Журнал обчисл. прикл. матем.
Вип. 1, 2003, стор. 43–65

J. Comput. Appl. Math.
No. 88, 2003, pp. 43–65

MODIFIED THREE-POINT DIFFERENCE SCHEMES OF HIGH ACCURACY ORDER FOR SECOND ORDER MONOTONE ORDINARY DIFFERENTIAL EQUATIONS WITH DERIVATIVE IN THE RIGHT-HAND SIDE

UDC 519.62

L. B. GNATIV AND M. V. KUTNIV

ABSTRACT. For nonlinear monotone boundary value problems, the new algorithmic implementation exact three-point difference schemes on irregular grid in term three-point difference schemes of rank $\bar{m} = 2[(m+1)/2]$ ($[\cdot]$ is the integer part) is proposed. To construct a three-point difference schemes has $\bar{m}$th order of accurate that approximate the function $u(x)$ and its flux kdu/dx at the nodes of grid.

1 Introduction

Consider a nonlinear boundary value problem of the form

$$\frac{d}{dx}\left[k(x)\frac{du}{dx}\right] = -f\left(x, u(x), \frac{du}{dx}\right), \qquad x \in (0,1), \qquad u(0) = \mu_1, \qquad u(1) = \mu_2. \tag{1}$$

Three-point difference schemes (TPDS) of any accuracy on the uniform grid for problem (1), where the right-hand side of the differential equation does not depend from du/dx are constructed in [2]-[4]. In this paper for boundary value problem (1) with a monotone operator on a non-uniform grid the exact three-point difference scheme (ETPDS) is constructed. The existence and unique solution for weaker conditions than in [3] is proved, new more effective implementation of ETPDS in terms TPDS order of accuracy $\bar{m} = 2\left[(m+1)/2\right]$ ($[\cdot]$ is the integer part) is developed and analysed. To construct an $\bar{m}$th order accurate TPDS at each point x_j of the grid $\hat{\omega}_h$, it is necessary to solve two nonlinear initial value problems on the intervals $[x_{j-1}, x_j]$ (forward) and $[x_j, x_{j+1}]$ (backward). Each initial value problem is solved in one step by a one-step method (Taylor series expansion or Runge-Kutta method). The efficiency of a six-order accurate TPDS is illustrated by a numerical example.

Since the problem is nonlinear, our analysis is based on the method of monotone operators (see, e.g. [1]).

Theorem 1. *Let conditions*

$$0 < c_1 \le k(x) \le c_2, \forall x \in [0,1], \qquad k(x) \in Q^1[0,1], \tag{2}$$

$$f_{u\xi}(x) \equiv f(x,u,\xi) \in Q^0[0,1], \qquad \forall u, \xi \in R^1,$$
$$f_x(u,\xi) \equiv f(x,u,\xi) \in C\left(R^2\right), \qquad \forall x \in [0,1], \tag{3}$$

$$|f(x,u,\xi)| \le g(x) + c(|u| + |\xi|), \forall x \in [0,1], u, \xi \in R^1, \tag{4}$$

$$[f(x,u,\xi) - f(x,v,\eta)](u-v) \le c_3\left(|u-v|^2 + |\xi-\eta|^2\right),$$
$$\forall x \in [0,1], u, \xi, v, \eta \in R^1, \tag{5}$$

Key words and phrases. Nonlinear monotone boundary value problems, exact three-point difference schemes, irregular grid.

$$c_3 < \frac{\pi^2}{\pi^2+1} c_1 \qquad (6)$$

be fulfilled. That problem (1) has a unique solution $u(x)$.

Here $g(x) \in L_2(0,1)$ and c is a nonnegative constant, $Q^p[0,1]$ is the class of functions with piesewise continuous derivative up to the pth order inclusive with a finite of descontinuity points of the first kind.

Proof. By virtue of (3), (4) the function $f\left(x, u, \frac{du}{dx}\right)$ satisfies the Caratheodory conditions [1], and belongs to $L_2(0,1)$. In view of this, the operator $A(x,u)$ is defined by the relation

$$(A(x,u), v) = \int_0^1 k(x) \frac{du(x)}{dx} \frac{dv(x)}{dx} dx - \int_0^1 f\left(x, u(x), \frac{du}{dx}\right) v(x)\, dx,$$

which holds for $\forall u \in W_2^1(0,1), \forall v \in \overset{0}{W}{}_2^1(0,1) = \{v(x) \,|\, v(x) \in W_2^1(0,1), v(0) = = v(1) = 0\}$.

$u(x)$ is said to be a weak solution to problem (1) if

$$(A(x,u), v) = 0, \qquad \forall v(x) \in \overset{0}{W}{}_2^1(0,1).$$

Let us show that the operator $A(x,u)$ is bounded. Using the Cauchy-Schwarz inequality and taking into account (2) and (4), we obtain

$$|(A(x,u), v)| \le \left\{\int_0^1 \left[k(x)\frac{du(x)}{dx}\right]^2 dx\right\}^{1/2} \left\{\int_0^1 \left[\frac{dv(x)}{dx}\right]^2 dx\right\}^{1/2} +$$

$$+ \left[\int_0^1 f^2\left(x, u, \frac{du}{dx}\right) dx\right]^{1/2} \left[\int_0^1 v^2(x)\, dx\right]^{1/2} \le$$

$$\le \left[c_2 \|u\|_{1,2,(0,1)} + \|f\|_{0,2,(0,1)}\right] \|v\|_{1,2,(0,1)} \le$$

$$\le \left[(c_2 + c) \|u\|_{1,2,(0,1)} + \|g\|_{0,2,(0,1)}\right] \|v\|_{1,2,(0,1)}.$$

If $u_n \to u_0$ in $W_2^1(0,1)$, then

$$f\left(x, u_n, \frac{du_n}{dx}\right) \to f\left(x, u_0, \frac{du_0}{dx}\right), \qquad k(x)\frac{du_n(x)}{dx} \to k(x)\frac{du_0(x)}{dx}$$

in $L_2(0,1)$ (see [1]). Thus, $\forall v \in \overset{0}{W}{}_2^1(0,1)$

$$\lim_{n\to\infty} (A(x,u_n), v) = \lim_{n\to\infty} \left[\int_0^1 k(x) \frac{du_n(x)}{dx}\frac{dv(x)}{dx} dx - \right.$$

$$\left. - \int_0^1 f\left(x, u_n(x), \frac{du_n(x)}{dx}\right) v(x)\, dx \right] = \int_0^1 k(x) \frac{du_0(x)}{dx}\frac{dv(x)}{dx} dx -$$

$$- \int_0^1 f\left(x, u_0(x), \frac{du_0(x)}{dx}\right) v(x)\, dx = (A(x,u_0), v),$$

i.e. $A(x,u)$ is a demicontinuous operator.

Let us show the operator $A(x,u)$ is strongly monotone. Then, using (2), (5) and $\|v\|_{0,2,(0,1)} \leq \frac{1}{\pi}\left\|\frac{dv}{dx}\right\|_{0,2,(0,1)}$ for $\forall v \in \overset{0}{W}{}_2^1(0,1)$, we obtain

$$(A(x,u) - A(x,v), u-v) = \int_0^1 k(x)\left[\frac{du(x)}{dx} - \frac{dv(x)}{dx}\right]^2 dx -$$

$$-\int_0^1 \left[f\left(x,u(x),\frac{du}{dx}\right) - f\left(x,v(x),\frac{dv}{dx}\right)\right][u(x)-v(x)]\,dx \geq$$

$$\geq c_1\left\|\frac{du}{dx} - \frac{dv}{dx}\right\|^2_{0,2(0,1)} - c_3\int_0^1 |u(x)-v(x)|^2 dx - c_3\int_0^1\left|\frac{du(x)}{dx} - \frac{dv(x)}{dx}\right|^2 dx =$$

$$= (c_1 - c_3)\left\|\frac{du}{dx} - \frac{dv}{dx}\right\|^2_{0,2,(0,1)} - c_3\|u-v\|^2_{0,2,(0,1)} \geq c_4\|u-v\|^2_{1,2,(0,1)},$$

where by virtue of (6) $c_4 = c_1 - c_3\frac{\pi^2+1}{\pi^2} > 0$. From strong monotonicity, it follows that $A(x,u)$ is a coercive operator.

Consequently, by the Browder theorem (see [1]) problem (1) has unique solution.

2 Existense of an exact three-point difference scheme

Define the non-uniform grid

$$\hat{\omega}_h = \left\{x_j \in (0,1), j = 1,2,\dots,N-1,\ h_j = x_j - x_{j-1} > 0, \sum_{j=1}^N h_j = 1\right\}.$$

The discontinuity points of functions $k(x)$ and $f\left(x,u,\frac{du}{dx}\right)$ coincide with nodes x_j of the grid $\hat{\omega}_h$. Denote by ρ set of all points of discontinuity and we shall assume, that N such that $\rho \subseteq \hat{\omega}_h$. Moreover, at the discontinuity points, usual agreement conditions must be satisfied

$$u(x_i - 0) = u(x_i + 0),\ k(x)\left.\frac{du}{dx}\right|_{x=x_i-0} = k(x)\left.\frac{du}{dx}\right|_{x=x_i+0},\ \forall x_i \in \rho.$$

Define the function

$$Y_\alpha^j(x,u) = \hat{u}(x) + w_\alpha^j(x,u) - \frac{V_\alpha^j(x)}{V_\alpha^j(x_j)} w_\alpha^j(x_j,u),$$

$$x \in [x_{j-2+\alpha}, x_{j-1+\alpha}], \qquad \alpha = 1,2, \qquad j = 1,2,\dots,N-1,$$

where

$$\hat{u}(x) = \left[u(x_j)V_1^j(x) + u(x_{j-1})V_2^{j-1}(x)\right]\left[V_1^j(x_j)\right]^{-1}, \qquad x \in [x_{j-1}, x_j],$$

$$V_1^j(x) = \int_{x_{j-1}}^x \frac{dt}{k(t)}, \qquad V_2^j(x) = \int_x^{x_{j+1}} \frac{dt}{k(t)},$$

the functions $w_\alpha^j(x,u)$ and $l_\alpha^j(x,u)$ with $\alpha = 1,2$ that are the solutions to the initial value problems

$$\frac{dw_\alpha^j(x,u)}{dx} = \frac{l_\alpha^j(x,u)}{k(x)}, \quad \frac{dl_\alpha^j(x,u)}{dx} = -f\left(x, Y_\alpha^j(x,u), \frac{dY_\alpha^j(x,u)}{dx}\right),$$
$$x_{j-2+\alpha} < x < x_{j-1+\alpha},$$
$$w_\alpha^j\left(x_{j+(-1)^\alpha}, u\right) = l_\alpha^j\left(x_{j+(-1)^\alpha}, u\right) = 0, \ \alpha = 1,2, \ j = 1,2,\dots,N-1. \tag{7}$$

Lemma 1. *Let conditions (3)-(6) be fulfilled. Then problems (7) have an unique solution* $w_\alpha^j(x,u), l_\alpha^j(x,u), \alpha = 1,2$. *Moreover, the solution to problem (1) can be represented as*

$$u(x) = Y_\alpha^j(x,u) = \hat{u}(x) + w_\alpha^j(x,u) - \frac{V_\alpha^j(x)}{V_\alpha^j(x_j)} w_\alpha^j(x_j,u),$$
$$x \in [x_{j-2+\alpha}, x_{j-1+\alpha}], \qquad \alpha = 1,2, \qquad j = 1,2,\dots,N-1. \tag{8}$$

Proof. Since

$$\frac{dY_\alpha^j(x,u)}{dx} = \frac{1}{k(x)}\left[\frac{h_{j-1+\alpha}u_{\bar{x},j-1+\alpha} + (-1)^\alpha w_\alpha^j(x_j,u)}{V_\alpha^j(x_j)} + l_\alpha^j(x,u)\right],$$

and

$$\frac{d}{dx}\left[k(x)\frac{dY_\alpha^j(x,u)}{dx}\right] = \frac{dl_\alpha^j(x,u)}{dx} = -f\left(x, Y_\alpha^j(x,u), \frac{dY_\alpha^j(x,u)}{dx}\right),$$

we conclude that the functions $Y_\alpha^j(x,u), \alpha = 1,2$ are a solutions to the boundary value problems

$$\frac{d}{dx}\left(k(x)\frac{dY_\alpha^j(x,u)}{dx}\right) = -f\left(x, Y_\alpha^j(x), \frac{dY_\alpha^j(x,u)}{dx}\right), x_{j-2+\alpha} < x < x_{j-1+\alpha},$$
$$Y_\alpha^j(x_{j-2+\alpha}, u) = u(x_{j-2+\alpha}), Y_\alpha^j(x_{j-1+\alpha}, u) = u(x_{j-1+\alpha}),$$
$$\alpha = 1,2, \qquad j = 1,2,\dots N-1. \tag{9}$$

By virtue of (3) and (4) the function $f\left(x, u, \frac{du}{dx}\right)$ satisfies the Caratheodory conditions and is element of $L_2(0,1)$ (see [1]). In view of this, the nonlinear operator $A_\alpha^j\left(x, Y_\alpha^j(x,u)\right)$ is defined by the relation

$$\left(A_\alpha^j\left(x, Y_\alpha^j(x,u)\right), v\right) = \int_{x_{j-2+\alpha}}^{x_{j-1+\alpha}} k(x)\frac{dY_\alpha^j(x,u)}{dx}\frac{dv(x)}{dx} -$$
$$- \int_{x_{j-2+\alpha}}^{x_{j-1+\alpha}} f\left(x, Y_\alpha^j(x,u), \frac{dY_\alpha^j(x,u)}{dx}\right) v(x)\,dx,$$

which holds for

$$\forall Y_\alpha^j(x,u) \in W_2^1(e_\alpha), \forall v \in \overset{0}{W}{}_2^1(e_\alpha), e_\alpha = (x_{j-2+\alpha}, x_{j-1+\alpha}),$$

$$\overset{0}{W}{}_2^1(e_\alpha) = \left\{v(x) \,\middle|\, v(x) \in W_2^1(e_\alpha), \quad v(x_{j-2+\alpha}) = 0, v(x_{j-1+\alpha}) = 0, \quad \alpha = 1,2\right\}.$$

$Y_\alpha^j(x,u)$ is said to be a weak solution to problem (9) if

$$\left(A_\alpha^j\left(x, Y_\alpha^j(x)\right), v\right) = 0, \quad \forall v(x) \in \overset{0}{W}{}_2^1(e_\alpha).$$

Let us show that the operator $A_\alpha^j\left(x, Y_\alpha^j(x,u)\right)$ is bounded. Using the Cauchy-Schwarz inequality and taking into account (2) and (4), we obtain

$$A_\alpha^j\left(x, Y_\alpha^j(x,u)\right) \le \left\{\int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}}\left[k(x)\frac{dY_\alpha^j(x,u)}{dx}\right]^2 dx\right\}^{1/2}\left\{\int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}}\left[\frac{dv(x)}{dx}\right]^2 dx\right\}^{1/2} +$$

$$+\left\{\int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}} f^2\left(x, Y_\alpha^j(x,u), \frac{dY_\alpha^j(x,u)}{dx}\right) dx\right\}^{1/2}\left\{\int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}}[v(x)]^2 dx\right\}^{1/2} =$$

$$= c_2\left\|Y_\alpha^j\right\|_{1,2,e_\alpha}\|v\|_{1,2,e_\alpha} + \|f\|_{0,2,e_\alpha}\|v\|_{1,2,e_\alpha} =$$

$$= c_2\left\|Y_\alpha^j\right\|_{1,2,e_\alpha}\|v\|_{1,2,e_\alpha} + \left(\|g\|_{0,2,e_\alpha} + c\left\|Y_\alpha^j\right\|_{1,2,e_\alpha}\right)\|v\|_{1,2,e_\alpha} =$$

$$= \left[(c_2+c)\left\|Y_\alpha^j\right\|_{1,2,e_\alpha} + \|g\|_{0,2,e_\alpha}\right]\|v\|_{1,2,e_\alpha}.$$

The fact that $A_\alpha^j\left(x, Y_\alpha^j(x,u)\right)$ is demicontinuous follows from (2) and (3). Indeed (see [1]), if $Y_{\alpha n}^j(x,u) \to Y_{\alpha 0}^j(x,u)$ in $W_2^1(e_\alpha)$, then

$$f\left(x, Y_{\alpha n}^j(x,u), \frac{dY_{\alpha n}^j(x,u)}{dx}\right) \to f\left(x, Y_{\alpha 0}^j(x,u), \frac{dY_{\alpha 0}^j(x,u)}{dx}\right),$$

$$k(x)\frac{dY_{\alpha n}^j(x,u)}{dx} \to k(x)\frac{dY_{\alpha 0}^j(x,u)}{dx}$$

in $L_2(e_\alpha)$. Thus, for $\forall v(x) \in \overset{0}{W}{}_2^1(e_\alpha)$

$$\lim_{n\to\infty}\left(A_\alpha^j\left(x, Y_{\alpha n}^j(x,u)\right), v\right) = \lim_{n\to\infty}\left\{\int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}} k(x)\frac{dY_{\alpha n}^j(x,u)}{dx}\frac{dv(x)}{dx}dx -\right.$$

$$\left.-\int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}} f\left(x, Y_{\alpha n}^j(x,u), \frac{dY_{\alpha n}^j(x,u)}{dx}\right) v(x)\,dx\right\} = \int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}} k(x)\frac{dY_{\alpha 0}^j(x,u)}{dx}\frac{dv(x)}{dx}dx -$$

$$-\int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}} f\left(x, Y_{\alpha 0}^j(x,u), \frac{dY_{\alpha 0}^j(x,u)}{dx}\right) v(x)\,dx = \left(A_\alpha^j\left(x, Y_{\alpha 0}^j(x,u)\right), v\right).$$

Let us show the operator $A_\alpha^j\left(x, Y_\alpha^j(x,u)\right)$ is strongly monotone. Then, taking into account conditions (3) and (5), the inequality $\|v\|_{0,2,e_\alpha} \le \frac{1}{\pi}\left\|\frac{dv}{dx}\right\|_{0,2,e_\alpha}$, we have

$$\left(A_\alpha^j\left(x, Y_\alpha^j(x,u)\right) - A_\alpha^j\left(x, \tilde{Y}_\alpha^j(x,u)\right), Y_\alpha^j(x,u) - \tilde{Y}_\alpha^j(x,u)\right) =$$

$$= \int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}} k(x)\left(\frac{dY_\alpha^j(x,u)}{dx} - \frac{d\tilde{Y}_\alpha^j(x,u)}{dx}\right)^2 dx -$$

$$-\int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}}\left[f\left(x, Y_\alpha^j(x,u), \frac{dY_\alpha^j(x,u)}{dx}\right) - f\left(x, \tilde{Y}_\alpha^j(x,u), \frac{d\tilde{Y}_\alpha^j(x,u)}{dx}\right)\right]\times$$

$$\times\left[Y_\alpha^j(x,u)-\tilde{Y}_\alpha^j(x,u)\right]dx\ge c_1\left\|\frac{dY_\alpha^j}{dx}-\frac{d\tilde{Y}_\alpha^j}{dx}\right\|_{0,2,e_\alpha}^2-$$

$$-c_3\int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}}\left[Y_\alpha^j(x,u)-\tilde{Y}_\alpha^j(x,u)\right]^2dx-c_3\int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}}\left[\frac{dY_\alpha^j(x,u)}{dx}-\frac{d\tilde{Y}_\alpha^j(x,u)}{dx}\right]^2dx=$$

$$=(c_1-c_3)\left\|\frac{dY_\alpha^j}{dx}-\frac{d\tilde{Y}_\alpha^j}{dx}\right\|_{0,2,e_\alpha}^2-c_3\left\|Y_\alpha^j-\tilde{Y}_\alpha^j\right\|_{0,2,e_\alpha}^2\ge c_4\left\|Y_\alpha^j-\tilde{Y}_\alpha^j\right\|_{1,2,e_\alpha}^2.$$

Strong monotonocity implies that $A_\alpha^j\left(x,Y_\alpha^j(x,u)\right)$ is a coercive operator.

Consequently, by the Browder theorem [1], problems (9) have a unique solution.

Using the lemma, one can prove the next result.

Theorem 2. *Suppose that the conditions of Lemma 1 are fulfilled. Then, for problem (1)-(6), there exists an ETPDS*

$$(au_{\bar{x}})_{\hat{x}}=-\hat{T}^x\left(f\left(\xi,u(\xi),\frac{du(\xi)}{d\xi}\right)\right),\qquad x\in\hat{\omega}_h,\qquad u(0)=\mu_1,\qquad u(1)=\mu_2,\tag{10}$$

which has a unique solution $u(x)$ that is the projection of the solution to problem (1) onto the grid $\hat{\omega}_h$. Here

$$u_{\bar{x},j}=\frac{u_j-u_{j-1}}{h_j},\quad u_{\hat{x},j}=\frac{u_{j+1}-u_j}{\hbar_j},\quad \hbar_j=\frac{h_j+h_{j+1}}{2},$$

$$a(x_j)=\left[\frac{1}{h_j}V_1^j(x_j)\right]^{-1},\tag{11}$$

$$\hat{T}^{x_j}(w(\xi))=\left[\hbar_jV_1^j(x_j)\right]^{-1}\int\limits_{x_{j-1}}^{x_j}V_1^j(\xi)w(\xi)\,d\xi+\left[\hbar_jV_2^j(x_j)\right]^{-1}\int\limits_{x_j}^{x_{j+1}}V_2^j(\xi)w(\xi)\,d\xi,$$

and the function $u(x)$ on the right-hand side of (10) is defined by (8) and depends only on $u(x_j)$, $j=0,1,\dots,N$.

Proof. Applyinge the operator $\hat{T}^{x_j}$ to (1) gives

$$\hat{T}^{x_j}\left(\frac{d}{d\xi}\left(k(\xi)\frac{du(\xi)}{d\xi}\right)\right)=-\hat{T}^{x_j}\left(f\left(\xi,u(\xi),\frac{du(\xi)}{d\xi}\right)\right),$$

where

$$\hat{T}^{x_j}\left(\frac{d}{d\xi}\left(k(\xi)\frac{du(\xi)}{d\xi}\right)\right)=\left[\hbar_jV_1^j(x_j)\right]^{-1}\int\limits_{x_{j-1}}^{x_j}V_1^j(\xi)\frac{d}{d\xi}\left[k(\xi)\frac{du(\xi)}{d\xi}\right]d\xi+$$

$$+\left[\hbar_jV_2^j(x_j)\right]^{-1}\int\limits_{x_j}^{x_{j+1}}V_2^j(\xi)\frac{d}{d\xi}\left[k(\xi)\frac{du(\xi)}{d\xi}\right]d\xi.$$

Integrating the integrals by parts yields (see, e.g., [5])

$$\hat{T}^{x_j}\left(\frac{d}{d\xi}\left(k(\xi)\frac{du(\xi)}{d\xi}\right)\right)=(au_{\bar{x}})_{\hat{x},j}$$

which, in view of (8), proves the existence of the ETPDS given by (10) and (11).

To prove the uniqueness of the solution to the ETPDS given by (10) and (11), we consider the operator

$$A_h(x,u) = -(au_{\bar{x}})_{\hat{x}} - \hat{T}^x\left(f\left(\xi, u(\xi), \frac{du(\xi)}{d\xi}\right)\right)$$

which is defined in the finite-dimensional Hilbert space $L_2(\hat{\omega}_h)$which the scalar products

$$(u,v)_{\hat{\omega}_h} = \sum_{\xi\in\hat{\omega}_h} \hbar(\xi)\, u(\xi)\, v(\xi), \qquad (u,v)_{\hat{\omega}_h^+} = \sum_{\xi\in\hat{\omega}_h^+} h(\xi)\, u(\xi)\, v(\xi)$$

and with the norms $\|u\|_{0,2,\hat{\omega}_h} = \sqrt{(u,u)_{\hat{\omega}_h}}$, $\|u\|_{0,2,\hat{\omega}_h^+} = \sqrt{(u,u)_{\hat{\omega}_h^+}}$. By virtue of (2) and (3), the operator $A_h(x,u)$ is continuous. Let us show that $A_h(x,u)$ is strongly monotone. Indeed, in view of the equation

$$\sum_{\xi\in\hat{\omega}_h} \hbar(\xi)\, \hat{T}^\xi(w(\eta))\, g(\xi) = \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} \hat{g}(\eta)\, w(\eta)\, d\eta = \int_0^1 \hat{g}(\eta)\, w(\eta)\, d\eta,$$

$$\hat{g}(\eta) = g(x_j)\frac{V_1^j(\eta)}{V_1^j(x_j)} + g(x_{j-1})\frac{V_2^{j-1}(\eta)}{V_1^j(x_j)}, \quad x_{j-1}\le\eta\le x_j,$$

we have

$$\left(\hat{T}^x\left(f\left(\eta, u(\eta), \frac{du(\eta)}{d\eta}\right) - f\left(\eta, v(\eta), \frac{dv(\eta)}{d\eta}\right)\right), u - v\right)_{\hat{\omega}_h} =$$

$$= \sum_{\xi\in\hat{\omega}_h} \hbar(\xi)\, \hat{T}^\xi\left(f\left(\eta, u(\eta), \frac{du(\eta)}{d\eta}\right) - f\left(\eta, v(\eta), \frac{dv(\eta)}{d\eta}\right)\right)[u(\xi) - v(\xi)] =$$

$$= \int_0^1 [\hat{u}(\eta) - \hat{v}(\eta)]\left[f\left(\eta, u(\eta), \frac{du(\eta)}{d\eta}\right) - f\left(\eta, v(\eta), \frac{dv(\eta)}{d\eta}\right)\right] d\eta,$$

where the functions $u(x)$ and $v(x)$ are defined by (8). Then, using (5), we obtain

$$\left(\hat{T}^x\left(f\left(\eta, u(\eta), \frac{du(\eta)}{d\eta}\right) - f\left(\eta, v(\eta), \frac{dv(\eta)}{d\eta}\right)\right), u - v\right)_{\hat{\omega}_h} =$$

$$= \int_0^1 [u(\eta) - v(\eta)]\left[f\left(\eta, u(\eta), \frac{du(\eta)}{d\eta}\right) - f\left(\eta, u(\eta), \frac{du(\eta)}{d\eta}\right)\right] d\eta +$$

$$+ \int_0^1 [\hat{u}(\eta) - \hat{v}(\eta) - u(\eta) + v(\eta)]\left[f\left(\eta, u(\eta), \frac{du(\eta)}{d\eta}\right) -\right.$$

$$\left. - f\left(\eta, v(\eta), \frac{dv(\eta)}{d\eta}\right)\right] d\eta \le -\int_0^1 [\hat{u}(\eta) - \hat{v}(\eta)]\frac{d}{d\eta}\left\{k(\eta)\frac{d}{d\eta}[u(\eta) - v(\eta)]\right\} d\eta +$$

$$+ \int_0^1 [u(\eta) - v(\eta)]\frac{d}{d\eta}\left\{k(\eta)\frac{d}{d\eta}[u(\eta) - v(\eta)]\right\} d\eta + c_3\|u - v\|_{1,2,(0,1)}^2 =$$

$$= -\sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} [\hat{u}(\eta) - \hat{v}(\eta)]\frac{d}{d\eta}\left\{k(\eta)\frac{d}{d\eta}[u(\eta) - v(\eta)]\right\} d\eta -$$

$$-\int_0^1 k(\eta)\left[\frac{d}{d\eta}\left[u(\eta)-v(\eta)\right]\right]^2 d\eta + c_3\left\|u-v\right\|^2_{1,2,(0,1)} \le$$

$$\le \sum_{j=1}^{N}\int_{x_{j-1}}^{x_j} k(\eta)\left[\frac{d\hat{u}(\eta)}{d\eta}-\frac{d\hat{v}(\eta)}{d\eta}\right]\left[\frac{du(\eta)}{d\eta}-\frac{dv(\eta)}{d\eta}\right]d\eta -$$

$$-c_1\left\|\frac{du}{dx}-\frac{dv}{dx}\right\|^2_{0,2,(0,1)} + c_3\left\|u-v\right\|^2_{1,2,(0,1)} =$$

$$= \left(a\left(u_{\bar{x}}-v_{\bar{x}}\right), u_{\bar{x}}-v_{\bar{x}}\right)_{\hat{\omega}_h^+} - c_1\left\|\frac{du}{dx}-\frac{dv}{dx}\right\|^2_{0,2,(0,1)} + c_3\left\|u-v\right\|^2_{1,2,(0,1)}.$$

Hence,

$$\begin{gathered}\left(\hat{T}^x\left(f\left(\eta,u(\eta),\frac{du(\eta)}{d\eta}\right)-f\left(\eta,v(\eta),\frac{dv(\eta)}{d\eta}\right)\right),u-v\right)_{\hat{\omega}_h} \le \\ \le \left(a\left(u_{\bar{x}}-v_{\bar{x}}\right), u_{\bar{x}}-v_{\bar{x}}\right)_{\hat{\omega}_h^+} - c_1\left\|\frac{du}{dx}-\frac{dv}{dx}\right\|^2_{0,2,(0,1)} + c_3\left\|u-v\right\|^2_{1,2,0,1}.\end{gathered} \tag{12}$$

In view of (3), we obtain

$$\left(A_h(x,u)-A_h(x,v),u-v\right)_{\hat{\omega}_h} = \left(a\left(u_{\bar{x}}-v_{\bar{x}}\right), u_{\bar{x}}-v_{\bar{x}}\right)_{\hat{\omega}_h^+} -$$

$$-\left(\hat{T}^x\left(f\left(\eta,u(\eta),\frac{du(\eta)}{d\eta}\right)-f\left(\eta,v(\eta),\frac{dv(\eta)}{d\eta}\right)\right),u-v\right)_{\hat{\omega}_h} \ge$$

$$\ge c_1\left\|\frac{du}{dx}-\frac{dv}{dx}\right\|^2_{0,2,(0,1)} - c_3\left\|u-v\right\|^2_{1,2,(0,1)} \ge$$

$$\ge (c_1-c_3)\left\|\frac{du}{dx}-\frac{dv}{dx}\right\|^2_{0,2,(0,1)} - c_3\left\|u-v\right\|^2_{0,2,(0,1)} \ge c_4\left\|\frac{du}{dx}-\frac{dv}{dx}\right\|^2_{0,2,(0,1)},$$

where $c_4 = c_1 - c_3\dfrac{1+\pi^2}{\pi^2} > 0$. By virtue of

$$\int_0^1 k(\eta)\left\{\frac{d}{d\eta}\left[\hat{u}(\eta)-\hat{v}(\eta)-u(\eta)+v(\eta)\right]\right\}^2 d\eta = -\sum_{j=1}^{N} h_j a(x_j)\left(u_{\bar{x},j}-v_{\bar{x},j}\right)^2 +$$

$$+\int_0^1 k(\eta)\left\{\frac{d}{d\eta}\left[u(\eta)-v(\eta)\right]\right\}^2 d\eta = \left(a\left(u_{\bar{x}}-v_{\bar{x}}\right), u_{\bar{x}}-v_{\bar{x}}\right)_{\hat{\omega}_h^+} +$$

$$+\left(k(x)\left(\frac{du}{dx}-\frac{dv}{dx}\right),\frac{du}{dx}-\frac{dv}{dx}\right)_{\hat{\omega}_h} \ge 0$$

we have

$$\left\|\frac{du}{dx}-\frac{dv}{dx}\right\|^2_{0,2,(0,1)} \ge \frac{1}{c_2}\left(a\left(u_{\bar{x}}-v_{\bar{x}}\right), u_{\bar{x}}-v_{\bar{x}}\right)_{\hat{\omega}_h^+}.$$

Consequently

$$\begin{gathered}\left(A_h(x,u)-A_h(x,v),u-v\right)_{\hat{\omega}_h} \ge \frac{c_4}{c_2}\left(a\left(u_{\bar{x}}-v_{\bar{x}}\right), u_{\bar{x}}-v_{\bar{x}}\right)_{\hat{\omega}_h^+} \ge \\ \ge \frac{c_4c_1}{c_2}\left\|u_{\bar{x}}-v_{\bar{x}}\right\|^2_{\hat{\omega}_h^+} \ge 8\frac{c_4c_1}{c_2}\left\|u-v\right\|^2_{0,2,\hat{\omega}_h},\end{gathered} \tag{13}$$

i.e., $A_h(x,u)$ is strongly monotone. It follows (see [7]) that the equation $A_h(x,u)=0$ has a unique solution.

Lemma 2. *Let the conditions of Lemma 1 be fulfilled, and*

$$|f(x,u,\xi)-f(x,v,\eta)| \le L\{|u-v|+|\xi-\eta|\},\ \forall x\in(0,1), u,v,\xi,\eta\in R^1.$$

Then the iterative method

$$B_h\frac{u^{(n)}-u^{(n-1)}}{\tau}+A_h\left(x,u^{(n-1)}\right)=0,\quad x\in\hat\omega_h, \tag{14}$$

$$u^{(n)}(0)=\mu_1,\ u^{(n)}(1)=\mu_2,\ n=1,2,\dots,\ u^{(0)}(x)=\frac{V_2(x)}{V_1(1)}\mu_1+\frac{V_1(x)}{V_1(1)}\mu_2,$$
$$B_hu=-(au_{\bar x})_{\hat x},\ A_h(x,u)=B_hu-\hat T^x\left(f\left(\xi,u(\xi),\frac{du(\xi)}{d\xi}\right)\right)$$

with

$$\tau=\tau_0=\frac{c_4}{c_2}\left(1+\frac{L(1+2\sqrt{2})}{4c_1}\left(2+\frac{c_2}{c_1}\right)\left(1+\frac{L(\pi^2+1)}{\pi^2c_4}\right)\right)^{-2}$$

converges in the energy space H_{B_h}, and the error satisfies the estimate

$$\left\|u^{(n)}-u\right\|_{B_h}\le q^n\left\|u^{(0)}-u\right\|_{B_h}, \tag{15}$$

$$q=\sqrt{1-\frac{c_4}{c_2}\tau_0},\quad c_4=c_1-c_3\frac{1+\pi^2}{\pi^2}>0,$$

where $\|u\|_{B_h}=(B_hu,u)^{1/2}_{\hat\omega_h}$.

Proof. It follows from (13) that

$$(A_h(x,u)-A_h(x,v),u-v)_{\hat\omega_h}\ge\frac{c_4}{c_2}\|u-v\|^2_{B_h}. \tag{16}$$

Using the Cauchy-Schwarz inequality, we sequentially find

$$(A_h(x,u)-A_h(x,v),z)_{\hat\omega_h}=(B_hu-B_hv,z)_{\hat\omega_h}-$$
$$-\sum_{\xi\in\omega_h}\hbar(\xi)T^\xi\left(f\left(\eta,u(\eta),\frac{du(\eta)}{d\eta}\right)-f\left(\eta,v(\eta),\frac{dv(\eta)}{d\eta}\right)\right)z(\eta)=$$
$$=(B_hu-B_hv,z)_{\hat\omega_h}-\int_0^1\left[f\left(\eta,u(\eta),\frac{du(\eta)}{d\eta}\right)-f\left(\eta,v(\eta),\frac{dv(\eta)}{d\eta}\right)\right]\hat z(\eta)\,d\eta\le$$
$$\le\|u-v\|_{B_h}\|z\|_{B_h}+\left\{\int_0^1\left[f\left(\eta,u(\eta),\frac{du(\eta)}{d\eta}\right)-f\left(\eta,v(\eta),\frac{dv(\eta)}{d\eta}\right)\right]^2d\eta\right\}^{1/2}\times$$
$$\times\left\{\int_0^1[\hat z(\eta)]^2d\eta\right\}^{1/2}\le\|u-v\|_{B_h}\|z\|_{B_h}+L\|u-v\|_{1,2,(0,1)}\|\hat z\|_{0,2,(0,1)}.$$

Since $V_1^j(x) \le V_1^j(x_j)$ and $V_2^{j-1}(x) \le V_1^j(x_j)$, $\forall x \in [x_{j-1}, x_j]$, we conclude that

$$\|\hat{z}\|_{0,2,(0,1)}^2 = \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} \left[z_j \frac{V_1^j(x)}{V_1^j(x_j)} + z_{j-1} \frac{V_2^{j-1}(x)}{V_1^j(x_j)} \right]^2 dx \le$$
$$\le 2 \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} \left\{ z_j^2 \left[\frac{V_1^j(x)}{V_1^j(x_j)} \right]^2 + z_{j-1}^2 \left[\frac{V_2^{j-1}(x)}{V_1^j(x_j)} \right]^2 \right\} dx \le 4 \|z\|_{0,2,\hat{\omega}_h}^2 . \tag{17}$$

Let us show that

$$\|u - v\|_{1,2,(0,1)} \le \left(2 + \frac{c_2}{c_1}\right) \left(1 + \frac{L(\pi^2 + 1)}{\pi^2 c_4}\right) \|u - v\|_{1,2,\hat{\omega}_h} . \tag{18}$$

By the substitution $u(x) = \tilde{u}(x) + \hat{u}(x)$, $x_{j-1} \le x \le x_{j+1}$, the problems

$$\frac{d}{dx}\left[k(x)\frac{du}{dx}\right] = -f\left(x, u(x), \frac{du(x)}{dx}\right), \ x \in (x_{j-2+\alpha}, x_{j-1+\alpha}),$$
$$u(x_{j-2+\alpha}) = u_{j-2+\alpha}, \quad u(x_{j-1+\alpha}) = u_{j-1+\alpha}, \ \alpha = 1, 2.$$

is reduced to the form

$$\frac{d}{dx}\left[k(x)\frac{d\tilde{u}}{dx}\right] = -f\left(x, \tilde{u}(x) + \hat{u}(x), \frac{d\tilde{u}}{dx} + \frac{d\hat{u}}{dx}\right), \ x \in (x_{j-2+\alpha}, x_{j-1+\alpha}),$$
$$\tilde{u}(x_{j-2+\alpha}) = 0, \ \tilde{u}(x_{j-1+\alpha}) = 0, \ \alpha = 1, 2.$$

Then, by virtue of (2),(5) and the Lipschitz condition $\forall \tilde{u}(x), \tilde{v}(x) \in \overset{0}{W}{}_2^1(e)$, we have

$$c_1 \frac{\pi^2}{1+\pi^2} \|\tilde{u} - \tilde{v}\|_{1,2,e_\alpha}^2 \le c_1 \left\| \frac{d\tilde{u}}{dx} - \frac{d\tilde{v}}{dx} \right\|_{0,2,e_\alpha}^2 \le \int_{x_{j-2+\alpha}}^{x_{j-1+\alpha}} k(x) \left[\frac{d\tilde{u}}{dx} - \frac{d\tilde{v}}{dx} \right]^2 dx =$$
$$= - \int_{x_{j-2+\alpha}}^{x_{j-1+\alpha}} \frac{d}{dx}\left(k(x)\frac{d(\tilde{u} - \tilde{v})}{dx}\right)(\tilde{u} - \tilde{v})\, dx = \int_{x_{j-2+\alpha}}^{x_{j-1+\alpha}} \left[f\left(x, \tilde{u} + \hat{u}, \frac{d\tilde{u}}{dx} + \frac{d\hat{u}}{dx}\right) - \right.$$
$$\left. - f\left(x, \tilde{v} + \hat{v}, \frac{d\tilde{v}}{dx} + \frac{d\hat{v}}{dx}\right) \right] [\tilde{u}(x) - \tilde{v}(x)]\, dx = \int_{x_{j-2+\alpha}}^{x_{j-1+\alpha}} \left[f\left(x, \tilde{u} + \hat{u}, \frac{d\tilde{u}}{dx} + \frac{d\hat{u}}{dx}\right) - \right.$$
$$\left. - f\left(x, \tilde{v} + \hat{u}, \frac{d\tilde{v}}{dx} + \frac{d\hat{u}}{dx}\right) \right] [\tilde{u}(x) + \hat{u}(x) - \hat{u}(x) - \tilde{v}(x)]\, dx +$$
$$+ \int_{x_{j-2+\alpha}}^{x_{j-1+\alpha}} \left[f\left(x, \tilde{v} + \hat{u}, \frac{d\tilde{v}}{dx} + \frac{d\hat{u}}{dx}\right) - f\left(x, \tilde{v} + \hat{v}, \frac{d\tilde{v}}{dx} + \frac{d\hat{v}}{dx}\right) \right] [\tilde{u}(x) - \tilde{v}(x)]\, dx \le$$
$$\le c_3 \|\tilde{u} - \tilde{v}\|_{1,2,e_\alpha}^2 + \left\{ \int_{x_{j-2+\alpha}}^{x_{j-1+\alpha}} \left[f\left(x, \tilde{v} + \hat{u}, \ fracd\tilde{v}dx + \frac{d\hat{u}}{dx}\right) - \right. \right.$$
$$\left. \left. - f\left(x, \tilde{v} + \hat{v}, \frac{d\tilde{v}}{dx} + \frac{d\hat{v}}{dx}\right) \right]^2 dx \right\}^{1/2} \left\{ \int_{x_{j-2+\alpha}}^{x_{j-1+\alpha}} [\tilde{u}(x) - \tilde{v}(x)]^2 dx \right\}^{1/2} \le$$
$$\le L \|\hat{u} - \hat{v}\|_{1,2,e_\alpha} \|\tilde{u} - \tilde{v}\|_{1,2,e_\alpha} + c_3 \|\tilde{u} - \tilde{v}\|_{1,2,e_\alpha}^2 .$$

It follows that

$$\|\tilde{u}-\tilde{v}\|_{1,2,e_\alpha} \leq \frac{L(\pi^2+1)}{\pi^2 c_4}\|\hat{u}-\hat{v}\|_{1,2,e_\alpha}.$$

Consequently,

$$\|u-v\|_{1,2,(0,1)} \leq \|\tilde{u}-\tilde{v}\|_{1,2,(0,1)} + \|\hat{u}-\hat{v}\|_{1,2,(0,1)} \leq$$
$$\leq \left(1+\frac{L(\pi^2+1)}{\pi^2 c_4}\right)\|\hat{u}-\hat{v}\|_{1,2,(0,1)} \leq \left(2+\frac{c_2}{c_1}\right)\left(1+\frac{L(\pi^2+1)}{\pi^2 c_4}\right)\|u-v\|_{1,2,\hat{\omega}_h}.$$

In view of (17) and (18), we obtain

$$(A_h(x,u)-A_h(x,v),z)_{\hat{\omega}_h} \leq \|u-v\|_{B_h}\|z\|_{B_h} +$$
$$+2L\left(2+\frac{c_2}{c_1}\right)\left(1+\frac{L(\pi^2+1)}{\pi^2 c_4}\right)\|u-v\|_{1,2,\hat{\omega}_h}\|z\|_{0,2,\hat{\omega}_h} \leq \|u-v\|_{B_h}\|z\|_{B_h} +$$
$$+\frac{L(1+2\sqrt{2})}{4}\left(2+\frac{c_2}{c_1}\right)\left(1+\frac{L(\pi^2+1)}{\pi^2 c_4}\right)\|u_{\bar{x}}-v_{\bar{x}}\|_{0,2,\hat{\omega}_h^+}\|z_{\bar{x}}\|_{0,2,\hat{\omega}_h^+} \leq$$
$$\leq \left(1+\frac{L(1+2\sqrt{2})}{4c_1}\left(2+\frac{c_2}{c_1}\right)\left(1+\frac{L(\pi^2+1)}{\pi^2 c_4}\right)\right)\|u-v\|_{B_h}\|z\|_{B_h}.$$

Setting $z = B_h^{-1}(A_h(x,u)-A_h(x,v))$ yields

$$\left\|B_h^{-1}(A_h(x,u)-A_h(x,v))\right\|_{B_h} \leq$$
$$\leq \left(1+\frac{L(1+2\sqrt{2})}{4c_1}\left(2+\frac{c_2}{c_1}\right)\left(1+\frac{L(\pi^2+1)}{\pi^2 c_4}\right)\right)\|u-v\|_{B_h} \tag{19}$$

It follows from (19) and (16) that

$$\left(A_h(x,u)-A_h(x,v), B_h^{-1}(A_h(x,u)-A_h(x,v))\right)_{\hat{\omega}_h} \leq$$
$$\leq \left(1+\frac{L(1+2\sqrt{2})}{4c_1}\left(2+\frac{c_2}{c_1}\right)\left(1+\frac{L(\pi^2+1)}{\pi^2 c_4}\right)\right)^2\|u-v\|_{B_h}^2 \leq$$
$$\leq \frac{c_2}{c_4}\left(1+\frac{L(1+2\sqrt{2})}{4c_1}\left(2+\frac{c_2}{c_1}\right)\left(1+\frac{L(\pi^2+1)}{\pi^2 c_4}\right)\right)^2(A_h(x,u)-A_h(x,v),u-v)_{\hat{\omega}_h}.$$

Then (see [6]) the iterative method (14) converges in H_{B_h}, end the error is estimated by (15).

Note that the space H_{B_h} coincides with $\overset{0}{W}{}_2^1(\hat{\omega}_h)$ and we have the norm equivalence relations

$$\gamma_1\|u\|_{1,2,\hat{\omega}_h} \leq \|u\|_{B_h} \leq \gamma_2\|u\|_{1,2,\hat{\omega}_h},$$

where

$$\|u\|_{1,2,\hat{\omega}_h} = \left(\|u\|_{0,2,\hat{\omega}_h}^2 + \|u_{\bar{x}}\|_{0,2,\hat{\omega}_h^+}^2\right)^{1/2}.$$

Lemma 3. *Let the conditions of Lemma 2 be fulfilled. Then the iterative method (14) converges, its error satisfies (15), and the estimate*

$$\left\| k\frac{du^{(n)}}{dx} - k\frac{du}{dx} \right\|_{0,2,\hat{\omega}_h} \le M \left\| u^{(n)} - u \right\|_{1,2,\hat{\omega}_h} \le Mq^n$$

holds.

Proof. In view of (7), we obtain

$$\left\| k\frac{du^{(n)}}{dx} - k\frac{du}{dx} \right\|_{0,2,\hat{\omega}_h} = \left\{ \sum_{j=1}^{N-1} \hbar_j \left[\frac{h_j}{V_1^j(x_j)} \left(u_{\bar{x},j}^{(n)} - u_{\bar{x},j} \right) - \right.\right.$$

$$\left.\left. - \frac{1}{V_1^j(x_j)} \left(w_1^j \left(x_j, u^{(n)}\right) - w_1^j(x_j, u) \right) + l_1^j \left(x, u^{(n)}\right) - l_1^j(x, u) \right]^2 \right\}^{1/2} \le$$

$$\le \left\{ \sum_{j=1}^{N-1} \hbar_j \left[\frac{h_j}{V_1^j(x_j)} \right]^2 \left[u_{\bar{x},j}^{(n)} - u_{\bar{x},j} \right]^2 \right\}^{1/2} +$$

$$+ \left\{ \sum_{j=1}^{N-1} \hbar_j \left[\frac{1}{V_1^j(x_j)} \right]^2 \left[w_1^j \left(x_j, u^{(n)}\right) - w_1^j(x_j, u) \right]^2 \right\}^{1/2} +$$

$$+ \left\{ \sum_{j=1}^{N-1} \hbar_j \left[l_1^j \left(x, u^{(n)}\right) - l_1^j(x, u) \right]^2 \right\}^{1/2} \le c_2 \left\| u_{\bar{x}}^{(n)} - u_{\bar{x}} \right\|_{0,2,\hat{\omega}_h^+} +$$

$$+ c_2 \left\{ \sum_{j=1}^{N-1} \frac{\hbar_j}{h_j^2} \left[w_1^j \left(x_j, u^{(n)}\right) - w_1^j(x_j, u) \right]^2 \right\}^{1/2} +$$

$$+ \left\{ \sum_{j=1}^{N-1} \hbar_j \left[l_1^j \left(x, u^{(n)}\right) - l_1^j(x, u) \right]^2 \right\}^{1/2}.$$

Using the Cauchy-Schwarz inequality and the Lipschitz condition, we estimate the quantities

$$\left[l_1^j \left(x, u^{(n)}\right) - l_1^j(x, u) \right]^2 = \left\{ \int_{x_{j-1}}^{x} \left[f\left(t, u^{(n)}, \frac{du^{(n)}}{dt}\right) - f\left(t, u, \frac{du}{dt}\right) \right] dt \right\}^2 \le$$

$$\le \int_{x_{j-1}}^{x} dt \int_{x_{j-1}}^{x} \left[f\left(t, u^{(n)}, \frac{du^{(n)}}{dt}\right) - f\left(t, u, \frac{du}{dt}\right) \right]^2 dt \le$$

$$\le h_j L^2 \int_{x_{j-1}}^{x_j} \left[\left| u^{(n)}(t) - u(t) \right| + \left| \frac{du^{(n)}(t)}{dt} - \frac{du(t)}{dt} \right| \right]^2 dt \le$$

$$\le 2h_j L^2 \left[\int_{x_{j-1}}^{x_j} \left[u^{(n)}(t) - u(t) \right]^2 dt + \int_{x_{j-1}}^{x_j} \left[\frac{du^{(n)}(t)}{dt} - \frac{du(t)}{dt} \right]^2 dt \right].$$

$$\left[w_1^j\left(x_j,u^{(n)}\right)-w_1^j\left(x_j,u\right)\right]^2=\left[\int\limits_{x_{j-1}}^{x_j}\frac{\left[l_1^j\left(x,u^{(n)}\right)-l_1^j\left(x,u\right)\right]}{k\left(x\right)}dx\right]^2\leq$$

$$\leq\int\limits_{x_{j-1}}^{x_j}\frac{dx}{k^2\left(x\right)}\int\limits_{x_{j-1}}^{x_j}\left[l_1^j\left(x,u^{(n)}\right)-l_1^j\left(x,u\right)\right]^2dx\leq$$

$$\leq\frac{2h_j^3L^2}{c_1^2}\left[\int\limits_{x_{j-1}}^{x_j}\left[u^{(n)}(t)-u(t)\right]^2dt+\int\limits_{x_{j-1}}^{x_j}\left[\frac{du^{(n)}}{dt}-\frac{du}{dt}\right]^2dt\right].$$

In view of (18),

$$c_2\left\{\sum_{j=1}^{N-1}\frac{\hbar_j}{h_j^2}\left[w_1^j\left(x_j,u^{(n)}\right)-w_1^j\left(x_j,u\right)\right]^2\right\}^{1/2}\leq\sqrt{2}L\left|h\right|\frac{c_2}{c_1}\left\|u^{(n)}-u\right\|_{1,2,(0,1)}\leq$$

$$\leq\sqrt{2}L\frac{c_2}{c_1}\left(2+\frac{c_2}{c_1}\right)\left(1+\frac{L\left(\pi^2+1\right)}{\pi^2c_1}\right)\left\|u^{(n)}-u\right\|_{1,2,\hat{\omega}_h}.$$

$$\left\{\sum_{j=1}^{N-1}\left[l_1^j\left(x,u^{(n)}\right)-l_1^j\left(x,u\right)\right]^2\right\}^{1/2}=\sqrt{2}\left|h\right|L\left\|u^{(n)}-u\right\|_{1,2,(0,1)}\leq$$

$$\leq\sqrt{2}L\left(2+\frac{c_2}{c_1}\right)\left(1+\frac{L\left(\pi^2+1\right)}{\pi^2c_1}\right)\left\|u^{(n)}-u\right\|_{1,2,\hat{\omega}_h}.$$

Consequently,

$$\left\|k\frac{du^{(n)}}{dx}-k\frac{du}{dx}\right\|_{0,2,\hat{\omega}_h}\leq c_2\left\|u_{\bar{x},j}^{(n)}-u_{\bar{x},j}\right\|_{0,2,\hat{\omega}_h^+}+$$

$$+\sqrt{2}L\left(1+\frac{L\left(\pi^2+1\right)}{\pi^2c_1}\right)\left(2+3\frac{c_2}{c_1}+\frac{c_2^2}{c_1^2}\right)\left\|u^{(n)}-u\right\|_{1,2,\hat{\omega}_h}\leq$$

$$\leq max\left\{c_2,\sqrt{2}L\left(1+\frac{L\left(\pi^2+1\right)}{\pi^2c_1}\right)\left(2+3\frac{c_2}{c_1}+\frac{c_2^2}{c_1^2}\right)\right\}\left\|u^{(n)}-u\right\|_{1,2,\hat{\omega}_h}=$$

$$=M\left\|u^{(n)}-u\right\|_{1,2,\hat{\omega}_h}\leq Mq^n.$$

Lemma is proved.

3 Algorithmic implementation of the ETPDS

First, we take into account that, because of

$$(-1)^{\alpha+1}\int\limits_{x_{j+(-1)^\alpha}}^{x_j}V_\alpha^j\left(\xi\right)f\left(\xi,u,\frac{du}{d\xi}\right)d\xi=(-1)^\alpha V_\alpha^j\left(x_j\right)l_\alpha^j\left(x_j,u\right)+w_\alpha^j\left(x_j,u\right),$$

we have

$$\varphi\left(x_j,u\right)=\hat{T}^{x_j}\left(f\left(\xi,u\left(\xi\right),\frac{du\left(\xi\right)}{d\xi}\right)\right)=$$
$$=\hbar_j^{-1}\sum_{\alpha=1}^2(-1)^\alpha\left[l_\alpha^j\left(x_j,u\right)+(-1)^\alpha\frac{w_\alpha^j\left(x_j,u\right)}{V_\alpha^j\left(x_j\right)}\right],\qquad\alpha=1,2. \tag{20}$$

Therefore, to construct the ETDS defined (10), (11), and (20) $\forall x_j \in \hat{\omega}_h$, it is necessary to solve two initial value problems of (7): one ($\alpha = 1$) in the forward direction on the closed interval $[x_{j-1}, x_j]$, and the other ($\alpha = 2$) in the backward direction on $[x_j, x_{j+1}]$; both problems have the smoth coefficients. To solve them numerically, we apply a one step method Teylor series expansion of $\bar{m}$th order of accuracy.

Let's notice, the functions $w_\alpha^j(x,u)$, $l_\alpha^j(x,u)$, $\alpha = 1,2$, which are the solutions to the initial value problems (7) depend on parameters $b_\alpha \equiv b_\alpha^j(u) \equiv w_\alpha^j(x_j,u)$, i.e. $w_\alpha^j(x,u) \equiv w_\alpha^j(x,u,b_\alpha)$, $l_\alpha^j(x,u) \equiv l_\alpha^j(x,u,b_\alpha)$, $\alpha = 1,2$.

The algorithm for the solution of problems (7) works as follows.

1. By successive differentiation (7) we find derivatives

$$\frac{d^p w_\alpha^j(x,u,b_\alpha)}{dx^p}, p = 2,3,...,\bar{m}, \qquad \frac{d^p l_\alpha^j(x,u,b_\alpha)}{dx^p}, p = 2,3,...,m.$$

2. Evaluate the approximate values of parameters b_α, $\alpha = 1,2$ from formulas

$$b_\alpha^{(1)} = 0, \quad b_\alpha^{(s-1)} \equiv b_\alpha^{(s-1)j}(u) = w_\alpha^{(s-1)j}(x_j,u) =$$

$$= \sum_{p=2}^{s-1} \frac{\left[(-1)^{\alpha+1} h_{j-1+\alpha}\right]^p}{p!} \frac{d^p w_\alpha^j\left(x_{j+(-1)^\alpha}, u, b_\alpha^{(s-2)}\right)}{dx^p}, \; s = 3,4,...,\bar{m}. \tag{21}$$

3. Calculate now the approximate solutions of problems (7)

$$w_\alpha^{(\bar{m})j}(x_j,u) = \sum_{p=2}^{\bar{m}} \frac{\left[(-1)^{\alpha+1} h_{j-1+\alpha}\right]^p}{p!} \frac{d^p w_\alpha^j\left(x_{j+(-1)^\alpha}, u, b_\alpha^{(\bar{m}-1)}\right)}{dx^p}, \tag{22}$$

$$l_\alpha^{(m)j}(x_j,u) = \sum_{p=1}^{m} \frac{\left[(-1)^{\alpha+1} h_{j-1+\alpha}\right]^p}{p!} \frac{d^p l_\alpha^j\left(x_{j+(-1)^\alpha}, u, b_\alpha^{(\bar{m}-1)}\right)}{dx^p}. \tag{23}$$

Lemma 4. *Let*

$$0 < c_1 \le k(x), \quad \forall x \in [0,1], \; k(x) \in Q^{m+1}[0,1],$$
$$f(x,u,\xi) \in \bigcup_{j=1}^{N} C^m\left([x_{j-1},x_j] \times R^2\right)$$

Then

$$V_\alpha^{(m)j}(x_j) = (-1)^{\alpha+1} \sum_{p=1}^{m} \frac{\left[(-1)^{\alpha+1} h_{j-1+\alpha}\right]^p}{p!} \left[\frac{d^{p-1}}{dx^{p-1}} \frac{1}{k(x)}\right]_{x=x_{j+(-1)^\alpha}} = \tag{24}$$
$$= V_\alpha^j(x_j) + O\left(h_{j-1+\alpha}^{m+1}\right),$$

$$w_\alpha^j(x_j,u) = w_\alpha^{(\bar{m})j}(x_j,u) + (1 - \bar{m} + m) \times$$

$$\times \frac{\left[(-1)^{\alpha+1} h_{j-1+\alpha}\right]^{m+1}}{(m+1)!} \frac{d^{m+1} w_\alpha^j\left(x_{j+(-1)^\alpha}, u\right)}{dx^{m+1}} + O\left(h_{j-1+\alpha}^{m+2}\right), \tag{25}$$

$$l_\alpha^j(x_j,u) = l_\alpha^{(m)j}(x_j,u) +$$

$$+ \frac{\left[(-1)^{\alpha+1} h_{j-1+\alpha}\right]^{m+1}}{(m+1)!} \frac{d^{m+1} l_\alpha^j\left(x_{j+(-1)^\alpha}, u\right)}{dx^{m+1}} + O\left(h_{j-1+\alpha}^{m+2}\right). \tag{26}$$

Proof. To prove (24), we expand the functions on the left-hand side of (24) into Teylor series at points $x_{j+(-1)^\alpha}$ representing the residual term in the integral form:

$$V_\alpha^j(x_j) = \sum_{p=1}^{m} \frac{\left[(-1)^{\alpha+1} h_{j-1+\alpha}\right]^p}{p!} \frac{d^p V_\alpha^j\left(x_{j+(-1)^\alpha}\right)}{dx^p} +$$

$$+\frac{1}{m!} \int\limits_{x_{j+(-1)^\alpha}}^{x_j} (x_j - t)^m \frac{d^{m+1} V_\alpha^j(t)}{dt^{m+1}} dt = V_\alpha^{(m)j}(x_j) + O\left(h_{j-1+\alpha}^{m+1}\right).$$

Since

$$\frac{d^p V_\alpha^j\left(x_{j+(-1)^\alpha}\right)}{dx^p} = (-1)^{\alpha+1} \left[\frac{d^{p-1}}{dx^{p-1}} \frac{1}{k(x)}\right]_{x_{j+(-1)^\alpha}},$$

we arrive at (24).

Let as prove by induction the equalities

$$b_\alpha^{(s-1)} = b_\alpha + O\left(h_{j-1+\alpha}^s\right), \quad s = 2, 3, \ldots, \bar{m}. \tag{27}$$

Let $s = 2$ then we have

$$b_\alpha = b_\alpha^{(1)} + O\left(h_{j-1+\alpha}^2\right).$$

Assume that relations (27) are valid for $s = q - 1$ and prove them for $s = q$. Since

$$b_\alpha^{(q)} = w_\alpha^{(q)j}(x_j, u) = \sum_{p=2}^{q} \frac{\left[(-1)^{\alpha+1} h_{j-1+\alpha}\right]^p}{p!} \frac{d^p w_\alpha^j\left(x_{j+(-1)^\alpha}, u, b_\alpha^{(q-1)}\right)}{dx^p} =$$

$$= \sum_{p=2}^{q} \frac{\left[(-1)^{\alpha+1} h_{j-1+\alpha}\right]^p}{p!} \frac{d^p w_\alpha^j\left(x_{j+(-1)^\alpha}, u, b_\alpha + O\left(h_{j-1+\alpha}^q\right)\right)}{dx^p} =$$

$$= \sum_{p=2}^{q} \frac{\left[(-1)^{\alpha+1} h_{j-1+\alpha}\right]^p}{p!} \frac{d^p w_\alpha^j\left(x_{j+(-1)^\alpha}, u, b_\alpha\right)}{dx^p} + O\left(h_{j-1+\alpha}^{q+3}\right),$$

we obtain

$$b_\alpha = \sum_{p=2}^{q} \frac{\left[(-1)^{\alpha+1} h_{j-1+\alpha}\right]^p}{p!} \frac{d^p w_\alpha^j\left(x_{j+(-1)^\alpha}, u, b_\alpha\right)}{dx^p} +$$

$$+\frac{1}{q!} \int\limits_{x_{j+(-1)^\alpha}}^{x_j} (x_j - t)^q \frac{d^{q+1} w_\alpha^j(t, u)}{dt^{q+1}} dt = w_\alpha^{(q)j}(x_j, u) + O\left(h_{j-1+\alpha}^{q+1}\right) =$$

$$= b_\alpha^{(q)} + O\left(h_{j-1+\alpha}^{q+1}\right).$$

Moreover,

$$w_\alpha^{(m+1)j}(x_j,u)=\sum_{p=2}^{m+1}\frac{\left[(-1)^{\alpha+1}h_{j-1+\alpha}\right]^p}{p!}\frac{d^p w_\alpha^j\left(x_{j+(-1)^\alpha},u,b_\alpha^{(\bar m)}\right)}{dx^p}=$$

$$+\sum_{p=2}^{m+1}\frac{\left[(-1)^{\alpha+1}h_{j-1+\alpha}\right]^p}{p!}\frac{d^p w_\alpha^j\left(x_{j+(-1)^\alpha},u,b_\alpha\right)}{dx^p}+O\left(h_{j-1+\alpha}^{\bar m+3}\right),$$

$$l_\alpha^{(m)j}(x_j,u)=\sum_{p=1}^{m}\frac{\left[(-1)^{\alpha+1}h_{j-1+\alpha}\right]^p}{p!}\frac{d^p l_\alpha^j\left(x_{j+(-1)^\alpha},u,b_\alpha^{\bar m-1}\right)}{dx^p}=$$

$$=\sum_{p=1}^{m}\frac{\left[(-1)^{\alpha+1}h_{j-1+\alpha}\right]^p}{p!}\frac{d^p l_\alpha^j\left(x_{j+(-1)^\alpha},u,b_\alpha\right)}{dx^p}+O\left(h_{j-1+\alpha}^{\bar m+2}\right).$$

Therefore,

$$w_\alpha^j(x_j,u)=\sum_{p=2}^{m+1}\frac{\left[(-1)^{\alpha+1}h_{j-1+\alpha}\right]^p}{p!}\frac{d^p w_\alpha^j\left(x_{j+(-1)^\alpha},u,b_\alpha\right)}{dx^p}+$$

$$+\frac{1}{(m+1)!}\int_{x_{j+(-1)^\alpha}}^{x_j}(x_j-t)^{m+1}\frac{d^{m+2}w_\alpha^j(t,u)}{dt^{m+2}}dt=w_\alpha^{(m+1)j}(x_j,u)+O\left(h_{j-1+\alpha}^{m+2}\right),$$

$$l_\alpha^j(x_j,u)=\sum_{p=1}^{m}\frac{\left[(-1)^{\alpha+1}h_{j-1+\alpha}\right]^p}{p!}\frac{d^p l_\alpha^j\left(x_{j+(-1)^\alpha},u,b_\alpha\right)}{dx^p}+$$

$$+\frac{1}{(m+1)!}\int_{x_{j+(-1)^\alpha}}^{x_j}(x_j-t)^{m}\frac{d^{m+2}l_\alpha^j(t,u)}{dt^{m+2}}dt=l_\alpha^{(m)j}(x_j,u)+$$

$$+\frac{\left[(-1)^{\alpha+1}h_{j-1+\alpha}\right]^{m+1}}{(m+1)!}\frac{d^{m+1}l_\alpha^j\left(x_{j+(-1)^\alpha},u,b_\alpha\right)}{dt^{m+1}}+O\left(h_{j-1+\alpha}^{m+2}\right).$$

Let's prove relations (25). If m is odd, then $\bar m=m+1$ and it follows from (27) that

$$w_\alpha^j(x_j,u)=w_\alpha^{(\bar m)j}(x_j,u)+O\left(h_{j-1+\alpha}^{m+2}\right).$$

If $m=\bar m$ is even, we have

$$w_\alpha^j(x_j,u)=\sum_{p=2}^{m+1}\frac{\left[(-1)^{\alpha+1}h_{j-1+\alpha}\right]^p}{p!}\frac{d^p w_\alpha^j\left(x_{j+(-1)^\alpha},u\right)}{dx^p}+$$

$$+\frac{1}{(m+1)!}\int_{x_{j+(-1)^\alpha}}^{x_j}(x_j-t)^{m+1}\frac{d^{m+2}w_\alpha^j(t,u)}{dt^{m+2}}dt=w_\alpha^{(\bar m)j}(x_j,u)+$$

$$+\frac{\left[(-1)^{\alpha+1}h_{j-1+\alpha}\right]^p}{(m+1)!}\frac{d^{m+1}w_\alpha^j\left(x_{j+(-1)^\alpha},u\right)}{dx^{m+1}}+O\left(h_{j-1+\alpha}^{m+2}\right).$$

The lemma is proved.

Instead of the ETPDS given by (10), (11) and (20), we can use an TPDS of rank $\bar{m}$ of the form

$$\left(a^{(\bar{m})}y_{\bar{x}}^{(\bar{m})}\right)_{\hat{x}} = -\varphi^{(\bar{m})}\left(x, y^{(\bar{m})}\right), x \in \hat{\omega}_h,$$
$$y^{(\bar{m})}(0) = \mu_1,\ y^{(\bar{m})}(1) = \mu_2, a^{(\bar{m})}(x_j) = \left[\frac{1}{h_j}V_1^{(\bar{m})j}(x_j)\right]^{-1}, \tag{28}$$
$$\varphi^{(\bar{m})}(x_j, u) = \hbar_j^{-1}\sum_{\alpha=1}^{2}(-1)^{\alpha}\left[l_{\alpha}^{(m)j}(x_j, u) + (-1)^{\alpha}\frac{w_{\alpha}^{(\bar{m})j}(x_j, u)}{V_{\alpha}^{j}(x_j)}\right].$$

We need the following lemma to prove the existence and uniqueness of a solution to TPDS (28) and to establish its accuracy.

Lemma 5. *Let the conditions of Lemma 4 be fulfilled. Then*

$$\left|a^{(\bar{m})}(x_j) - a(x_j)\right| \le M|h|^{\bar{m}}, \tag{29}$$

$$\varphi^{(m)}(x_j, u) - \varphi(x_j, u) =$$

$$\left\{\frac{h_j^{m+1}}{(m+1)!}\left.\frac{d^{m+1}l_2^{j-1}(x, u)}{dx^{m+1}}\right|_{x_j-0}\right\}_{\hat{x}} + O\left(\frac{h_j^{m+2} + h_{j+1}^{m+2}}{\hbar_j}\right), \tag{30}$$

if m is odd, and

$$\varphi^{(m)}(x_j, u) - \varphi(x_j, u) =$$

$$\left\{\frac{h_j^{m}k(x_j-0)}{(m+1)!}\frac{d^m}{dx^m}\left.\left[\frac{l_2^{j-1}(x, u)}{k(x)}\right]\right|_{x_j-0}\right\}_{\hat{x}} + O\left(\frac{h_j^{m+1} + h_{j+1}^{m+1}}{\hbar_j}\right), \tag{31}$$

if m is even.

Proof. Inequality (29) follows from (11). Prove (30) and (31). Note that

$$\varphi^{(\bar{m})}(x_j, u) - \varphi(x_j, u) = \hbar_j^{-1}\sum_{\alpha=1}^{2}(-1)^{\alpha}\left\{l_{\alpha}^{(m)j}(x_j, u) - l_{\alpha}^{j}(x_j, u) + \right.$$
$$\left. + (-1)^{\alpha}\left[\frac{w_{\alpha}^{(\bar{m})j}(x_j, u)}{V_{\alpha}^{(\bar{m})j}(x_j)} - \frac{w_{\alpha}^{j}(x_j, u)}{V_{\alpha}^{j}(x_j)}\right]\right\}. \tag{32}$$

By Lemma 4, we have

$$l_{\alpha}^{j}(x_j, u) - l_{\alpha}^{(m)j}(x_j, u) =$$
$$= \frac{\left[(-1)^{\alpha+1}h_{j-1+\alpha}\right]^{m+1}}{(m+1)!}\frac{d^{m+1}l_{\alpha}^{j}\left(x_{j+(-1)^{\alpha}}, u\right)}{dx^{m+1}} + O\left(h_{j-1+\alpha}^{m+2}\right),$$
$$\frac{w_{\alpha}^{j}(x_j, u)}{V_{\alpha}^{j}(x_j)} - \frac{w_{\alpha}^{(\bar{m})j}(x_j, u)}{V_{\alpha}^{(\bar{m})j}(x_j)} = (1 - \bar{m} + m)\frac{\left[(-1)^{\alpha+1}h_{j-1+\alpha}\right]^{m+1}}{(m+1)!}\times \tag{33}$$
$$\times\frac{d^{m+1}w_{\alpha}^{j}\left(x_{j+(-1)^{\alpha}}, u\right)}{dx^{m+1}}\frac{1}{V_{\alpha}^{j}(x_j)} + O\left(h_{j-1+\alpha}^{m+2}\right).$$

Combining (32), (33), $V_1^j(x_j) = \dfrac{h_j}{k_{j-1}} + O(h_j^2)$ and $V_2^j(x_j) = \dfrac{h_{j+1}}{k_{j+1}} + O(h_{j+1}^2)$ we obtain

$$\varphi^{(m)}(x_j,u) - \varphi(x_j,u) = \frac{1}{(m+1)!\hbar_j}\left[h_{j+1}^{m+1}\frac{d^{m+1}l_2^j(x_{j+1},u)}{dx^{m+1}} - h_j^{m+1}\frac{d^{m+1}l_1^j(x_{j-1},u)}{dx^{m+1}}\right] + O\left(\frac{h_j^{m+2}+h_{j+1}^{m+2}}{\hbar_j}\right) \tag{34}$$

if m is odd, and for m is even, we come to the relation

$$\varphi^{(m)}(x_j,u) - \varphi(x_j,u) = \frac{1}{(m+1)!\hbar_j}\left[h_{j+1}^{m}k_{j+1}\frac{d^{m+1}w_2^j(x_{j+1},u)}{dx^{m+1}} - h_j^{m}k_{j-1}\frac{d^{m}w_1^j(x_{j-1},u)}{dx^{m+1}}\right] + O\left(\frac{h_j^{m+1}+h_{j+1}^{m+1}}{\hbar_j}\right). \tag{35}$$

From (8) we have

$$u(x) = \hat{u}(x) + w_1^j(x,u) - \frac{V_1^j(x)}{V_1^j(x_j)}w_1^j(x_j,u) =$$

$$= \hat{u}(x) + w_2^{j-1}(x,u) - \frac{V_2^{j-1}(x)}{V_2^{j-1}(x_{j-1})}w_2^{j-1}(x_j,u), \quad x \in [x_{j-1},x_j].$$

This reasoning yields the relation

$$w_1^j(x,u) = w_2^{j-1}(x,u) + \frac{V_1^j(x)}{V_1^j(x_j)}w_1^j(x_j,u) - \frac{V_2^{j-1}(x)}{V_2^{j-1}(x_{j-1})}w_2^{j-1}(x_{j-1},u),$$
$$x \in [x_{j-1},x_j].$$

Differentiating last equality and multiplying it on $k(x)$, we obtain

$$l_1^j(x,u) = l_2^{j-1}(x,u) + \frac{w_1^j(x_j,u) + w_2^{j-1}(x_{j-1},u)}{V_1^j(x_j)}, \quad x \in [x_{j-1},x_j].$$

Since under conditions of Lemma

$$\frac{d^{m+1}l_1^j(x,u)}{dx^{m+1}} = \frac{d^{m+1}l_2^{j-1}(x,u)}{dx^{m+1}},$$

$$\frac{d^{m+1}w_1^j(x,u)}{dx^{m+1}} = \frac{d^{m+1}w_2^{j-1}(x,u)}{dx^{m+1}} + \frac{d^m}{dx^m}\left[\frac{1}{k(x)}\right]\frac{w_1^j(x_j,u) + w_2^{j-1}(x_{j-1},u)}{V_1^j(x_j)} =$$

$$= \frac{d^{m+1}w_2^{j-1}(x,u)}{dx^{m+1}} + O(h_j), \quad x \in [x_{j-1},x_j],$$

the by virtue of $k_{j-1} = k_j + O(h_j)$, we obtain

$$\frac{d^{m+1}l_1^j(x_{j-1}+0,u)}{dx^{m+1}} = \frac{d^{m+1}l_2^{j-1}(x_j-0,u)}{dx^{m+1}} + O(h_j),$$
$$k(x_{j-1}+0)\frac{d^{m+1}w_1^j(x_{j-1}+0,u)}{dx^{m+1}} = k(x_j-0)\frac{d^{m+1}w_2^{j-1}(x_j-0,u)}{dx^{m+1}} + O(h_j). \tag{36}$$

Combining (36), (34) and (35) yields (30) and (31).

Theorem 3. *Let the conditions of Lemmas 1 and 4 be fulfilled. Then* $\exists h_0 > 0$ *such that* $\forall \{h_j\}_{j=1}^N : |h| = \max_{1\le j\le N} h_j \le h_0$, *the TPDS given by (28), (24)-(27) has a unique solution, whose accuracy is characterized by the estimate*

$$\left\| y^{(\bar m)} - u \right\|^*_{1,2,\hat\omega_h} = \left[\left\| y^{(\bar m)} - u \right\|^2_{0,2,\hat\omega_h} + \left\| k\frac{dy^{(\bar m)}}{dx} - k\frac{du}{dx} \right\|^2_{0,2,\hat\omega_h} \right]^{1/2} \le M\,|h|^{\bar m},$$

where

$$k(x_j)\,\frac{dy^{(\bar m)}(x_j)}{dx} = \frac{h_j y^{(\bar m)}_{\bar x,j} - w_1^{(\bar m)j}\left(x_j, y^{(\bar m)}\right)}{V_1^{(\bar m)j}(x_j)} + l_1^{(m)j}\left(x_j, y^{(\bar m)}\right),$$

M is a constant independent of $|h|$.

Proof. We consider the operator

$$A_h^{(\bar m)}(x,u) = B_h^{(\bar m)}u - \varphi^{(\bar m)}(x,u),\; B_h^{(\bar m)}u = -\left(a^{(\bar m)}u_{\bar x}\right)_{\hat x},$$

In view of (30) and (31), we obtain

$$\left(A_h^{(\bar m)}(x,u) - A_h^{(\bar m)}(x,v), u - v\right)_{\hat\omega_h} = \left(a^{(\bar m)}(u_{\bar x} - v_{\bar x}), u_{\bar x} - v_{\bar x}\right)_{\hat\omega_h} -$$
$$- \left(\varphi^{(\bar m)}(x,u) - \varphi^{(\bar m)}(x,v), u - v\right)_{\hat\omega_h} = \left(A_h(x,u) - A_h(x,v), u - v\right)_{\hat\omega_h} + O\left(|h|^{\bar m}\right).$$

Then, by virtue of (13) $\exists h_0 > 0$ such that $\forall \{h_j\}_{j=1}^N : |h| \le h_0$, $0 < \tilde c_1 \le a^{(\bar m)}(x)$ and

$$\left(A_h^{(\bar m)}(x,u) - A_h^{(\bar m)}(x,v), u - v\right)_{\hat\omega_h} \ge \|u - v\|^2_{B_h^{(\bar m)}} \ge 8\frac{c_4 c_1}{c_2}\|u - v\|^2_{0,2,\hat\omega_h}. \tag{37}$$

Hence $A_h^{(\bar m)}(x,u)$ for $|h| \le h_0$ is a strongly monotone operator and for $|h| \le h_0$ TPDS by (28) has a unique solution $y^{(\bar m)}(x)$, $x \in \hat\omega_h$ (see [7]).

For the error $z(x) = y^{(\bar m)}(x) - u(x)$, $x \in \hat\omega_h$, we have the problem

$$\left[a^{(\bar m)}(x)\,z_{\bar x}(x)\right]_x + \varphi^{(\bar m)}\left(x, y^{(\bar m)}\right) - \varphi^{(\bar m)}(x,u) = \varphi(x,u) -$$
$$-\varphi^{(\bar m)}(x,u) + \left[\left(a(x) - a^{(\bar m)}(x)\right)u_{\bar x}(x)\right]_x,\quad z(0) = z(1) = 0. \tag{38}$$

It follows from (38) that

$$\left(A_h^{(\bar m)}(x,u) - A_h^{(\bar m)}(x,v), z\right)_{\hat\omega_h} =$$
$$= \left(\varphi^{(\bar m)}(x,u) - \varphi(x,u), z\right) + \left(\left(a^{(\bar m)} - a\right)u_{\bar x}, z_{\bar x}\right). \tag{39}$$

Taking into account (37), we have estimate

$$\left(A_h^{(\bar m)}(x,u) - A_h^{(\bar m)}(x,v), u - v\right)_{\hat\omega_h} \ge \|z\|^2_{B_h^{(\bar m)}} \tag{40}$$

By using the Cauchy-Schwarz inequality, (32) and (33), the right-hand side of (39) is estimated as follows:

$$\left(\left(a - a^{(\bar m)}\right)u_{\bar x}, z_{\bar x}\right)_{\hat\omega_h} \le \left\|a^{(\bar m)} - a\right\|_{0,2,\hat\omega_h} \|u_{\bar x}\|_{0,2,\omega_h^+} \|z_{\bar x}\|_{0,2,\hat\omega_h^+} \le$$
$$\le M\,|h|^{\bar m}\,\|z_{\bar x}\|_{0,2,\hat\omega_h^+} \le \frac{M\,|h|^{\bar m}}{\tilde c_1}\|z\|_{B_h^{(\bar m)}}, \tag{41}$$

$$\left(\varphi^{(m)}(x,u)-\varphi(x,u),z\right)_{\hat{\omega}_h} \le M|h|^{m+1}\|z_{\bar{x}}\|_{0,2,\hat{\omega}_h^+} \le \frac{M|h|^{m+1}}{\tilde{c}_1}\|z\|_{B_h^{(\bar{m})}}, \tag{42}$$

for odd m and

$$\left(\varphi(x,u)-\varphi^{(m)}(x,u),z\right)_{\hat{\omega}_h} \le M|h|^{m}\|z_{\bar{x}}\|_{0,2,\hat{\omega}_h^+} \le \frac{M|h|^{m}}{\tilde{c}_1}\|z\|_{B_h^{(\bar{m})}}, \tag{43}$$

for even m. From (41) and (43), it follows that $\|z\|_{B_h^{(\bar{m})}} \le M|h|^{\bar{m}}$. Since the norms $\|\cdot\|_{1,2,\hat{\omega}_h}$ and $\|\cdot\|_{B_h^{(\bar{m})}}$ are equivalent, we obtain $\|z\|_{1,2,\hat{\omega}_h} \le M|h|^{\bar{m}}$. Since

$$\left\|k(x)\frac{dz}{dx}\right\|_{0,2,\hat{\omega}_h} \le \left\|\frac{1}{V_1^{(\bar{m})j}}-\frac{1}{V_1^j}\right\|_{0,2,\hat{\omega}_h}\left[|h|\,\|u_{\bar{x}}\|_{0,2,\hat{\omega}_h^+}+\left\|w_1^j(x,u)\right\|_{0,2,\hat{\omega}_h}\right]+$$
$$+\frac{1}{\left\|V_1^{(\bar{m})j}\right\|_{0,2,\hat{\omega}_h}}\left[|h|\left\|y_{\bar{x}}^{(\bar{m})}-u_{\bar{x}}\right\|_{0,2,\hat{\omega}_h^+}+\left\|w_1^{(\bar{m})j}\left(x,y^{(\bar{m})}\right)-w_1^j\left(x,y^{(\bar{m})}\right)\right\|_{0,2,\hat{\omega}_h}+\right.$$
$$\left.+\left\|w_1^j\left(x,y^{(\bar{m})}\right)-w_1^j(x,u)\right\|_{0,2,\hat{\omega}_h}\right]+\left\|l_1^{(m)j}\left(x,y^{(\bar{m})}\right)-l_1^j\left(x,y^{(\bar{m})}\right)\right\|_{0,2,\hat{\omega}_h}+$$
$$+\left\|l_1^j\left(x,y^{(\bar{m})}\right)-l_1^j(x,u)\right\|_{0,2,\hat{\omega}_h} \le M_1|h|^{\bar{m}}+$$
$$+\left[\frac{\left\|\frac{\partial}{\partial u}w_1^j(x,u)\Big|_{u=\tilde{u}}\right\|_{0,2,\hat{\omega}_h}}{\left\|V_1^{(\bar{m})j}\right\|_{0,2,\hat{\omega}_h}}+\left\|\frac{\partial}{\partial u}l_1^j(x,u)\Big|_{u=\tilde{u}}\right\|_{0,2,\hat{\omega}_h}\right]\|z\|_{0,2,\hat{\omega}_h} \le M|h|^{\bar{m}},$$

we use (29) and (31) to obtain $\|z\|^*_{1,\infty,\hat{\omega}_h} \le Mh^{\bar{m}}$ Theorem is proved.

The $\bar{m}$th-order accurate three-point difference scheme given by (28) is solved by iteration.

Theorem 4. *Let the conditions of Theorem 3 be fulfilled. Then*

$$\left|\varphi^{(\bar{m})}(x,u)-\varphi^{(\bar{m})}(x,v)\right| \le \tilde{L}|u-v|,$$

$\exists h_0>0$ *and* $\forall\{h_j\}_{j=1}^N : |h|\le h_0,\ 0<\tilde{c}_1\le a^{(\bar{m})}(x)$, *and the iterative method*

$$B_h^{(\bar{m})}\frac{y^{(\bar{m},n)}-y^{(\bar{m},n-1)}}{\tau}+A_h^{(\bar{m})}\left(x,y^{(\bar{m},n-1)}\right)=0,\ x\in\hat{\omega}_h, \tag{44}$$

$$y^{(\bar{m},n)}(0)=\mu_1,\ y^{(\bar{m},n)}(1)=\mu_2,\ n=1,2,\ldots,\ y^{(\bar{m},0)}(x)=\frac{V_2(x)}{V_1(1)}\mu_1+\frac{V_1(x)}{V_1(1)}\mu_2,$$
$$B_h^{(\bar{m})}y=-\left(a^{(\bar{m})}y_{\bar{x}}\right)_{\hat{x}},\ A_h^{(\bar{m})}(x,y)=B_h^{(\bar{m})}y-\varphi^{(\bar{m})}(x,y)$$

with $\tau=\tau_0=\left(1+\dfrac{\tilde{L}}{8\tilde{c}_1}\right)^{-2}$ *converges and its error is estimated as*

$$\left\|y^{(m,n)}-u\right\|^*_{1,2,\hat{\omega}_h} \le M\left(h^{\bar{m}}+q^n\right),\quad q=\sqrt{1-\tau_0}, \tag{45}$$

where

$$k(x_j)\frac{dy^{(\bar{m},n)}(x_j)}{dx}=\frac{h_jy_{\bar{x},j}^{(\bar{m},n)}-w_1^{(\bar{m})j}\left(x_j,y^{(\bar{m},n)}\right)}{V_1^{(\bar{m})j}(x_j)}+l_1^{(m)j}\left(x_j,y^{(\bar{m},n)}\right),$$

M is a constant independent of $|h|, m,$ *or* n.

Proof. According to Theorem 3,

$$\left\|y^{(m,n)}-u\right\|^{*}_{1,2,\hat{\omega}_h} \leq \left\|y^{(m)}-u\right\|^{*}_{1,2,\hat{\omega}_h} + \left\|y^{(m,n)}-y^{(m)}\right\|^{*}_{1,2,\hat{\omega}_h} \leq \leq Mh^m + \left\|y^{(m,n)}-y^{(m)}\right\|^{*}_{1,2,\hat{\omega}_h}. \tag{46}$$

Since $f(x,u,\xi) \in \bigcup_{j=1}^{N} C^{\bar{m}}\left([x_{j-1},x_j]\times R^2\right)$, we obtain

$$\left|\varphi^{(\bar{m})}(x,u)-\varphi^{(\bar{m})}(x,v)\right| \leq \tilde{L}\,|u-v|.$$

Applying the Cauchy-Schwarz inequality, we estimate

$$\left(A^{(\bar{m})}_h(x,u)-A^{(\bar{m})}_h(x,v),w\right)_{\hat{\omega}_h} \leq \|u-v\|_{B^{(\bar{m})}_h}\|w\|_{B^{(\bar{m})}_h} + \left\|\varphi^{(\bar{m})}(x,u)-\varphi^{(\bar{m})}(x,v)\right\|_{0,2,\hat{\omega}_h}\|w\|_{0,2,\hat{\omega}_h} \leq \|u-v\|_{B^{(\bar{m})}_h}\|w\|_{B^{(\bar{m})}_h} + \tilde{L}\|u-v\|_{0,2,\hat{\omega}_h}\|w\|_{0,2,\hat{\omega}_h} \leq \|u-v\|_{B^{(\bar{m})}_h}\|w\|_{B^{(\bar{m})}_h} + \frac{\tilde{L}}{8}\|u_{\bar{x}}-v_{\bar{x}}\|_{0,2,\hat{\omega}^+_h}\|w_{\bar{x}}\|_{0,2,\hat{\omega}^+_h} \leq \left(1+\frac{\tilde{L}}{8\tilde{c}_1}\right)\|u-v\|_{B^{(\bar{m})}_h}\|w\|_{B^{(\bar{m})}_h}.$$

Setting $w = B^{(\bar{m})^{-1}}_h\left(A^{(\bar{m})}_h(x,u)-A^{(\bar{m})}_h(x,v)\right)$ gives

$$\left\|B^{(\bar{m})^{-1}}_h\left(A^{(\bar{m})}_h(x,u)-A^{(\bar{m})}_h(x,v)\right)\right\|_{B^{(\bar{m})}_h} \leq \left(1+\frac{\tilde{L}}{8\tilde{c}_1}\right)\|u-v\|_{B^{(\bar{m})}_h}. \tag{47}$$

We use (38) and (47) to derive

$$\left(A^{(\bar{m})}_h(x,u)-A^{(\bar{m})}_h(x,v),B^{(\bar{m})^{-1}}_h\left(A^{(\bar{m})}_h(x,u)-A^{(\bar{m})}{}_h(x,v)\right)\right)_{\hat{\omega}_h} \leq \leq \left(1+\frac{\tilde{L}}{8\tilde{c}_1}\right)^2\|u-v\|^2_{B^{(\bar{m})}_h} \leq \left(1+\frac{\tilde{L}}{8\tilde{c}_1}\right)^2\left(A^{(\bar{m})}_h(x,u)-A^{(\bar{m})}_h(x,v),u-v\right)_{\hat{\omega}_h}.$$

Consequently (see [6]), the iterative method (44) converges in the energy space $H_{B^{(\bar{m})}_h}$, which coincides with $\overset{0}{W}{}^1_2(\hat{\omega}_h)$, and the error is estimated as

$$\left\|y^{(\bar{m},n)}-y^{(\bar{m})}\right\|_{1,2,\hat{\omega}_h} \leq M_1 q^n.$$

Moreover,

$$\left\|k(x)\frac{dy^{(\bar{m},n)}(x)}{dx}-k(x)\frac{dy^{(\bar{m})}(x)}{dx}\right\|_{0,2,\hat{\omega}_h} \leq \frac{1}{\left\|V^{(\bar{m})j}_1\right\|_{0,2,\hat{\omega}_h}}\left[|h|\left\|y^{(\bar{m},n)}_{\bar{x}}-y^{(\bar{m})}_{\bar{x}}\right\|_{0,2,\hat{\omega}^+_h} + \left\|w^{(\bar{m})j}_1\left(x,y^{(\bar{m},n)}\right)-w^{(\bar{m})j}_1\left(x,y^{(\bar{m})}\right)\right\|_{0,2,\hat{\omega}_h}\right] + \left\|l^{(m)j}_1\left(x,y^{(\bar{m},n)}\right)-l^{(m)j}_1\left(x,y^{(\bar{m})}\right)\right\|_{0,2,\hat{\omega}_h} \leq$$

$$\leq M_1 \left\| y_{\bar{x}}^{(\bar{m},n)} - y_{\bar{x}}^{(\bar{m})} \right\|_{0,2,\hat{\omega}_h^+} \left[\frac{1}{\left\| V_1^{(\bar{m})j} \right\|_{0,2,\hat{\omega}_h}} \left\| \frac{\partial}{\partial u} w_1^{(\bar{m})j}(x,u) \Big|_{u=\bar{y}^{(m)}} \right\|_{0,2,\hat{\omega}_h} + \right.$$

$$\left. + \left\| \frac{\partial}{\partial u} l_1^{(m)j}(x,u) \Big|_{u=\bar{y}^{(m)}} \right\|_{0,2,\hat{\omega}_h} \right] \left\| y^{(\bar{m},n)} - y^{(\bar{m})} \right\|_{0,2,\hat{\omega}_h} \leq$$

$$\leq M \left\| y^{(\bar{m},n)} - y^{(\bar{m})} \right\|_{1,2,\hat{\omega}_h} \leq M_2 q^n.$$

It follows that

$$\left\| y^{(\bar{m},n)} - y^{(\bar{m})} \right\|_{1,2,\hat{\omega}_h}^* \leq M q^n. \tag{48}$$

Combining (46) and (48) yields (45).

4 Numerical examples

Consider a nonlinear boundary value problem

$$\frac{d^2u}{dx^2} = \frac{1}{2} cos^2 \left(\frac{du}{dx} \right), \quad u(0) = 0, \quad u(1) = arctan\left(\frac{1}{2}\right) - ln\frac{5}{4},$$

with the exact solution $u(x) = x \cdot arctan\left(\frac{1}{2}x\right) - ln\left|1 + \frac{1}{4}x^2\right|$. To estimate the convergence rate in practice, the following quantities are introduced

$$er = \left\| z^{(6)} \right\|_{1,2,\omega_h}^* = \left\| y^{(6)} - u \right\|_{1,2,\omega_h}^*, \qquad p = log_2 \frac{\left\| z^{(6)} \right\|_{1,2,\omega_h}^*}{\left\| z^{(6)} \right\|_{1,2,\omega_{h/2}}^*}.$$

In the Table 1 we present the results of a numerical computations for a six-order ($m = 5$) accurate method. Consequently, the numerical experiment confirm our theoretical analysis.

Table 1.

N	er	p
8	$0.5765E-08$	
16	$0.1078E-09$	$.57E+01$
32	$0.1825E-11$	$.59E+01$
64	$0.2899E-13$	$.60E+01$

Bibliography

1. A. Kufner, S. Fucik, *Nonlinear Differential Equations*, Elsevier Scientific Publishing Company,, Amsterdam-Oxford-New York, 1980.
2. M.V. Kutniv, V.L. Makarov, A.A. Samarskii, *Accurate three-point difference schemes for second-order nonlinear ordinary differential equations and their implementation*, Comput. Math. Math. Phys. **39**, No 1 (1999), p. 45-60.
3. M.V. Kutniv, *Accurate three-point difference schemes for second-order monotone ordinary differential equations and their implementation*, Comput. Math. Math. Phys. **40**. **No 3** (2000), p. 368-382 No. 3, pp..
4. V.L. Makarov and A.A. Samarskii, *Accurate Three-Point Difference Schemes for Second-Order Nonlinear Ordinary Differential Equations and Their Implementation*, Soviet Math. Dokl. **312**, No **4** (1990), p. 795-800.
5. A.A. Samarskii, R.D. Lazarov, V.L. Makarov, *Difference Schemes for Differential Equations with Generalized Solutions*, Nauka, Moscow, 1987. (in Russian)

6. A.A. Samarskii, E.S. Nikolaev, ***Methods for Solving Grid Equations***, Nauka, Moscow, 1978. (in Russian)
7. V.A. Trenogin, ***Functional Analysis***, Nauka, Moscow, 1980. (in Russian)

NATIONAL UNIVERSITY "LVIVS'KA POLITEHNIKA", 12 ST. BANDERA STR., LVIV 79013 UKRAINE
E-mail address: kutniv@yahoo.com

Received 3.02.2003