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ALGORITHMS WITHOUT ACCURACY SATURATION AND EXPONENTIAL CONVERGENT ALGORITHMS FOR OPERATOR EQUATIONS

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ABSTRACT. We consider the question, how many real (rational) numbers are necessary in order to represent an arbitrary element of a class of functions with a given tolerance ε. This number is the complexity measure for optimal algorithms for solution of various equations. We show that this measure is polynomial with respect to $\frac{1}{\varepsilon}$ or $\log \frac{1}{\varepsilon}$ provided that the solution under consideration belongs to usual function classes (Sobolev classes, classes of piece-wise analytical functions etc.) arising in applications. The algorithms arriving at this optimal measure has possess the accuracy automatically depending on the smoothness of the solution (algorithms without accuracy saturation) or an exponential accuracy. We give examples of such algorithms.

1 Why algorithms without accuracy saturation or exponential convergent algorithms?

The problem class which we are interesting in are various operator, differential and integral equations from various applied fields. A suitable meta-model to describe such equations are differential equations with operator coefficients in Hilbert or Banach spaces. These operator coefficients can be, for example, partial differential or integral operators, Poincaré-Steklov operators etc.

Let X be a Banach space of vector valued functions $u : \mathbb{R}_+ \to X,\ t \to u(t)$ and A be an operator in X. We can consider the first order differential equation

$$u'(t) + Au(t) = 0, u(0) = u_0. \tag{1.1}$$

The solution operator for this equation is per definition the operator exponential or semigroup with symbolic notation $T(t) = e^{-At}$. Given the solution operator $T(t)$ the solution of the initial value problem (1.1) can be represented by $u(t) = T(t)u_0$.

One of the simplest examples of a partial differential equation from the class (1.1) is the heat equation (of parabolic type)

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \ + \textit{initial and boundary conditions}. \tag{1.2}$$

This problem is of the type (1.1) if we define

$$\begin{gathered} D(A) = \{v \in H^2(0,1) :\ v(0) = 0, v(1) = 0\}, \\ Av = -\frac{d^2 v}{dx^2} \forall v \in D(A). \end{gathered} \tag{1.3}$$

For second order differential equations it is possible to consider boundary or initial value problems.

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Let us consider the following boundary value problem

$$\frac{d^2u}{dx^2} - Au = 0, \quad u(0) = 0, \quad u(1) = u_1 \tag{1.4}$$

for which the solution operator is per definition the so called normalized hyperbolic operator sine family

$$E(x) \equiv E(x; A) = \sinh^{-1}(\sqrt{A}) \sinh(x\sqrt{A}),$$

so that $u(x) = E(x)u_1$.

An example of a partial differential equation from the class (1.4) is the Laplace equation (of elliptic type)

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0, \ x \in [0,1], \ y \in [0,1],$$

$$u(0,y) = 0, \ u(1,y) = \phi(y)$$

$$u(x,c) = 0, \ u(x,d) = 0$$

which can be written down in the form (1.4) with the same operator A as above.

One can consider also the initial value problem

$$u''(t) + Au(t) = 0, u(0) = u_0, u'(0) = 0, \tag{1.5}$$

for which the solution operator is the operator cosine family $C(t) = C(t; A) = \cos\sqrt{A}t$, so that the solution of (1.5) can be given by $u(t) = C(t)u_0$. The classical example is the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, t \in (0,1) \ + \textit{initial and boundary conditions}$$

where the operator A is defined by (1.3).

In order to give a motivation for our future considerations let us turn to the simple question: what do we do when solving problems of above types numerically? In fact, the problem is to represent an element of, in general, infinite dimensional space X (continuum), say a function $f(t)$ through some n real numbers, i.e. by an element of $\mathbb{R}^n$. There are many ways to do it. For example, one can get the first n Fourier coefficients of f, values of n linear independent functionals on f etc. But the most widely widespread approximation method is to get n values of the function on some grid. Let $X \in C(\overline{D})$ be a compact in the function space and $\omega = \{t_1, t_2, ..., t_n\} \in D$ be a grid on D, then we want to represent any function $f(t) \in X$ by the vector $\xi = (f(t_1), ..., f(t_n))$ with a given tolerance ϵ. Now, the questions are: a) given a grid ω and the vector ξ how we can characterize the accuracy of such approximation?; b)given ϵ, how many numbers (grid points) we need to arrive at the given tolerance ϵ?

Let $\phi : X \to \mathbb{R}^n, f \to \xi = (f(t_1), ..., f(t_n))$ be the mapping corresponding to the grid ω. Then $d[\phi^{-1}(\xi)] = \sup_{g,h\in\phi^{-1}(\xi)} |g - h|_\infty$ can be a measure for the goodness of $\xi = \phi(f)$ as an approximation of f, the number $\mathcal{E}(X, \mathbb{R}^n; \phi) = \sup_{f\in X} d[\phi^{-1} \circ \phi(f)]$ characterizes the accuracy of the approximation in the worst case and finally the quantity $\Delta_n(X, C(\overline{D})) = inf_\phi \mathcal{E}(X, \mathbb{R}^n; \phi)$ shows how good an arbitrary element of X can be extremely well represented by its grid values. We call this quantity the grid n-width of X (see [4]). The asymptotics for n-widthes of various classes of functions arising in practical applications are well known (see, for example [4]).

Let $X = W_p^r(M; I)$ be the Sobolev class of functions defined on the interval $I = [0,1]$ which possess generalized derivatives up to the order r which are bounded by the constant M in the Chebyshev norm. It is known [4], p.232 that for this class it holds asymptotically $\Delta_n\{W_p^r(M; I)\} \asymp cn^{-r+p^{-1}}$ with some constant c independent of n (in the$l$-dimensional case of anisotropic Sobolev space $W_p^{\vec{r}}$ the parameter $\rho = (\sum_{j=1}^{l} r_j^{-1})^{-1}$ characterizes the so called effective smoothness [4], p.81). Thus, in order to approximate an arbitrary function of this class with a given tolerance ε we need $n_\varepsilon^{(opt)} \asymp const \left(\frac{1}{\varepsilon}\right)^{\frac{1}{-r+p^{-1}}}$ numbers (grid points).

Let us look how many numbers (grid points) we need in order to arrive at the same tolerance when solving the simple initial value problem

$$u''(t) = f(t), t \in (0,1); u(0) = u'(0) = 0$$

with a solution from the class $W_2^r(M; I)$ by a finite difference or finite element method. We introduce the equidistant grid

$$\omega_\tau = \{t_i = i\tau | i = 1, ..., n, \tau = \frac{1}{n}\}$$

with n points and consider the following finite difference scheme

$$y_\tau \equiv \frac{y(t+\tau) - y(t)}{\tau} = f(t + \tau/2), t \in \omega_\tau. \tag{1.6}$$

The following accuracy estimate is known [20], p.100

$$\|y - u\| C(\omega) \le c\tau |u|_{W_2^2(0,1)} = cn^{-1} |u|_{W_2^2(0,1)} (\asymp \varepsilon). \tag{1.7}$$

From the last equality we receive the asymptotical number of grid points we need to get the solution with the given tolerance ε:

$$n_\varepsilon^{FD} \asymp const \left(\frac{1}{\varepsilon}\right) \gg n_\varepsilon^{(opt)} \asymp const \left(\frac{1}{\varepsilon}\right)^{\frac{2}{3}}.$$

One can see that with the difference method we loose a lot of information about the solution and the distance between n_ε^{FD} and $n_\varepsilon^{(opt)}$ increases if the smoothness of the solution increases. This is due to the fact that the accuracy of the finite difference method beginning with a some threshold order remains constant independent of the smoothness of the solution.This effect is called the accuracy saturation [4]. This discrepancy between the theoretically needed number of grid points and the ones needed in a difference or FEM method in order to arrive at given tolerance is more critical for analytical solutions.

Let $K = I_0 = [-1, 1]$ be a real interval and E_r be the domain enveloped by the ellipse with the focal points -1 and 1 whose sum of semi-axes is equal to $r > 1$. Let $X(E_r, I_0; M)$ be the compact of continuous on I_0 functions with the usual Chebyshev norm $\|\cdot\|_\infty$ which can be extended analytically into E_r and are bounded by a positive constant M. For this class of functions it holds [4], p.260

$$\Delta_n(X(E_r, I_0; M)) \asymp Mr^{-n} \tag{1.8}$$

so that

$$n_\varepsilon^{(opt)} \asymp \log \varepsilon^{-1} \tag{1.9}$$

points (coordinates)are required to represent an arbitrary analytic function with the tolerance ε.

Solutions of many applied problems are analytic or piece wise analytic. If an algorithm for solution of such a problem uses a constant account of arithmetical operations per coordinate then the measure for complexity of this algorithm is of the order $\log \varepsilon^{-1}$. Let us suppose that an algorithm to find an analytic function $u(x)$ as the solution of an applied problem proceeds a vector y of n_a numbers where in some norm it holds $\|u - y\| \le \phi(n_a)$. In order to arrive at the tolerance ε with the (asymptotically) optimal coordinate numbers $n_a \asymp \log \frac{1}{\varepsilon}$ the function $\phi(n_a)$ must be exponential. On the other hand, in order to be able to keep the estimate $n_{opt} \asymp n_a \asymp \log \frac{1}{\varepsilon}$ the algorithm must possess a complexity

$C = C(n_a)$ of the order $n_a \asymp \log\frac{1}{\varepsilon}$ with respect to the account of arithmetical operations (we say the linear complexity). If $C(n_a)$ is proportional to $n_a log^\alpha n_a$ with α independent of n_a then we say that the algorithm possesses almost linear complexity. Thus, an optimal algorithm for analytic solutions has to be exponential convergent and possess the linear complexity and the value $\log\varepsilon^{-1}$ is a near optimal complexity measure. As a rule, the most frequent operations which an algorithm for solving PDE or integral equations has to carry out is the matrix-vector multiplication and the inversion of a matrix . It is known that the complexity of the last operation is asymptotically equal to the complexity of the matrix-matrix multiplication [1]. There exists a hypothesis [4] that there exists an algorithm for inversion of a general $(n \times n)-$ matrix with the complexity $\asymp n^{2+\epsilon}$ (multiplications) with an arbitrarily small positive ϵ , i.e. in general algorithms with linear or almost linear complexity are impossible. But there are special cases important for applied problems described by partial differential equations , for example, the FFT algorithm of almost linear complexity [1]. A new algorithmic approach with linear or almost linear complexity in BEM and FEM represent the so called hierarchical $\mathcal{H}-$ matrix [32, 32I,34].

In the analysis above we have assumed that algorithms can operate with real numbers. Indeed, it is not the case and computer algorithms operate with rational numbers only and we need a measure to estimate the goodness of a representation of functions from various classes by n rational numbers.

Let X be a compact in a functional space B with the norm $\|\cdot\|$, and f be an element of X. Let T_f be the word from an alphabet A_0 representing the element $f \in X$ and U be the encoding algorithm $U : T_f \to g_f, g_f \in B$ which proceeds an approximation g_f for f. We call the pare (T_f, U) the table of the element f and the quantity $\varepsilon_f = \|f - g_f\|$ the accuracy of the table (T_f, U). Indeed, the length of tables which we deal with in the praxis is bounded by some positive integer N. If we consider the set $\{T_f : f \in X,\ l(T_f) \le N\}$ of all tables with *length* $\le N$, then $\varepsilon_N = \sup_{f\in X}\varepsilon_f$ characterizes the accuracy of the representation of elements from X by tables of the length less or equal N.

Let $N(\varepsilon; X)$ be the minimal capacity of2ε - coverings of X by closed sets, then the value $H(\varepsilon; X) = \log N(\varepsilon; X)$ is called the ε-entropy of X (see [4], p.245).

It is known that a method using tables of the maximal length N for representation of elements of a Banach space possesses an accuracy ε if $N \ge H(\varepsilon; X)$ (see [4], p. 250), i.e. $H(\varepsilon; X)$ is the length of an optimal table to represent X with the given tolerance ε.

For the class $X(E_r, I_0; M)$ of analytic functions it holds [4], p.256

$$H(\varepsilon; X(E_r, I_0; M)) = \frac{1}{\log r}\log^2\frac{M}{\varepsilon} + \mathcal{O}(\log\frac{M}{\varepsilon}\log\log\frac{M}{\varepsilon}). \tag{1.10}$$

Since $\Delta_n\{X(E_r, I_0; M)\} \asymp Mr^{-n}$ one needs

$$n_\Delta(\varepsilon; X) = \frac{1}{\log r}\log\frac{M}{\varepsilon} + \mathcal{O}(1)$$

real numbers to represent an arbitrary element of X with the tolerance ε. Thus,

$$\frac{H(\varepsilon; X(E_r, I_0; M))}{n_\Delta(\varepsilon; X(E_r, I_0; M))} = \log\frac{M}{\varepsilon} + \mathcal{O}(\log\log\frac{M}{\varepsilon}) \tag{1.11}$$

bits per coordinate are necessary to represent an arbitrary element of $X(E_r, I_0; M)$ with a given tolerance ε. Under the assumption that we need at least one arithmetical operation per bit the optimal account of arithmetical operations for representation of an analytic function by n rational numbers of the maximal length N is asymptotically equal $\log^2 \frac{1}{\varepsilon}$. This quantity is also the minimal measure for the account of arithmetical operations for an arbitrary algorithm computing an approximation for this function.

Taking into account that for the anisotropic Sobolev space $X = W_p^{\bar{r}}$ it holds [4], p.267

$$c_1(\frac{1}{\varepsilon})^{1/\rho} \leq H(\varepsilon, X) \leq c_2[(\frac{1}{\varepsilon})^{1/\rho} + \log \frac{1}{\varepsilon}] \tag{1.12}$$

and [4], p.234

$$n_\Delta \asymp (\frac{1}{\varepsilon})^{p/(\rho p-1)}, \rho p > 1, 1 \leq p \leq \infty \tag{1.13}$$

we get similarly that not more then

$$\varepsilon^{1/(\rho(\rho p-1))} + \varepsilon^{p/(\rho p-1)} \log \frac{1}{\varepsilon} \tag{1.14}$$

bits per coordinate are necessary to represent an arbitrary element of $X = W_p^{\bar{r}}$ with a given tolerance ε.

Considering differential equations with operator coefficients one has often to analyze related operator equations (see e.g. [28]) which are also of independent interest. A classical example of such equation is the Lyapunov or Silvester equation

$$UX + XV = Y \tag{1.15}$$

with given operators U, V, Y and unknown operator X. One can also in this case reasonably define the smoothness of solutions of such operator equations and expect algorithms without accuracy saturation for their numerical approximations (see e.g. [12,28]).

The consideration above is the motivation for our aims: 1. Construction of algorithms without accuracy saturation. 2. Construction of exponentially convergent algorithms for analytic solutions with a complexity which is polynomial with respect to $\log \frac{1}{\varepsilon}$.

Below we give a short but not complete survey on various results about algorithms without accuracy saturation or exponentially convergent algorithms developed with a significant contribution by V.L.Makarov.

2 Classes of operators

We begin this section with the definition of positive and strongly positive operators which play a fundamental role in the theory of the first order differential equations with an operator coefficient (see e.g. [8, 35]).

Definition 2.1. We say that an operator A is ***positive***, if

$$\Sigma^+ = \left\{z \in \mathbb{C} : 0 < \varphi \leq |\arg z| \leq \pi\right\} \cup \left\{z \in \mathbb{C} : |z| \leq \gamma\right\} \subset \varrho(A)$$

and

$$\|(z-A)^{-1}\| \leq \frac{M}{1+|z|} \qquad \text{for all} \quad z \in \Sigma^+$$

for some positive constants φ, γ and M. The lower bound of all such φ, for which the relations above hold, is called the ***spectral angle*** of the positive operator A and will be denoted by $\varphi(A; E)$ or simply $\varphi(A)$.

A positive operator A is called ***strongly positive*** if $\varphi(A) < \frac{\pi}{2}$.

Let Γ be a closed path in the complex plane $\mathbb{C}$ which consists of two rays

$$S(\pm\varphi) = \left\{\varrho e^{\pm i\varphi} : \gamma \leq \varrho \leq +\infty\right\}$$

and of the circular arc

$$\left\{z \in \mathbb{C} : |z| = \gamma, \ |\arg z| \leq \varphi, \ \varphi(A) < \varphi < \frac{\pi}{2}\right\}.$$

The domain Ω_Γ bounded by Γ contains the spectrum of A. If $M = 1$ and $\varphi = \frac{\pi}{2}$, then $-A$ is the infinitesimal generator of a C_0-semigroup [35], p. 69. If $\varphi(A) < \frac{\pi}{2}$, i.e. the operator A is strongly positive, then $-A$ is the infinitesimal generator of an analytic semigroup [35], p. 69. For an analytic function $f = f(z)$ in Ω_Γ one can define the operator $f(A)$ by

$$f(A) = \frac{1}{2\pi i}\int_\Gamma f(z)(z-A)^{-1}dz$$

where the orientation of Γ is chosen so that the spectrum of A lies on the left. In particular, for $\sigma > 0$ we have

$$A^{-\sigma} = \frac{1}{2\pi i}\int_\Gamma z^{-\sigma}(z-A)^{-1}dz$$

where $z^{-\sigma}$ is taken to be positive for real positive values of z. If $\sigma = n$ is an integer, then using the residue theorem it follows that the integral equals A^{-n}. Thus, for positive integer values of σ the definition of $A^{-\sigma}$ above coincides with the classical definition of $(A^{-1})^n$. The operator A^σ $(\sigma > 0)$ is defined as $(A^{-\sigma})^{-1}$.

Strongly P-positive operators were introduced in [9] and play an important role in the theory of the second order difference equations [22], evolution differential equations as well as the cosine operator family in a Banach space X [9] .

Let $A : X \to X$ be a linear, densely defined, closed operator in X with the spectral set $sp(A)$ and the resolvent set $\rho(A)$. Let $\Gamma_0 = \{z = \xi + i\eta : \xi = a\eta^2 + \gamma_0\}$ be a parabola, whose interior contains $sp(A)$. In what follows we suppose that the parabola lies in the right half-plane of the complex plane, i.e., $\gamma_0 > 0$. We denote by $\Omega_{\Gamma_0} = \{z = \xi + i\eta : \xi > a\eta^2 + \gamma_0\}$, $a > 0$, the domain inside of the parabola. Now, we are in the position to give the following definition.

Definition 2.2. We say that an operator $A : X \to X$ is strongly P-positive if its spectrum $sp(A)$ lies in the domain Ω_{Γ_0} and the estimate

$$\|(zI - A)^{-1}\|_{X\to X} \le \frac{M}{1+\sqrt{|z|}} \qquad \text{for all } z \in \mathbb{C}\backslash\Omega_{\Gamma_0} \tag{2.1}$$

holds true with a positive constant M.

In [9] it was be shown that inequality (2.1) holds true in $\mathbb{C}\backslash\Omega_{\Gamma_0}$ for an elliptic partial differential operator $\mathcal{L}$ and the parameters of the parabola Γ_δ enveloping the spectrum were determined by the coefficients of this differential operator (see the discussion in [[]pp. 330-331]9 and [14]). If the spectrum lies in the right half-plane then the coefficients of a parabola Γ_0 lying in the right half-plane and enveloping the spectrum can be determined by the parameters of Γ_δ.

Sometimes consideration of special subclasses of operators leads to an improvement of various algorithms. These classes are defined by functions describing the behavior of the resolvent and the spectral set at the infinity. Let Γ_S be a curve in the complex plane $z = \xi + i\eta$ defined by the equation $\xi = f_S(\eta)$ in the coordinates ξ, η. We denote by

$$\Omega_{\Gamma_S} := \{z = \xi + i\eta : \xi > f_S(\eta)\} \tag{2.2}$$

the domain inside of Γ_S. In what follows, we suppose that this curve lies in the right half-plane of the complex plane and contains $sp(A)$, i.e., $sp(A) \subset \mathbb{C}\backslash\Omega_{\Gamma_S}$. Now, we are in the position to give the following definition.

Definition 2.3. Given an operator $A : X \to X$. Let $f_S(\eta)$ and $f_R(z)$ be functions such that

$$\|(zI - A)^{-1}\|_{X\to X} \le f_R(z) \qquad \text{for all } z \in \mathbb{C}\backslash\Omega_{\Gamma_S}. \tag{2.3}$$

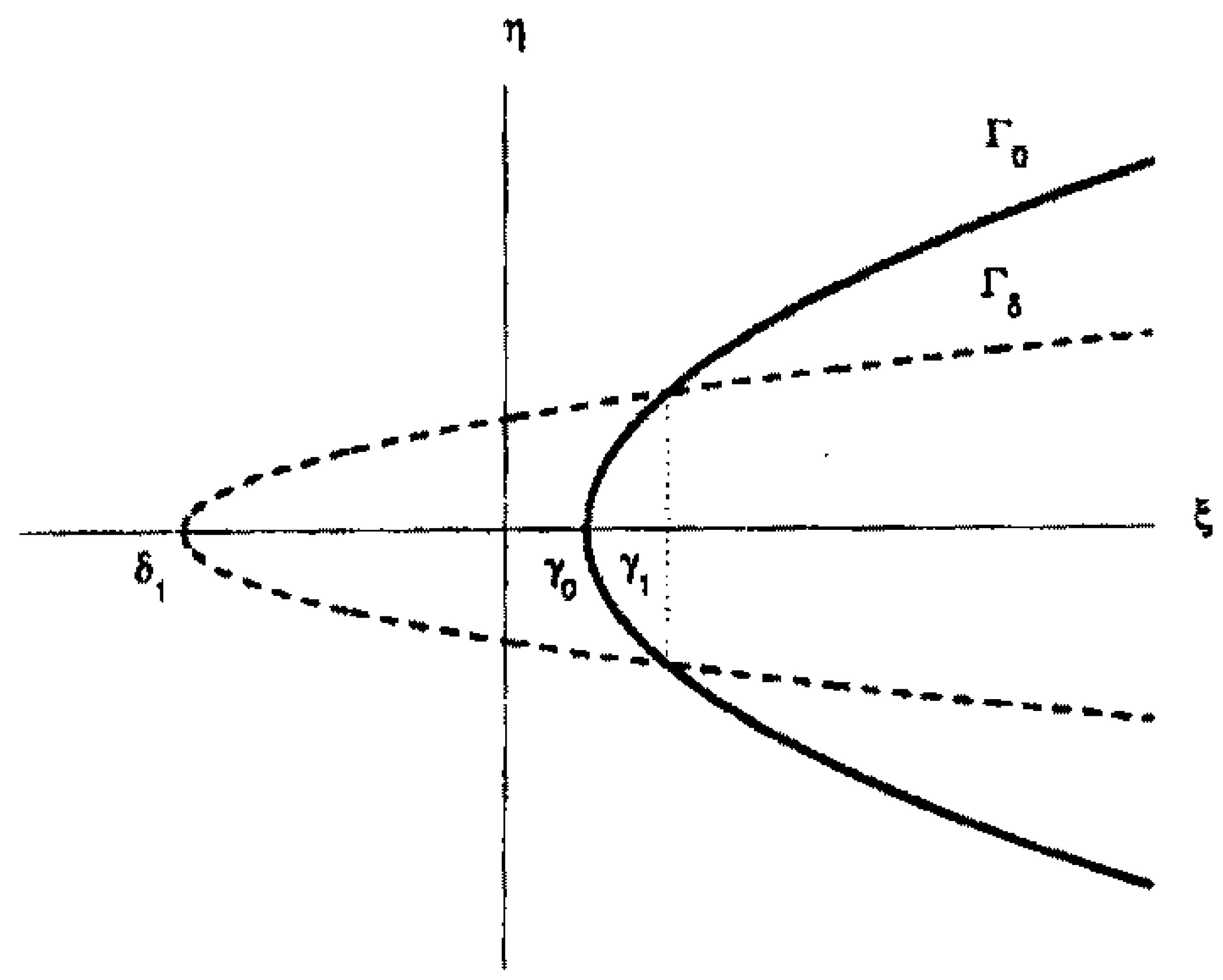

FIG. 1. Parabolae Γ_δ and Γ_0

Note that Γ_S is defined by means of f_S. Then we say that the operator $A : X \to X$ is of (f_S, f_R)-type.

Note that a strongly P-positive operator is also an operator of (f_S, f_R)-type with the special choice

$$f_S(\eta) = a\eta^2 + \gamma_0, \quad f_R(z) = M/(1+\sqrt{|z|}), \quad a > 0, \quad \gamma_0 > 0, \quad M > 0. \tag{2.4}$$

We use also the subclass of operators with an exponential function $f_S(\eta) = a \cosh b\eta$ in order to get exponentially convergent algorithms or to improve the exponential convergence rate.

The Cayley transform of an operator A

$$T_\gamma = (\gamma I - A)(\gamma I + A)^{-1}, \tag{2.5}$$

where I is the identity operator and γ is an arbitrary complex number, is well-known in operator theory and possesses many useful properties. For example, if A is a densely defined, strictly dissipative unbounded operator in some Hilbert space H, then the operator T_γ is contrative. In what follows we call also other rational transforms like (2.5) as the Cayley transforms.

3 Algorithms without accuracy saturation based on the Cayley transform

3.1 The first order differential equations with an operator coefficient

Let us consider the initial value problem

$$\frac{du}{dt} + Au = 0, \; u(0) = 0, \tag{3.1}$$

where $u(t)$ is a vector-valued functions with values in a Banach space X and A is an operator in X.

The discrete initial value problem

$$\begin{gathered} y_{\gamma,n+1} = T_\gamma^n y_{\gamma,n} \quad (n = 0, 1, \dots) \\ y_{\gamma,0} = x_0 \end{gathered} \tag{3.2}$$

with the Cayley transform $T_\gamma = T_\gamma(A) = (\gamma I - A)(\gamma I + A)^{-1}$ and an arbitrary positive real number γ is regarded together with problem (3.1). It was shown in [21, 31] that the solutions of these problems and the corresponding continuous and discrete semigroups

$\{T(t)\}_{t\geq 0}$ and $\{T_\gamma^n\}_{n\geq 0}$, respectively, are connected by the formulas

$$\begin{aligned} x(t) &= T(t)x_0 = \sum_{p=0}^{\infty}(-1)^p\varphi_p(2\gamma t)\, y_{\gamma,p} \\ y_{\gamma,p} &= T_\gamma^p y_{\gamma,0} = (-1)^{p+1}\left[\int_0^{+\infty}\psi_n(t)\, x\left(\frac{t}{2\gamma}\right)dt + x_0\right] \\ T(t) &= \sum_{p=0}^{+\infty}(-1)^p\varphi_p(2\gamma t)\, T_\gamma^p \\ T_\gamma^p &= (-1)^p\left[\int_0^{+\infty}\psi_p(t)\, T\left(\frac{t}{2\gamma}\right)dt + I\right] \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} \varphi_p(t) &= -\frac{t}{p}\, e^{-\frac{t}{2}} L_{p-1}^{(1)}(t), \quad |\varphi_p(t)| \leq 1 \text{ for all } p\geq 0 \\ \psi_p(t) &= -e^{-\frac{t}{2}} L_{p-1}^{(1)}(t) = e^{-\frac{t}{2}}\frac{d}{dt}\, L_p^{(0)}(t) \end{aligned} \tag{3.4}$$

with Laguerre polynomials $L_p^{(\alpha)}$. The approximate solution x^N of problem (3.1) (and the algorithm) is defined by

$$x^N(t) = \sum_{p=0}^{N}(-1)^p\varphi_p(2\gamma t)\, y_{\gamma,p}. \tag{3.5}$$

The following *error estimate* holds

$$\|x^N(t) - x(t)\| \leq c\, N^{-\sigma+\delta}\|A^\sigma x_0\| \tag{3.6}$$

uniformly in $t \in [0,\infty)$ provided that $x_0 \in D(A^\sigma), \sigma > 0$ and δ is an arbitrary number from the interval $(0,\sigma)$. This estimate shows that (3.5) represents an algorithm without accuracy saturation.

Assumptions: We suppose in (3.1) the operator A to be strongly positive.

3.2 The second order differential equations with an unbounded operator coefficient

3.2.1 The elliptic case. We consider the following second order boundary value problem (the elliptic case, a partial example is the Poisson equation)

$$\frac{d^2u}{dx^2} + Au = 0, u(0) = 0, u(1) = u_1. \tag{3.7}$$

The solution can be represented by

$$u(x) \equiv \sinh^{-1}\sqrt{A}\sinh^{-1}\sqrt{A}xu_1 = \sum_{k=0}^{\infty} v_k(x)y_k \tag{3.8}$$

with

$$y_k = Ty_{k-1} = T^k u_1 \tag{3.9}$$

where $T = (I+A)^{-1}A$ is the Cayley transform, $v_k(x) = \frac{P_{2k+1}(x)}{(2k+1)!}$ and $P_k(x)$ is the Meixner polynomial [19] .

The algorithm:

$$u_N(x) = \sum_{k=0}^{N} v_k(x) y_k. \tag{3.10}$$

The error estimate:

$$\sup_{x\in[0,1]} \|u(x) - u_N(x)\| \le cN^{-(\sigma-\varepsilon)} \|A^\sigma u_1\|. \tag{3.11}$$

Assumptions: A is strongly positive, $u_1 \in D(A^\sigma)$.

3.2.2 The hyperbolic case. We consider the following second order boundary value problem (the hyperbolic case, a partial example is the wave equation)

$$\frac{d^2u}{dt^2} + Au = 0, u(0) = u_0, u'(0) = 0. \tag{3.12}$$

The solution can be represented by

$$u(t) \equiv \cos\sqrt{A}tu_0 = \frac{1}{2} e^{-\delta t} \sum_{k=0}^{\infty} (L_k^{(0)}(t) - L_{k-1}^{(0)}(t)) y_k \tag{3.13}$$

with

$$y_k = 2T_1 y_{k-1} - T_2 y_{k-2} \tag{3.14}$$

where $T_1 = (A + (\delta-1)^2 I)^{-1}(A + \delta(\delta-1)I), T_2 = (A + (\delta-1)^2 I)^{-1}(A + \delta^2 I)$ are the Cayley transforms, $L_k^{(0)}(t)$ is the Laguerre orthogonal polynomial, δ is an arbitrary real number such that $\delta < 0.5$ [9].

The algorithm:

$$u(t) \approx u_N(t) = \frac{1}{2} e^{-\delta t} \sum_{k=0}^{N} (L_k^{(0)}(t) - L_{k-1}^{(0)}(t)) y_k. \tag{3.15}$$

The error estimate:

$$\sup_{t\in[0,T]} \|u(t) - u_N(t)\| \le cN^{-(\sigma-3/4\varepsilon)} \|A^\sigma u_0\|, \tag{3.16}$$

$\varepsilon > 0$ is arbitrarily small.

Assumptions: A is strongly positive, $u_0 \in D(A^\sigma)$.

3.3 The equation $X = A^\lambda$

Fractional powers of operators are used in various applications. The following representation holds true [12]

$$A^\lambda = \alpha^\lambda \sum_{m=0}^{\infty} (-1)^m C_{2m}^\lambda(0)\, T_\alpha^m, \quad \lambda \in (0,1), \tag{3.17}$$

where $C_n^\lambda(x)$ are the Gegenbauer (ultraspherical) polynomials,

$$T_\alpha = (A - \alpha)\, A^{-1}, \tag{3.18}$$

is the Cayley transform, α is defined by the spectral characteristics of the operator A (the last ones for an elliptic operator A can be explicit calculated through its coefficients).

From (3.17) we get

$$A^\lambda = \alpha^\lambda \sum_{m=0}^{\infty} \frac{(\lambda)_m}{m!} T_\alpha^m, \tag{3.19}$$

where

$$(\lambda)_m = \frac{\Gamma(\lambda+m)}{\Gamma(\lambda)} = \lambda(\lambda+1)\cdots(\lambda+m-1),$$

$\Gamma(z)$ is the gamma function, since

$$C_n^\lambda(0) = \begin{cases} 0 & \text{if} n = 2m+1, \\ \frac{(-1)^m(\lambda)_m}{m!} & \text{if} n = 2m. \end{cases}$$

The algorithm:

$$A^\lambda \approx A_N^\lambda = \alpha^\lambda \sum_{m=0}^{N} \frac{(\lambda)_m}{m!} T_\alpha^m. \tag{3.20}$$

The error estimate:

$$\|(A^\lambda - A_N^\lambda)x\| \le cN^{-(\sigma-\lambda-\varepsilon)}\|A^\sigma x\| \tag{3.21}$$

with an arbitrarily small $\varepsilon > 0$.

Assumptions: A is strongly positive, $x \in D(A^\sigma)$.

3.4 The equation $UX + XV = Y$

Let us consider the operator equation [28]

$$SX = Y \tag{3.22}$$

with the elementary operator S given by

$$SX = \sum_{j=1}^{M} U_j X V_j \tag{3.23}$$

where U_j, V_j are given commutative subsets of a complex Banach algebra of operators, Y is a given and X the unknown operator. Let the following expansion be valid

$$P^{-1}(\lambda,\mu) = \left(\sum_{j=1}^{M} \lambda_j \mu_j\right)^{-1} = \sum_{p=0}^{\infty} \sum_{|m|+|n|=p} d_{mn} z^m d^n, \tag{3.24}$$

where $m = (m_1, ..., m_M)$, $n = (n_1, ..., n_M)$,

$$|m| = \sum_{i=1}^{M} m_i, \quad |n| = \sum_{i=1}^{M} n_i, \quad z^m = z_1^{m_1} \cdots z_M^{m_M}.$$

Then the solution of (3.23) can be represented by

$$X = \sum_{p=0}^{\infty} (-1)^p \sum_{|m|+|n|=p} d_{mn} \prod_{j=1}^{M} T_{\gamma_j}^{m_j}(U_j) Y T_{\delta_j}^{n_j}(V_j), \tag{3.25}$$

where $T_{\gamma_j}(U_j) = (\gamma_j I - U_j)(\gamma_j I + U_j)^{-1}$, $T_{\delta_j}(V_j) = (\delta_j I - V_j)(\delta_j I + V_j)^{-1}$ with arbitrary positive numbers γ_j, δ_j are the Cayley transforms of the operators U_j and V_j respectively.

The algorithm:

$$X_N = \sum_{p=0}^{N} (-1)^p \sum_{|m|+|n|=p} d_{mn} \prod_{j=1}^{M} T_{\gamma_j}^{m_j}(U_j) Y T_{\delta_j}^{n_j}(V_j). \tag{3.26}$$

The error estimate:

$$\|X - X_N\| \le c\|Y\| N^{2M-1} q^N, q \in (0,1), \tag{3.27}$$

provided that U_j, V_j are bounded operators and

$$\|X - X_N\| \le cN^{-\sigma+1+\varepsilon}\|\tilde{Y}\|, \tag{3.28}$$

provided that $\tilde{Y} = (\prod_{j=1}^{M} U_j^{\sigma_j})Y(\prod_{j=1}^{M} V_j^{\theta_j})$ is a bounded operator (smoothness property!) for some positive σ_j, θ_j, where $\sigma = \min(\sigma_j, \theta_j)$ and $\varepsilon > 0$ is an arbitrarily small number.

Assumptions: U_j, V_j are strongly positive operators.

For the particular case of the Silvester and Ljapunov equations

$$UX + XVB = Y \tag{3.29}$$

we get

$$X = \frac{1}{2\gamma}\{Y + \hat{Y} + 2T_\gamma(U)YT_\gamma(V) + T_\gamma(U)\hat{Y}T_\gamma(V) \\ +2T_\gamma^2(U)YT_\gamma^2(V) + T_\gamma^2(U)\hat{Y}T_\gamma^2(V) + \cdots\}, \tag{3.30}$$

where $\hat{Y} = T_\gamma(U)Y + YT_\gamma(V)$. For X_N being the sum of the $N+1$ terms and for bounded U, V, Y we have

$$\|X - X_N\| \le 2\|Y\|\frac{q^{N+1}}{1-q} \tag{3.31}$$

with $q = \max\{\|T_\gamma(U)\|, \|T_\gamma(V)\|\} < 1$.

Exponentially convergent parallel algorithms for the Silvester equation were proposed in [17].

4 Two exponential convergent algorithms

In this section we describe two exponential convergent algorithms for approximation of solution operators for parabolic and elliptic PDEs.

4.1 Approximation of the operator exponential

Here we outline the description of the operator exponent with a strongly P-positive operator. As a particular case a second order elliptic differential operator will be considered. We derive the characteristics of this operator which are important for our representation and give the approximation results.

Let $\Gamma_0 = \{z = \xi + i\eta : \xi = a\eta^2 + \gamma_0\}$ be the spectral parabola defined as above and containing the spectrum $sp(\mathcal{L})$ of the strongly P-positive operator $\mathcal{L}$. The following assertion was proved in [24,14].

Lemma 4.1. *Choose a parabola (called the integration parabola)* $\Gamma = \{z = \xi + i\eta : \xi = \tilde{a}\eta^2 + b\}$ *with* $\tilde{a} \le a$, $b \le \gamma_0$. *Then the exponent* $\exp(-t\mathcal{L})$ *can be represented by the Dunford-Cauchy integral* [6]

$$\exp(-t\mathcal{L}) = \frac{1}{2\pi i}\int_\Gamma e^{-zt}(zI - \mathcal{L})^{-1}dz. \tag{4.1}$$

The parametrised integral (4.1) can be represented in the form

$$\exp(-t\mathcal{L}) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} F(\eta, t)d\eta \tag{4.2}$$

with

$$F(\eta, t) = e^{-zt}(zI - \mathcal{L})^{-1}\frac{dz}{d\eta}, \qquad z = \tilde{a}\eta^2 + b - i\eta,$$

for which we can use a Sinc-quadrature rule [37] with $2N+1$ nodes. This quadrature rule possesses the convergence rate $\mathcal{O}(e^{-sN^{2/3}})$, where the constant s depends on the parameters of the integration parabola. The exponential convergence of our quadrature rule allows to introduce the following algorithm for the approximation of the operator exponent at a given time value t.

Algorithm 4.2.

1. Given a tolerance ε *choose* $k > 1$, $N = \mathcal{O}(\log^{3/2} \frac{1}{\varepsilon})$ *and determine* $d = (1 - \frac{1}{\sqrt{k}})\frac{k}{2a}$, z_p $(p = -N, \dots, N)$ *by* $z_p = \frac{a}{k}(ph)^2 + b - iph$, *where* $h = \sqrt[3]{\frac{2\pi dk}{a}}(N+1)^{-2/3}$ *and* $b = \gamma_0 - \frac{k-1}{4a}$.

2. Find the resolvents $(z_p I - \mathcal{L})^{-1}$, $p = -N, \dots, N$ *(note that it can be done in parallel).*

3. Find the approximation $\exp_N(-t\mathcal{L})$ *for the operator exponent* $\exp(-t\mathcal{L})$ *in the form*

$$\exp_N(-t\mathcal{L}) = \frac{h}{2\pi i} \sum_{p=-N}^{N} e^{-tz_p}[2\frac{a}{k}ph - i](z_p I - \mathcal{L})^{-1}. \tag{4.3}$$

Remark 4.3. The above algorithm possesses *two sequential levels of parallelism*: first, one can compute all resolvents at Step 2 in parallel and, second, each operator exponent at different time values (provided that we apply the operator exponential for a given time vector $(t_1, t_2, \dots, t_M)$).

In the case when the spectrum of $\mathcal{L}$ lies inside of a curve like $\xi = a\cosh b\eta$ a similar algorithm of the complexity $\mathcal{O}(\log^{1+\delta} \frac{1}{\varepsilon})$ with an arbitrarily small positive δ was proposed in [16]. Combining the algorithm above with the approximation of resolvents by data sparse $\mathcal{H}$-matrix one can get data sparse approximations of the operator exponential with almost linear costs of matrix-vector multiplication [14].

4.2 Approximation of the normalized sinh

Let $\mathcal{L}$ be a linear, densely defined, closed, strongly P-positive operator in a Banach space X. The operator value function (hyperbolic sine family of bounded operators [19])

$$E(x) \equiv E(x;\mathcal{L}) := \sinh^{-1}(\sqrt{\mathcal{L}})\sinh(x\sqrt{\mathcal{L}})$$

satisfies the elliptic differential equation[1]

$$\frac{d^2E}{dx^2} - \mathcal{L}E = 0, \quad E(0) = \Theta, \quad E(1) = I \tag{4.4}$$

where I is the identity and Θ the zero operator. Given the normalized hyperbolic operator sine family $E(x)$, the solution of the elliptic differential equation (elliptic equation)

$$\frac{d^2u}{dx^2} - \mathcal{L}u = 0, \quad u(0) = 0, \quad u(1) = u_1 \tag{4.5}$$

with a given vector u_1 and unknown vector valued function $u(x) : (0,1) \to X$ can be represented as

$$u(x) = E(x;\mathcal{L})u_1. \tag{4.6}$$

Let $\Gamma_0 = \{z = \xi + i\eta : \xi = a\eta^2 + \gamma_0\}$ be the parabola (called the spectral parabola) defined as above and containing the spectrum $sp(\mathcal{L})$ of the strongly P-positive operator $\mathcal{L}$.

Lemma 4.4. *Choose a parabola (called the integration parabola)* $\Gamma = \{z = \xi + i\eta : \xi = \tilde{a}\eta^2 + b\}$ *with* $b \in (0, \gamma_0)$. *Then the operator family* $E(x;\mathcal{L})$ *can be represented by the*

[1]The operator $\sinh^{-1} A := (\sinh A)^{-1}$ means the inverse to the operator $\sinh A$

Dunford-Cauchy integral [6]

$$\begin{aligned} E(x;\mathcal{L}) &= \frac{1}{2\pi i}\int_\Gamma \sinh^{-1}(\sqrt{z})\sinh(x\sqrt{z})(zI-\mathcal{L})^{-1}dz \\ &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} F(\eta,x)d\eta, \end{aligned} \tag{4.7}$$

where

$$F(\eta,x) = -\sinh^{-1}(\sqrt{z})\sinh(x\sqrt{z})(2\tilde{a}\eta - i)(zI-\mathcal{L})^{-1}, \qquad z = \tilde{a}\eta^2 + b - i\eta. \tag{4.8}$$

Applying the quadrature rule T_N with the operator valued function

$$F(\eta,x;\mathcal{L}) = (2\tilde{a}\eta - i)\varphi(\eta)(\psi(\eta)I - \mathcal{L})^{-1}, \tag{4.9}$$

where

$$\varphi(\eta) = -\sinh^{-1}(\sqrt{\psi(\eta)})\sinh(x\sqrt{\psi(\eta)}), \quad \psi(\eta) = \tilde{a}\eta^2 + b - i\eta, \tag{4.10}$$

we obtain for the integral (4.1) that

$$E(x) \approx E_N(x) = h\sum_{k=-N}^{N} F(kh,x;\mathcal{L}). \tag{4.11}$$

The error analysis is given by the following Theorem (see [15] for the proof).

Theorem 4.5. *Choose $k>1$, $\tilde{a}=a/k$, $h = \frac{\sqrt{2\pi d}}{\sqrt[4]{\min\{\frac{a}{k},b\}}}N^{-1/2}$, $b=b(k)=\gamma_0-(k-1)/(4a)$ and the integration parabola $\Gamma_{b(k)} = \{z=\tilde{a}\eta^2+b(k)-i\eta : \eta\in(-\infty,\infty)\}$. Then there holds*

$$\|E(x)-E_N(x)\| \le c\left[\frac{e^{-s\sqrt{N}}}{1-e^{-s\sqrt{N}}} + he^{-s(1-x)\sqrt{N}}\right], \tag{4.12}$$

where

$$\begin{aligned} s &= \sqrt{2\pi d}\sqrt[4]{\min\{\frac{a}{k},b\}}, \\ d &= (1-\frac{1}{\sqrt{k}})\frac{k}{2a}, \end{aligned} \tag{4.13}$$

with some positive constant c independent of N.

The exponential convergence of our quadrature rule allows for a given tolerance ε to introduce the following algorithm for the approximation of the normalized hyperbolic sine family at a given space-variable value $x\in(0,1)$ with the complexity $\mathcal{O}(\log^2\frac{1}{\varepsilon})$.

Algorithm 4.6.

1. Give ε choose $k>1$, $N=\mathcal{O}(\log^2\frac{1}{\varepsilon})$. Determine $d=(1-\frac{1}{\sqrt{k}})\frac{k}{2a}, z_p$ ($p=-N,\dots,N$) by $z_p=\frac{a}{k}(ph)^2+b-iph$, where $h=\frac{\sqrt{2\pi d}}{\sqrt[4]{\min\{\frac{a}{k},b\}}}N^{-1/2}$ and $b=\gamma_0-\frac{k-1}{4a}$.

2. Find the resolvents $(z_pI-\mathcal{L})^{-1}$, $p=-N,\dots,N$ (note that it can be done in parallel).

3. Find the approximation $E_N(x;\mathcal{L})$ for the normalised hyperbolic operator sine $E(x;\mathcal{L})$ in the form

$$E_N(x;\mathcal{L}) = \frac{h}{2\pi i}\sum_{p=-N}^{N}\sinh^{-1}(\sqrt{z_p})\sinh(x\sqrt{z_p})\,[2\frac{a}{k}ph - i](z_pI-\mathcal{L})^{-1}. \tag{4.14}$$

Remark 4.7. The above algorithm possesses *two sequential levels of parallelism:* first, one can compute all resolvents at Step 2 in parallel and, second, each operator exponent at different values of x (provided that we apply the operator function for a given vector $(x_1,x_2,\dots,x_M)$).

The above approximation of the normalized hyperbolic sine family can be used in order to get explicit approximations to the Poincaré-Steklov operators [15]. Representing the resolvents by $\mathcal{H}$−matrix one can get data sparse approximations to all these operators (see [15]).

Further algorithms without accuracy saturation for various applied problems were proposed in [16, 2, 3, 10,13,18], [23]-[31]. Note, that methods developed in the papers cited above have also contributed to the solution of various other problems from the field of numerical analysis. It is worth mentioning the related (by investigation methods) paper [22] where for the first time the conditions of the stability of three-level difference schemes with unbounded operators coefficients were pointed out. The principal stability condition is the strong P-positivity of these operators.

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