulties to investigate its solution with a priori properties. Moreover, there is no way to define the left regulator in the general case. That is why to obtain the solutions two ways have been used.

The first way is to use a priori information on the sought function response. It is some approach of the left regulator for equation (7). This general-purpose way leads to the solution, which best corresponds to phenomenon in the problems with electrode shape complicated sufficiently.

Let us represent function $1/Q(\alpha,\beta;\alpha_0,\beta_0)$ as

$$1/Q(\alpha,\beta;\alpha_0,\beta_0)=E(\alpha,\beta;\alpha_0,\beta_0)+F(\alpha,\beta;\alpha_0,\beta_0),\quad(8)$$

where

$$E(\alpha,\beta;\alpha_0,\beta_0)=\begin{cases}\dfrac{(\Delta_1-|\alpha-\alpha_0|)\cdot(\Delta_2-|\beta-\beta_0|)}{\Delta_1\cdot\Delta_2\cdot Q(\alpha,\beta;\alpha_0,\beta_0)}, & (\alpha,\beta)\in S_0;\\ 0, & (\alpha,\beta)\notin S_0;\end{cases}$$

$$F(\alpha,\beta;\alpha_0,\beta_0)=\begin{cases}\dfrac{\Delta_1\cdot|\beta-\beta_0|+\Delta_2\cdot|\alpha-\alpha_0|-|\alpha-\alpha_0|\cdot|\beta-\beta_0|}{\Delta_1\cdot\Delta_2\cdot Q(\alpha,\beta;\alpha_0,\beta_0)}, & (\alpha,\beta)\in S_0;\\ 1/Q(\alpha,\beta;\alpha_0,\beta_0), & (\alpha,\beta)\notin S_0.\end{cases}$$

Here $\Delta_1,\Delta_2>0$ are some parameters, and

$$S_0=\left\{(\alpha,\beta)\in\mathcal{D}^2\;\middle|\;|\alpha-\alpha_0|\le\Delta_1,\ |\beta-\beta_0|\le\Delta_2\right\}.$$

Let function $\tau(\alpha,\beta)\cdot\Omega(\alpha,\beta)$ be not much modified in S_0 under Δ_1 and Δ_2, being small enough, at that

$$\Omega(\alpha,\beta)=\frac{\left[(1-\lambda\alpha)\cdot(1-\mu\beta)\right]^{1/2}}{(1-\lambda\alpha)^{\sigma}+(1-\mu\beta)^{\sigma}},\quad(9)$$

where $\lambda,\mu\in\{-1,1\}$, σ is some physical constant. The drawing of λ and μ depends on location of S_0 in $\mathcal{D}^2$. Note that function $\Omega(\alpha,\beta)$ shows the singular response of the sought solution closed to the contour and corner points of open surface S. Using (8) and (9), we get an approximate analogue of (7)

$$\tau(\alpha_0,\beta_0)\cdot\Omega(\alpha_0,\beta_0)\cdot I(\alpha_0,\beta_0)+$$

$$+\iint\limits_{\mathcal{D}^2}(D\cdot F+N)(\alpha,\beta;\alpha_0,\beta_0)\cdot\tau(\alpha,\beta)\,d\alpha\,d\beta=g(\alpha_0,\beta_0),\quad(\alpha_0,\beta_0)\in\mathcal{D}^2\quad(10)$$

where

$$I(\alpha_0, \beta_0) = \iint\limits_{S_0} \frac{(D \cdot F)(\alpha, \beta; \alpha_0, \beta)}{\Omega(\alpha_0, \beta_0)} d\alpha d\beta,$$

$D \cdot F + N$ is a function smooth enough. Integral equation (10) can be treated as equation of the second kind. Thus, the proposed way is named as self-regulating method. Its effectiveness has been shown on the test problems [9].

The second way [6, 10] to solve integral equation (7) is to apply collocation method approximating the sought function by piecewise-linear, piecewise-square and piecewise-cube bases. Its effectiveness rises with the simplicity of the boundary surface system $\{S_i\}_{i=1}^m$. Geometry of each single surface and their relative position are important. It guarantees the solution stability of the primary problem.

Note that the obtained linear system of algebraic equations has linear combinations of two-dimensional proper and improper integrals in place of coefficients. The use of the high order cubic formula to approximate the integrals slows down the process to solve boundary value problem.

The analysis of actual optoelectronic systems shows the simplicity of their components with respect to the making in most cases. That is why, this simulation implies that geometrically true surfaces are made to agree with the elements comprising the optoelectronic systems. There are rectangular and trapezoid plates and constructions made by cylindrical surfaces. According to these constraints, most of the integrals mentioned above here been determined analytically.

This way does not needs the procedures with the guaranteed accuracy to approximate integrals. The time to solve whole problem slows down together with missing error of numerical integration. Without loss of generality let us give examples to demonstrate its features.

To define the charge distribution on two parallel rectangular plates S_1 and S_2 and spherical surface Σ located between them with radius R, specify Cartesian coordinates (x_1, x_2, x_3) in $\mathbb{R}^3$ and assume that plates are parallel to the plane $x_3 = 0$ and the distance between them equals D. Let S_i be represented as

$$S_i = \Big\{ x_1^i(\alpha, \beta) \equiv \alpha,\ x_2^i(\alpha, \beta) \equiv \beta,\ x_3^i(\alpha, \beta) \equiv \hat{C}_i;$$
$$(\alpha, \beta) \in \overline{\Delta}_i = [a_i, b_i] \times [c_i, d_i],\ b_i > a_i,\ d_i > c_i;\ \overline{\Delta}_i \subset \mathbb{R}^2;\ i = 1, 2 \Big\}.$$

Obviously, in this case integral equation (5) is as follows

$$\sum_{i=1}^{2} \iint\limits_{S_i} G(P_j, M_i) \tau(M_i)\, dS_{M_i} = \hat{g}(P_j) = g(P_j) - U_1(P_j). \tag{11}$$

Here $M_i \in S_i$ is a point of integration; $P_j = \left(x_k^j(\alpha_0, \beta_0) \right)_{k=1}^3$, $(\alpha_0 \beta_0) \in \overline{\Delta}_j$ is an arbitrary settled point, belonging to S_j; g is a total potential value on $S_1 \bigcup S_2$: $g(P_j) = g_j = \text{const}$; τ is a total charge density on $S_1 \bigcup S_2$; $dS_{M_i} = \sigma(\alpha, \beta)\, d\alpha\, d\beta$ is an element of surface S_i with respect to the local coordinates (α, β). Green function for domain located outside the sphere with radius R has the form

$$G(P_j, M_i) = \frac{1}{4\pi} \left[\frac{1}{\text{dist}\,(P_j, M_i)} - \frac{R}{\text{dist}\,(P_j, 0)} \cdot \frac{1}{\text{dist}\,(\hat{P}_j, M_i)} \right],$$

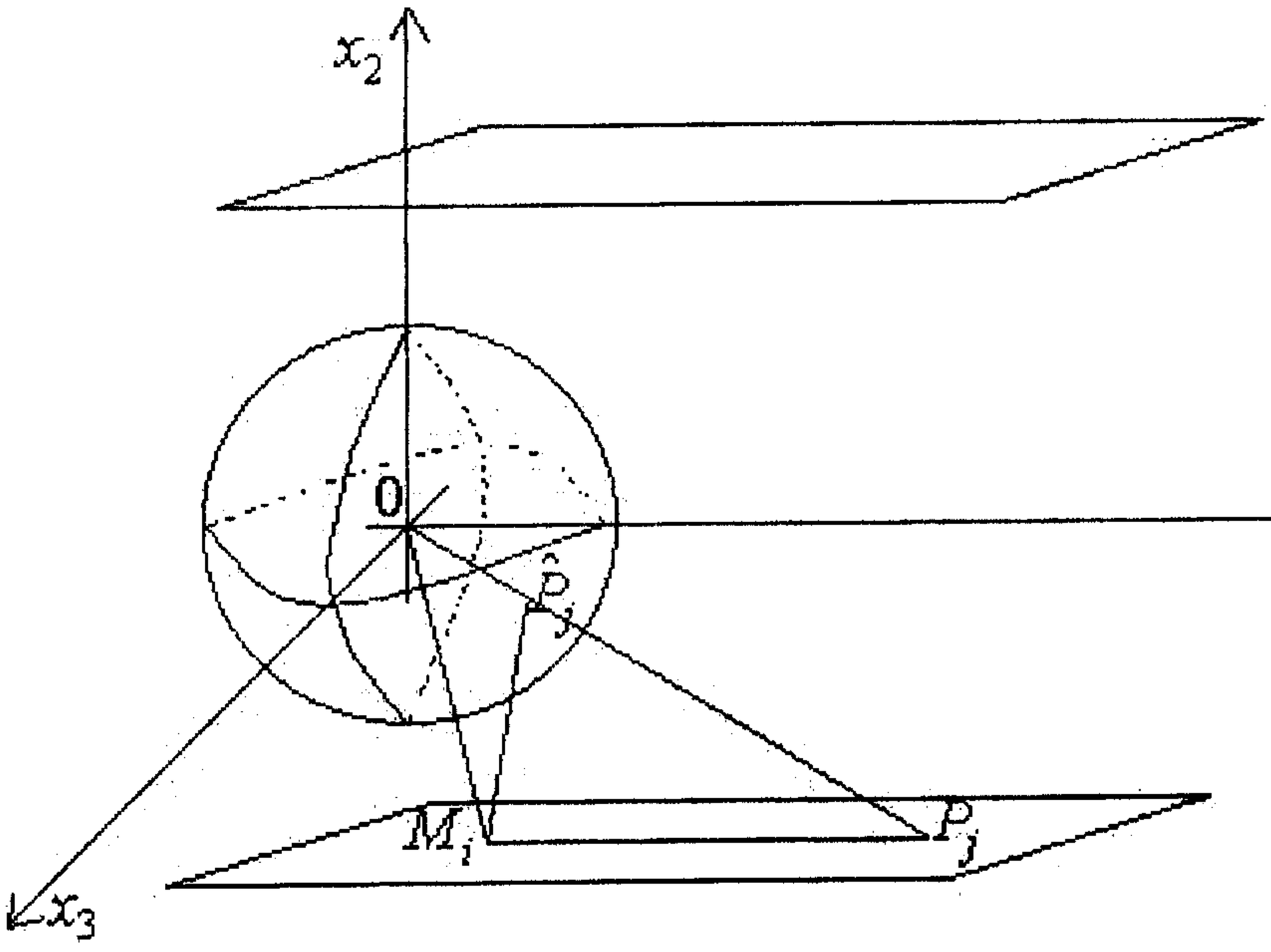

FIG. 1. The points to build Green function

where

$$\operatorname{dist}^2(P_j, M_i) = \sum_{k=1}^{3} \left[x_k^j(\alpha_0, \beta_0) - x_k^i(\alpha, \beta)\right]^2; \quad \operatorname{dist}^2(P_j, 0) = \sum_{k=1}^{3} \left[x_k^j(\alpha_0, \beta_0)\right]^2;$$

$$\operatorname{dist}^2(\hat{P}_j, M_i) = \sum_{k=1}^{3} \left[\hat{x}_k^j(\alpha_0, \beta_0) - x_k^i(\alpha, \beta)\right]^2; \quad \hat{x}_k^j(\alpha_0, \beta_0) = \frac{x_k^j(\alpha_0, \beta_0)}{\operatorname{dist}^2(P_j, 0)} \cdot R.$$

Assuming $\Delta_1 = \Delta_2$ and using the given parameterization, we have

$$\operatorname{dist}^2(P_i, M_i) = (\alpha - \alpha_0)^2 + (\beta - \beta_0)^2, \quad \operatorname{dist}^2(P_j, M_i) = (\alpha - \alpha_0)^2 + (\beta - \beta_0)^2 + D^2,$$

at that $D = \operatorname{dist}(S_1, S_2)$. To determine $U_1(P_j)$ at the right hand of integral equation (11) we use Poisson's integral

$$U(\rho_1, \theta_1, \varphi_1) = \frac{R}{4\pi} \int_0^{2\pi} \int_0^{\pi} \frac{\rho_1^2 - R^2}{\left(R^2 - 2\rho_1 R\cos(\gamma) + \rho_1^2\right)^{3/2}} g_0(\theta, \varphi) \sin(\theta)\, d\theta\, d\varphi,$$

where $(\rho_1, \theta_1, \varphi_1)$ are the spherical coordinates of point P_j; $g_0(\theta, \varphi) \equiv g_0 = \text{const}$ is a boundary potential value on sphere Σ with radius R, and

$$\cos(\gamma) = \cos(\theta)\cos(\theta_1) + \sin(\theta)\sin(\theta_1)\cos(\varphi - \varphi_1).$$

To solve equation (11) by collocation method by means of piecewise polynomial basic functions it is necessary to calculate improper integrals such as

$$I(m, n; a, b; x_0, y_0) = \int_{-1}^{1} \int_{-1}^{1} \frac{x^m y^n\, dx\, dy}{\sqrt{(ax - x_0)^2 + (by - y_0)^2 + d^2}}, \quad \begin{array}{l} m, n = 0, 1, 2, 3; \\ a, b > 0; d \geq 0. \end{array} \tag{12}$$

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To unify their shape let us make a change of variables, which corresponds to displacement of the frame center to the point (x_0, y_0), then

$$I(m,n;a,b;x_0,y_0) = a^{-(m+1)}b^{-(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n}\int_{e_1}^{e_2}\int_{f_1}^{f_2}\frac{u^i v^j\,du\,dv}{\sqrt{u^2+v^2+d^2}},$$

where $u = ax - x_0$, $v = by - y_0$, $e_1 = -a - x_0$, $f_1 = -b - y_0$, $e_2 = a - x_0$, $f_2 = b - y_0$, Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{ab},$$

and C_k^j are the coefficients of binomial formula. Thus, to determine integral (12) we have to calculate

$$J(m,n) = \int_{e_1}^{e_2}\int_{f_1}^{f_2}\frac{u^i v^j\,du\,dv}{\sqrt{u^2+v^2+d^2}}, \tag{13}$$

assuming $e_1 < e_2$, $f_1 < f_2$. With respect to (13) some formulas have been obtained [10]. For example,

$$J(3,1) = \frac{1}{15}\sum_{i,j=1}^{2}(-1)^{i+j}(3e_i^2 - 2f_j^2 - 2d^2)\cdot s_{ij}^3,$$

where

$$s_{ij} = \sqrt{e_i^2 + f_j^2 + d^2}.$$

Note that such a formula under $m = 2$ and $n = 0$ turned out to be most complicated

$$J(2,0) = \frac{1}{6}\sum_{i,j=1}^{2}(-1)^{i+j}e_i f_j s_{ij} +$$

$$+ \sum_{\begin{pmatrix} e & f & k \\ f & e & l \end{pmatrix}} k\sum_{i=1}^{2}(-1)^i e_i\left(s_{ii}^2 - f_i^2\right)\ln\left|\frac{f_2 + s_{i2}}{f_1 + s_{i1}}\right| - \frac{d^2}{3}e_{21}^{-}f_{21}^{-}I\left(0,0;e_{21}^{-},f_{21}^{-},e_{21}^{+},f_{21}^{+}\right),$$

where $k = 2$, $l = -1$, $e_{21}^{-} = (e_2 - e_1)/2$, $e_{21}^{+} = -(e_2 + e_1)/2$, $f_{21}^{-} = (f_2 - f_1)/2$, $f_{21}^{+} = -(f_2 + f_1)/2$, and

$$I\left(0,0;e_{21}^{-},f_{21}^{-},e_{21}^{+},f_{21}^{+}\right) = \frac{1}{ab}\sum_{\begin{pmatrix} a & b & x_0 & y_0 \\ b & a & y_0 & x_0 \end{pmatrix}}\sum_{i=1}^{2}a_i(x_0)\ln\left|\frac{b_2(y_0) + t_{i2}(x_0,y_0)}{-b_1(y_0) + t_{i1}(x_0,y_0)}\right| +$$

$$+2d\sum_{i,j=1}^{2}(-1)^{i+j}\operatorname{arctg}\left(\frac{1}{d}\left[(-1)^j b_j(y_0) - (-1)^i a_i(x_0) + t_{ij}(x_0,y_0)\right]\right),$$

$$a_i(x_0) = a + (-1)^i x_0, \quad b_j(y_0) = b - (-1)^j y_0, \quad t_{ij}(x_0,y_0) = \sqrt{a_i^2(x_0) + b_j^2(y_0) + d_2}.$$

Recurring to formulation of the problem, let us note that the described numerically analytical approach to solve two-dimensional integral equations of the first kind with weak singularity in the kernel has to be used, when charged plate S_i $(i = \overline{1,m})$ is an arbitrary quadrilateral. Really, let (x_{ij}, y_{ij}, z_{ij}), $j = \overline{1,4}$ be the coordinates of the plate corners. Then corresponding parametrical equations can be produced as

$$x_i(\alpha,\beta)=\frac{1}{4}\sum_{j=1}^{4}x_{ij}\varphi_j(\alpha,\beta),\quad \varphi_j(\alpha,\beta)=(1+(-1)^p\alpha)\cdot(1+(-1)^q\beta),$$

$$y_i(\alpha,\beta)=\frac{1}{4}\sum_{j=1}^{4}y_{ij}\varphi_j(\alpha,\beta),\quad p=\left[\frac{j}{2}\right]+1,\quad q=\left[\frac{j-1}{2}\right]+1,$$

$$z_i(\alpha,\beta)=\frac{1}{4}\sum_{j=1}^{4}z_{ij}\varphi_j(\alpha,\beta),\quad (\alpha,\beta)\in[-1,1]\times[-1,1].$$

Numerical experiments

Preferring the second way of solving the integral equation, let us estimate a potential of electrostatic field in $\mathbb{R}^3$-optoelectronic system (see Fig. 2). With respect to the electrodes potential value $g=0$ on cathode S_1, $g=-20$ on modulators S_2 and S_3, $g=15000$ on forming plates S_4 and S_5, $g=-1000$ on forming plates S_6 and S_7.

The obtained integral equation such as (7) has been solved by collocation method by means of bilinear and biquadratic basic functions. Each plate S_i ($i=\overline{1,7}$) has been divided into some elements, whose number is equal to 8×32, 4×8, 4×8, 8×16, 8×16, 8×16, 8×16 respectively.

Some of obtained potential values are given in Table 1. Distribution of the equipotential lines is shown in Fig. 3-5.

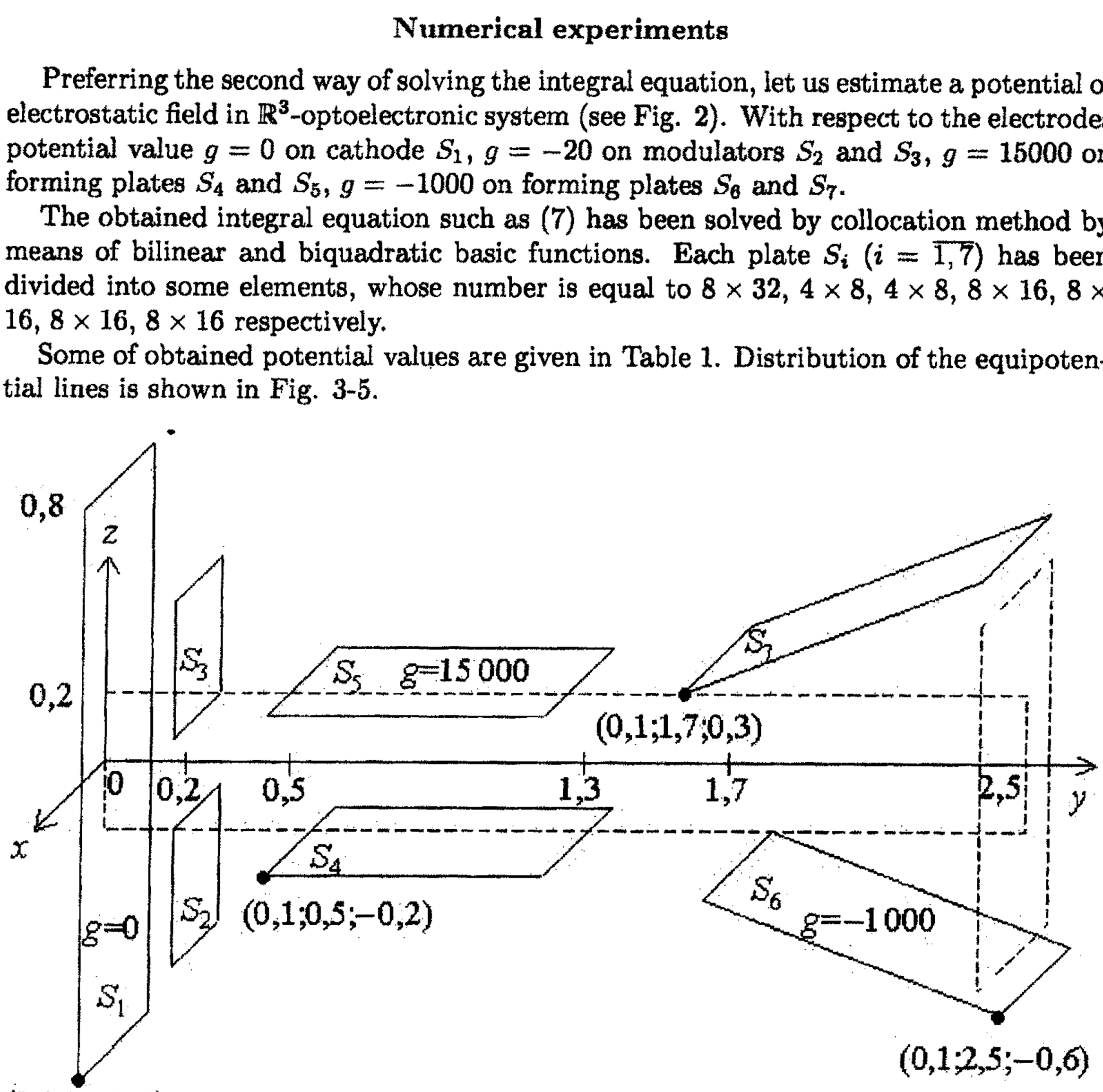

FIG. 2. Optoelectronic system

Conclusion

The proposed technique has prospects due to the possibility to expand it on cylindrical surfaces and well-known advantages of integral equation method.

The error estimation of charge density and its convergence complete the problem solving. The error is estimated a posteriori by means of the boundary condition satisfaction

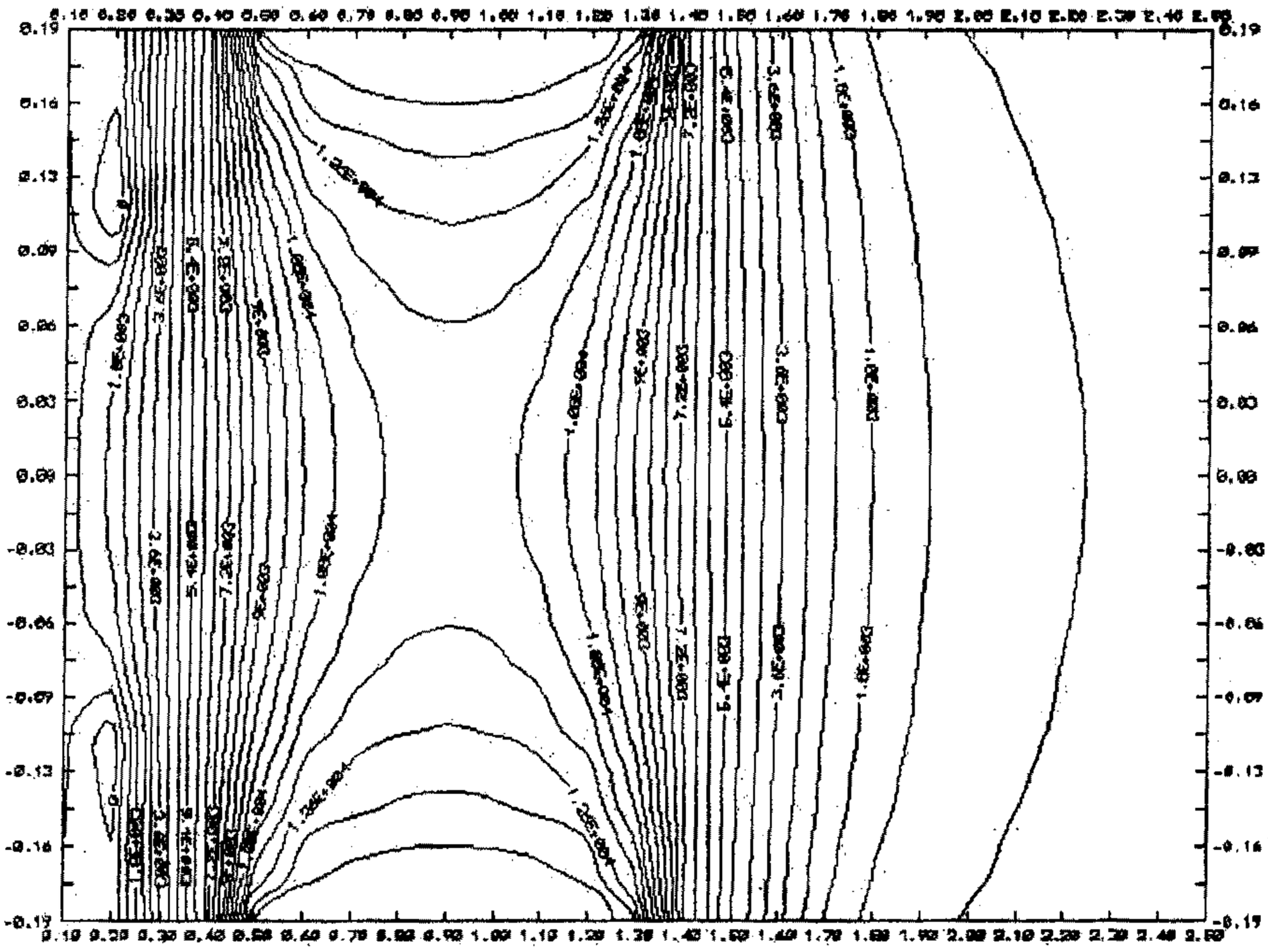

FIG. 3. Distribution of the equipotential lines along optoelectronic system

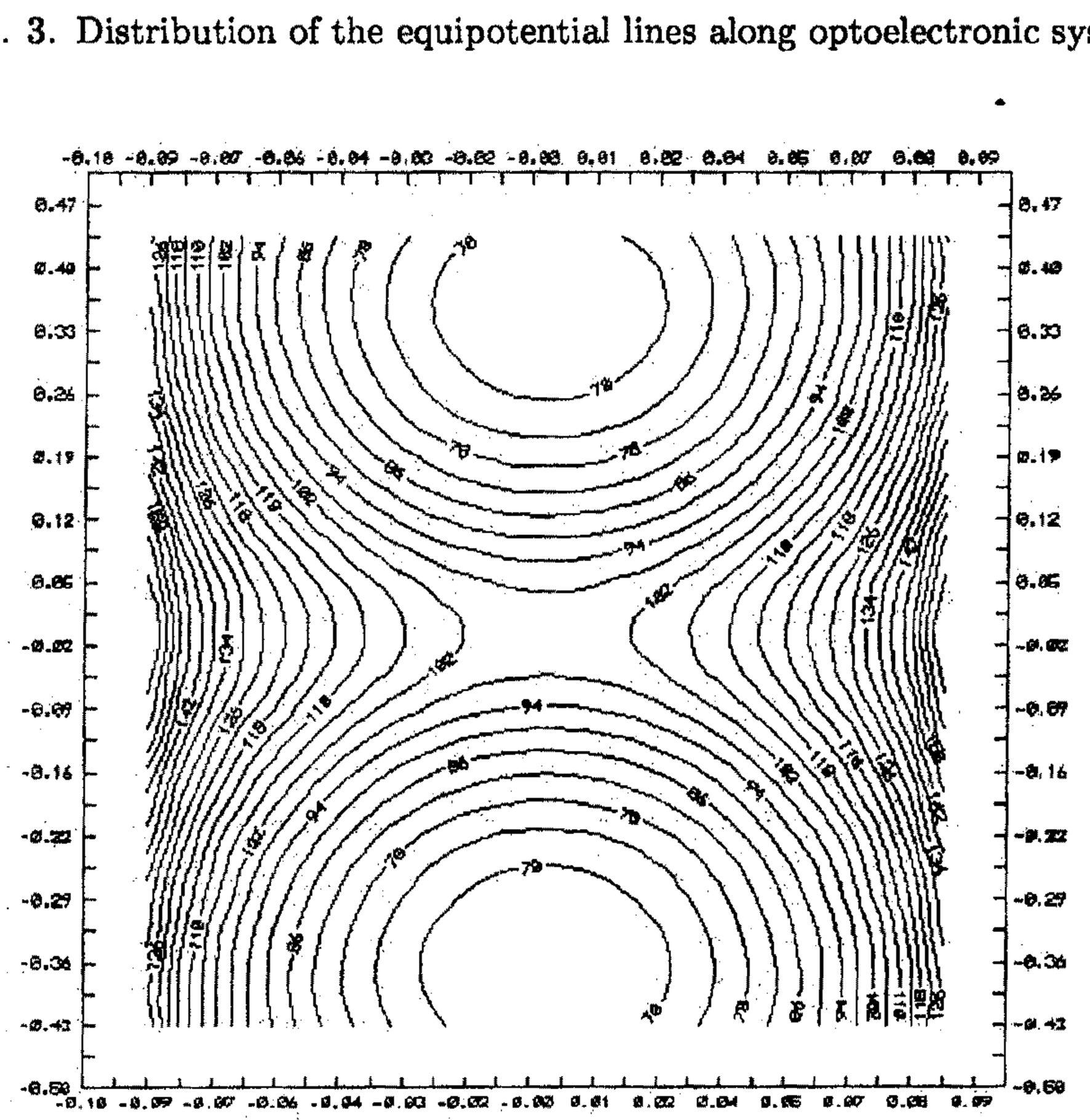

FIG. 4. Distribution of the equipotential lines close to cathode

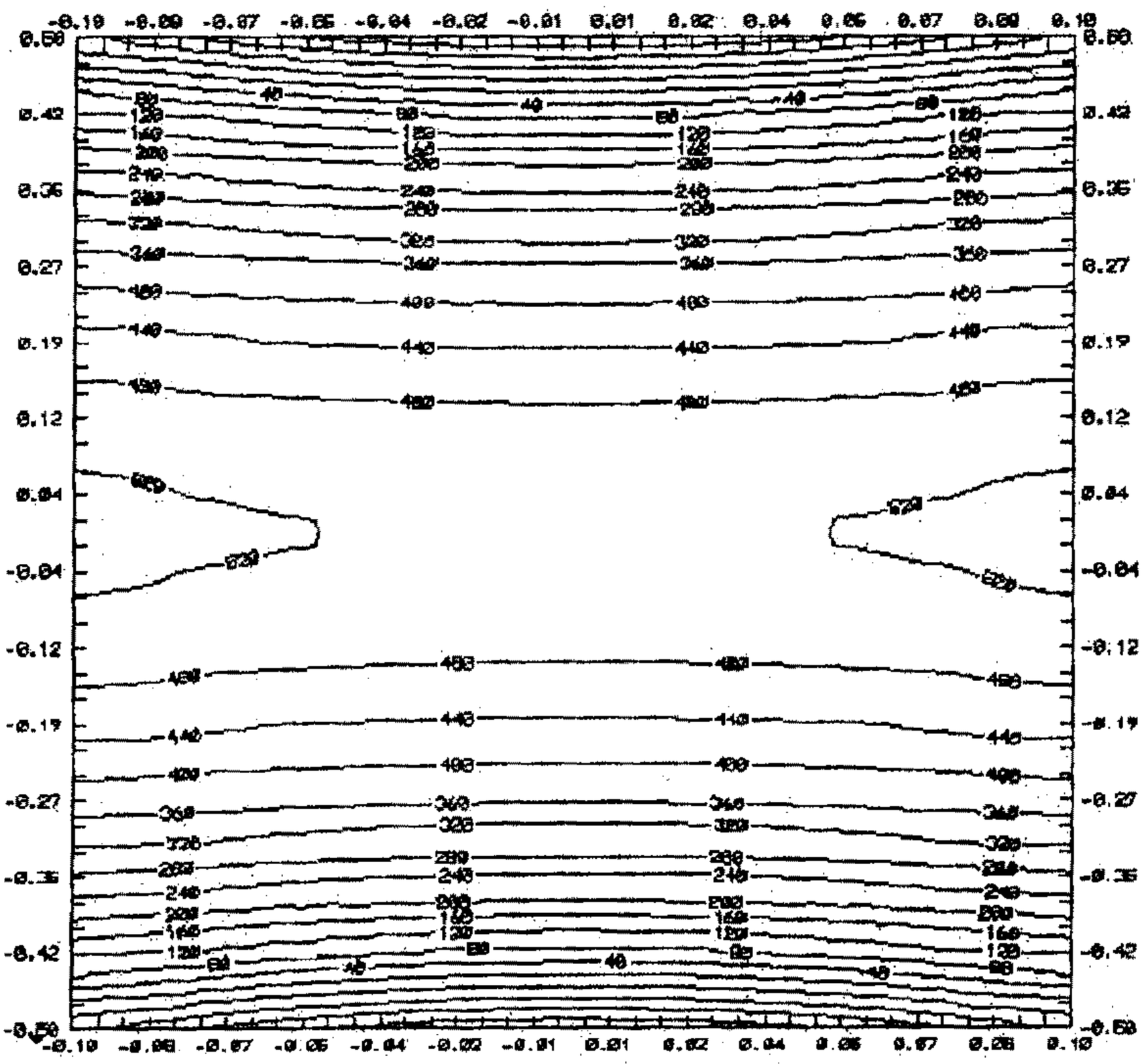

FIG. 5. Distribution of the equipotential lines in plane $y = 2.5$

TABLE 1. The potential of electrostatic field in the points of plane $x = 0$

y	z	$U(0, y, z)$	y	z	$U(0, y, z)$
0.100	−0.190	552.066	0.700	−0.190	14649.340
0.100	0.000	950.180	0.700	0.000	11133.542
0.200	−0.190	1.848	0.800	0.000	11540.765
0.200	0.000	1987.060	1.000	0.000	11566.096
0.300	−0.190	3475.094	1.300	0.000	8870.075
0.300	0.000	3915.523	1.500	0.000	5090.864
0.400	−0.190	6716.549	1.700	0.000	2524.879
0.400	0.000	6277.450	1.900	0.000	1248.640
0.500	−0.190	13809.470	2.100	0.000	744.238
0.500	0.000	8588.622	2.300	0.000	560.734
0.600	−0.190	14538.963	2.500	0.000	516.259
0.600	0.000	10247.443			

in the points located between the collocation points. As is well known, the approximation of integral operator and stability of approximate scheme determine the convergence order. When the boundary value problem is formulated correctly and surfaces are not distorted abruptly, the scheme stability is achieved under the problem solution on the sequence of thickened grids. Thus, the convergence order depends on the approximate order of the sought solution substantially.

Let us note that exhaustive theoretical substantiation of the collocation method being used here has not existed yet. Therefore, this task has to be carried out next.

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Received 3.02.2003