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ON ONE CLASS OF NONLOCAL IN TIME PROBLEMS FOR FIRST-ORDER EVOLUTION EQUATIONS

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ABSTRACT. Present paper studies nonlocal in time problems in abstract Hilbert spaces for first-order evolution equations. The sequence of classical problems is constructed so that their solutions converge to the solution of nonlocal problem. It is shown that in some cases nonlocal problem can be considered as an approximation to the corresponding classical problem.

1 Introduction

Nonlocal in time problems introduce the type of non-classical problems, which emerge while applying mathematical modeling to various phenomena of physics, biology and ecology. These problems are formulated for evolution equations and instead of classical initial conditions a combination of the initial value of the solution and its values for later times is given. Note that nonlocal in time problems are generalizations of periodical problem, which can be considered as a particular case of problem of controllability by the initial conditions.

A systematic investigation of one class of nonlocal boundary value problems first was carried out by A. Bitsadze and A. Samarskii [1]. Further, in [2-14] various types of spatial nonlocal boundary conditions for equations of mathematical physics were studied. Nonlocal in time conditions were first considered in [15], where discrete nonlocal in time problem for parabolic equation was studied. Later in [16-18] problems of this type with discrete nonlocal initial conditions were investigated for some linear equations of mathematical physics. Concrete discrete and integral nonlocal problems for Navier-Stokes equation were considered in [19, 20], where applying Brouwer's fixed-point theorem was proved an existence of weak solution.

It must be pointed out, that in papers devoted to investigation of nonlocal in time problems mainly consider concrete nonlocal initial conditions ([15-22]). In the present paper we study nonlocal in time problems for abstract first-order evolution equations with various nonlocal initial operators and generalize previously obtained results for the parabolic equations in [15-17]. More precisely, in section 2 we formulate nonlocal in time problem for abstract first-order evolution equation and under certain assumptions on the spectrum of the elliptic operator and nonlocal operator we prove the existence and uniqueness of the solution. We construct an iteration procedure and show that the solution of the nonlocal problem can be approximated by a sequence of solutions to classical problems. In the same section we consider nonlocal in time problem with arbitrary coercive operator and nonlocal operator with norm less than one and prove the existence and uniqueness of solution. In section 3 we consider some particular cases of the nonlocal in time problem stated in section 2 and obtain corresponding results for parabolic equations and systems.

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2 Statement of the problems and results in abstract spaces

As specified in the introduction, in this section we study nonlocal in time problems for abstract first-order evolution equation and prove existence and uniqueness of their solutions, when the nonlocal operator satisfies various conditions.

Let V and H be Hilbert spaces, V is dense in H and is continuously imbedded in it. The dual space of V is denoted with V' and H is identified with its dual with respect to the scalar product, then $V \subset H \subset V'$ with continuous and dense imbeddings. The dual relation between the spaces V' and V is denoted with $\langle .,. \rangle$. $\mathfrak{L}(X;Y)$ designates the space of continuous operators from X to Y, where X, Y are Banach spaces. Denote with $L^p(0,T;X)$, $1 \le p \le \infty$, the space of measurable vector-functions $g : (0,T) \to X$, for which $\int_0^T \|g(t)\|_X^p \, dt < \infty$, for $1 \le p < \infty$ and $\operatorname{supess}_{t \in (0,T)} \|g(t)\|_X < \infty$, when $p = +\infty$. Note that, each vector-function $g \in L^p(0,T;X)$ can be identified with the distribution in $(0,T)$ with values in X ([23]) and its generalized derivative is denoted with $g' = \frac{dg}{dt} \in D'((0,T);X) = \mathfrak{L}(\mathfrak{D}(0,T);X)$, where $\mathfrak{D}(O,T)$ is the space of infinitely differentiable functions with compact support in $(0,T)$. Assume that operator $A \in \mathfrak{L}(V;V')$ is coercive, i.e., bilinear form $a(v,w) = \langle Av, w \rangle$ satisfies the following conditions:

$$\begin{aligned} |a(v,w)| &\le c_a \|v\|_V \|w\|_V , \\ a(v,v) &\ge \alpha \|v\|_V^2 , \quad \forall v, w \in V, \end{aligned} \tag{2.1}$$

where $c_a, \alpha = \text{const} > 0$. Let us suppose that the set of eigenvectors $\{v_k\}_{k=1}^{\infty}$ of the operator A (v_k corresponds to the eigenvalue λ_k^2) is complete in V and is orthonormal in H.

Let us consider nonlocal in time problem for first-order evolution equation

$$\frac{du}{dt} + Au = f, \quad t \in (0,T),$$

the variational formulation of which is as follows: the unknown is the vector-function $u \in L^2(0,T;V)$, $u' \in L^2(0,T;V')$, which satisfies equation

$$\frac{d}{dt}(u(.),v)_H + a(u(.),v) = \langle f(.), v \rangle , \quad \forall v \in V, \tag{2.2}$$

in the sense of distributions in $(0,T)$ and the following nonlocal initial condition

$$u(0) = Bu + u_0, \tag{2.3}$$

where $u_0 \in H$, $f \in L^2(0,T;V')$, $B \in \mathfrak{L}(C^0([0,T];H);H)$, $C^0([0,T];H)$ is the space of continuous vector-functions on $[0,T]$ with values in H. Notice, that condition (2.3) is correct, because $u \in C^0([0,T];H)$, since $u \in L^2(0,T;V)$, $u' \in L^2(0,T;V')$ ([24, 25]).

Suppose that the nonlocal operator B satisfies the following conditions: there exist continuous functionals $b_{ij} \in \mathfrak{L}(C^0([0,T]);\mathbb{R})$, $i \ge j$, such that

$$B(\zeta v_p) = \sum_{i=p}^{\infty} b_{ip}(\zeta) v_i, \qquad \forall \zeta \in C^0([0,T]), \ p \in \mathbb{N}. \tag{2.4}$$

Denote with $\{e_i\}_{i=1}^p$ the orthonormal basis in p-dimensional space $\mathbb{R}^p$, with scalar product $(x,y)_p = \sum_{i=1}^p x_i y_i$, $x = \sum_{i=1}^p x_i e_i$, $y = \sum_{i=1}^p y_i e_i$, and norm with $\|x\|_p = \sqrt{(x,x)_p}$. The space of $p \times p$ matrices is denoted with $\mathbb{M}^p$, which is equipped with the norm $\|M\| = (\sum_{i=1}^p \sum_{j=1}^p M_{ij}^2)^{1/2}$, $\forall M \in \mathbb{M}^p$. For any matrix $M_1 \in \mathbb{M}^p$ denote with $\|M_1\|_S$

the spectral norm of M_1, $\|M_1\|_S = \sup\limits_{x\in\mathbb{R}^p} \dfrac{\|M_1 x\|_p}{\|x\|_p} = \sqrt{\lambda_{\max}(M_1 M_1^T)}$, where M_1^T is transposed matrix of M_1 and $\lambda_{\max}(M_1 M_1^T)$ is the maximal eigenvalue of $M_1 M_1^T$. By means of the functionals b_{ij} let us construct linear operators $\overline{B}_p$, $\widetilde{B}_p$, $p \geq 1$, $\overline{B}_p : [C^0([0,T])]^p \to \mathbb{R}^p$, $\widetilde{B}_p : C^0([0,T];\mathbb{M}^p) \to \mathbb{M}^p$, which are given by

$$\overline{B}_p(\vec{\zeta}) = \sum_{i=1}^{p}\sum_{j=1}^{i} b_{ij}(\zeta_j)\, e_i, \quad \forall\vec{\zeta} = (\zeta_j) = \left[C^0([0,T])\right]^p,$$

$$\widetilde{B}_p(F) = G,\ G = \{G_{ij}\},\ G_{ij} = \sum_{k=1}^{i} b_{ik}(F_{kj}), \quad \forall F = \{F_{ij}(t)\} \in C^0([0,T];\mathbb{M}^p).$$

The diagonal matrix of order p with (i,i) element λ_i^2 is denoted with A_p, $1 \leq i \leq p$.

For the formulated nonlocal problem the following existence and uniqueness theorem is true.

Theorem 2.1. *Assume that the linear continuous functionals b_{ij} and eigenvalues $\{\lambda_k^2\}_{k=1}^{\infty}$ are such that the following conditions are fulfilled:*

$$\left\| \left(I - \widetilde{B}_k(e^{-A_k t})\right)^{-1} \right\|_S < \beta, \qquad \beta = \text{const} > 0,\ \forall k \in \mathbb{N}. \tag{2.5}$$

If $u_0 \in H$, $f \in L^2(0,T;V')$, then the nonlocal in time problem (2.2), (2.3) *has a unique solution and the following estimate is valid:*

$$\max\left\{ \|u\|_{C^0([0,T];H)}, \|u\|_{L^2(0,T;V)} \right\} \leq c\left(\|u_0\|_H + \|f\|_{L^2(0,T;V')} \right).$$

Proof. First we prove existence of the solution. Denote with V_n the linear subspace of V defined by the vectors $v_1, ..., v_n$. Since V is dense in H, then $\bigcup\limits_{n\geq 1} V_n$ is dense in H. Therefore, the sequence $\{u_{0n}\}_{n\geq 1} \subset V$, where $u_{0n} \in V_n$, $u_{0n} = \sum\limits_{k=1}^{n} \varphi_k v_k$, $\varphi_k = (u_0, v_k)_H$ is such that $u_{0n} \to u_0$ in H, as $n \to \infty$.

Let us consider the sequence of approximations $\widetilde{u}^n(t) = \sum\limits_{k=1}^{n} u_k(t) v_k$ to u, which satisfy the following problem:

$$\begin{gathered}\left(\frac{d\widetilde{u}^n}{dt}, v_k\right)_H + a(\widetilde{u}^n(t), v_k) = \langle f(t), v_k \rangle, \quad t \in (0,T), \\ (\widetilde{u}^n(0), v_k)_H = (B\widetilde{u}^n, v_k)_H + \varphi_k, \quad k = 1, 2, ..., n.\end{gathered} \tag{2.6}$$

The latter problem is nonlocal problem for the system of ordinary differential equations

$$\begin{gathered}\frac{d\vec{u}_n}{dt} + A_n \vec{u}_n(t) = \vec{f}_n(t), \qquad t \in (0,T), \\ \vec{u}_n(0) = \overline{B}_n(\vec{u}_n) + \vec{u}_{0n},\end{gathered} \tag{2.7}$$

where $\vec{u}_n = (u_k)_{k=1}^n$, $\vec{u}_{0n} = (\varphi_k)_{k=1}^n$, $\vec{f}_n = (f_k)_{k=1}^n$, $f_k(t) = \langle f(t), v_k \rangle$, $k = \overline{1,n}$.

The problem (2.7) has a unique solution $\vec{u}_n \in C^0([0,T];\mathbb{R}^n)$, $\vec{u}_n' \in C^0([0,T];\mathbb{R}^n)$, which is given by

$$\vec{u}_n(t) = e^{-A_n t}\vec{d}_n + \int_0^t e^{-A_n(t-\tau)}\vec{f}_n(\tau)d\tau,$$

where, taking into account condition (2.5), $\vec{d}_n$ is determined from the initial condition (2.6)

$$\vec{d}_n = \left(I - \widetilde{B}_n(e^{-A_n t})\right)^{-1} \left(\overline{B}_n \left(\int_0^t e^{-A_n(t-\tau)} \vec{f}_n(\tau)\right) + \vec{u}_{0n}\right). \tag{2.8}$$

In order to show convergence of the sequence $\{\widetilde{u}^n\}_{n\geq 1}$, let us estimate norm of the vector $\vec{d}_n$. Since the operator A is coercive, from Lax-Milgram lemma we deduce that A is invertible, i.e., for almost all $t \in (0,T)$, there exists $v^{f(t)} \in V$ such that $Av^{f(t)} = f(t)$ and $A^{-1}f \in L^2(0,T;V)$. Hence,

$$\left|\int_0^t e^{-\lambda_k^2(t-\tau)} f_k(\tau)\, d\tau\right|^2 = \left|\int_0^t e^{-\lambda_k^2(t-\tau)} \langle Av^{f(\tau)}, v_k\rangle d\tau\right|^2 =$$

$$= \left|\int_0^t e^{-\lambda_k^2(t-\tau)} a(v^{f(\tau)}, v_k) d\tau\right|^2 = \left|\int_0^t e^{-\lambda_k^2(t-\tau)} \lambda_k^2 v_k^{f(\tau)} d\tau\right|^2 \leq$$

$$\leq \int_0^t \lambda_k^2 (v_k^{f(\tau)})^2 d\tau \int_0^t \lambda_k^2 e^{-2\lambda_k^2(t-\tau)} d\tau \leq \frac{1}{2}\int_0^T \lambda_k^2 (v_k^{f(\tau)})^2 d\tau,$$

where $v_k^{f(\tau)} = (v^{f(\tau)}, v_k)_H$, $k \in \mathbb{N}$. Therefore, the series $\sum_{k=1}^{\infty} \int_0^t e^{-\lambda_k^2(t-\tau)} f_k(\tau) d\tau v_k$ converges uniformly in H, since

$$\sum_{k=1}^{n} \int_0^T \lambda_k^2 (v_k^{f(\tau)})^2 d\tau = \int_0^T a\left(\sum_{k=1}^{n} v_k^{f(\tau)} v_k, \sum_{k=1}^{n} v_k^{f(\tau)} v_k\right) d\tau \leq \int_0^T a(v^{f(\tau)}, v^{f(\tau)}) d\tau,$$

and

$$\sum_{k=1}^{\infty} \left|\int_0^t e^{-\lambda_k^2(t-\tau)} f_k(\tau) d\tau\right|^2 \leq \int_0^T a(v^{f(\tau)}, v^{f(\tau)}) d\tau \leq c_a \int_0^T \left\|v^{f(\tau)}\right\|_V^2 d\tau =$$

$$= c_a \left\|A^{-1}f\right\|^2_{L^2(0,T;V)} \leq c_1 \left\|f\right\|^2_{L^2(0,T;V')}, \quad \forall t \in [0,T].$$

From the latter estimate, taking into account (2.5) and continuity of the operator B, we obtain

$$\left\|\vec{d}_n\right\| \leq \beta \left(\left\|\sum_{i=1}^{n} \sum_{j=1}^{i} b_{ij} \left(\int_0^t e^{-\lambda_j^2(t-\tau)} f_j(\tau)\, d\tau\right) v_i\right\|_H + \|u_0\|_H\right) \leq$$

$$\leq \beta \left(\left\|\sum_{i=1}^{\infty} \sum_{j=1}^{i} b_{ij} \left(\int_0^t e^{-\lambda_j^2(t-\tau)} f_j(\tau)\, d\tau\right) v_i\right\|_H + \|u_0\|_H\right) =$$

$$= \beta \left(\left\|B\left(\sum_{j=1}^{\infty} \int_0^t e^{-\lambda_j^2(t-\tau)} f_j(\tau)\, d\tau v_j\right)\right\|_H + \|u_0\|_H\right) \leq$$

$$\leq c_2 \left(\|f\|_{L^2(0,T;V')} + \|u_0\|_H\right).$$

Thus, the series $\sum_{k=1}^{\infty} u_k^2(t)$ is majorized by converging number series, since

$$|u_k(t)|^2 \leq 2 \left(|d_k|^2 + \int_0^T \lambda_k^2 \left(A^{-1} f(\tau), v_k\right)^2 d\tau \right), \quad k \geq 1.$$

So, the sequence $\{\widetilde{u}^n(t)\}_{n \geq 1}$ converges in the space $C^0([0,T]; H)$ and its limit, denoted with $w(t)$, is a solution to the problem (2.2), (2.3). Indeed, since $\widetilde{u}^n$ satisfies equation (2.6), then

$$\frac{1}{2} \|\widetilde{u}^n(t)\|_H^2 + \int_0^t a(\widetilde{u}^n(\tau), \widetilde{u}^n(\tau))\, d\tau = \int_0^t \langle f(\tau), \widetilde{u}^n(\tau)\rangle\, d\tau + \frac{1}{2} \|\widetilde{u}^n(0)\|_H^2, \quad n \geq 1,$$

and, by coerciveness of the bilinear form a, we have

$$\|\widetilde{u}^n(t)\|_H^2 + \int_0^t \|\widetilde{u}^n(\tau)\|_V^2\, d\tau \leq \widehat{c} \left(\|f\|^2_{L^2(0,T;V')} + \|u_0\|_H^2 \right), \ \forall t \in [0,T],\ n \geq 1,$$

where $\widehat{c} =$ const > 0. From the latter inequality we deduce that $\{\widetilde{u}^n\}_{n \geq 1}$ is bounded in $L^2(0,T;V)$ and hence

$$\widetilde{u}^n \to w \quad \text{weakly in } L^2(0,T;V), \text{ as } n \to \infty. \tag{2.9}$$

Also, from equation (2.6) we obtain

$$-\int_0^T (\widetilde{u}^n(\tau), z_n)_H\, \psi'(\tau)\, d\tau + \int_0^T a(\widetilde{u}^n(\tau), z_n)\, \psi(\tau)\, d\tau = \int_0^T \langle f(\tau), z_n\rangle\, \psi(\tau)\, d\tau, \tag{2.10}$$

where $\psi \in \mathfrak{D}(0,T)$, z_n is a linear combination of the vectors $v_1, \ldots, v_n$. Since $\{v_k\}_{k \geq 1}$ is complete in V, then for any $v \in V$ there exists sequence $\{z_n\}_{n \geq 1}$, such that $z_n \to v$ strongly in V, as $n \to \infty$. Then, letting $n \to \infty$ in (2.10) and applying (2.9), we have

$$-\int_0^T (w(\tau), v)_H\, \psi'(\tau)\, d\tau + \int_0^T a(w(\tau), v)\, \psi(\tau)\, d\tau = \int_0^T \langle f(\tau), v\rangle\, \psi(\tau) d\tau.$$

Thus, $w \in L^2(0,T;V)$ satisfies equation (2.2) in the sense of distributions in $(0,T)$ and as $A \in \mathfrak{L}(V;V')$, we infer that $w' \in L^2(0,T;V')$.

Let us show that w satisfies initial condition (2.3). Let $\psi_1 \in C^1([0,T])$, $\psi_1(T) = 0$, $\psi_1(0) \neq 0$. Since $\widetilde{u}^n$ is a solution to the problem (2.6), we obtain

$$-\int_0^T (\widetilde{u}^n(\tau), z_n)_H\, \psi_1'(\tau)\, d\tau + \int_0^T a(\widetilde{u}^n(\tau), z_n)\, \psi_1(\tau)\, d\tau = (B\widetilde{u}^n, z_n)_H \psi_1(0) +$$

$$+ (u_{0n}, z_n)_H \psi_1(0) + \int_0^T \langle f(\tau), z_n\rangle\, \psi_1(\tau)\, d\tau.$$

From the latter equality, by uniform convergence of the sequence $\{\widetilde{u}^n\}_{n \geq 1}$ in H, weak convergence in $L^2(0,T;V)$ and continuity of B, we infer that

$$-\int_0^T (w(\tau), v)_H\, \psi_1'(\tau)\, d\tau + \int_0^T a(w(\tau), v)\, \psi_1(\tau)\, d\tau = (Bw, v)_H \psi_1(0) +$$

$$+(u_0, v)_H \psi_1(0) + \int_0^T \langle f(\tau), v\rangle \psi_1(\tau)\, d\tau.$$

Since w is a solution to the equation (2.2), we have

$$-\int_0^T (w(\tau), v)_H \psi_1'(\tau)\, d\tau + \int_0^T a(w(\tau), v)\psi_1(\tau)\, d\tau = (w(0), v)_H \psi_1(0) +$$

$$+\int_0^T \langle f(\tau), v\rangle \psi_1(\tau)\, d\tau,$$

which, together with the latter equality, implies that

$$(w(0), v) = (Bw + u_0, v)_H, \quad \forall v \in V,$$

and consequently $w(0) = Bw + u_0$, since V is dense in H. So, w is a solution to the problem (2.2), (2.3) and hence

$$\|w(t)\|_H^2 + \int_0^t \|w(\tau)\|_V^2\, d\tau \le c\left(\|f\|^2_{L^2(0,T;V')} + \|u_0\|_H^2\right), \quad \forall t \in [0, T].$$

Now we prove uniqueness of the solution. Since the problem (2.2), (2.3) is linear, it suffices to show that $u \equiv 0$, when $f \equiv 0$ and $u_0 = 0$. Note that solution u of the homogeneous problem (2.2), (2.3) is a solution to the classical initial problem for equation (2.2) with initial condition $u(0) \in H$ and consequently $u(t) = \sum_{k=1}^{\infty} e^{-\lambda_k^2 t}\widetilde{c}_k v_k$. Since u satisfies homogeneous nonlocal in time condition (2.3), we obtain

$$\widetilde{c}_n = \sum_{k=1}^{n} b_{nk}(e^{-\lambda_k^2 t})\widetilde{c}_k, \quad n \ge 1,$$

from which, by the condition (2.5), we infer that $\widetilde{c}_k = 0$, $\forall k \in \mathbb{N}$. Hence, $u \equiv 0$ and the solution to the problem (2.2), (2.3) is unique.

Remark. From the proof of uniqueness of the solution to the nonlocal in time problem (2.2), (2.3) it follows that if instead of the condition (2.5) more weak condition $\|I - \widetilde{B}_k(e^{-A_k t})\|_S > 0$, $\forall k \in \mathbb{N}$ is fulfilled, then the problem (2.2), (2.3) has at most one solution.

Note that solution to the nonlocal in time problem can be approximated with solutions of the classical initial problem if the nonlocal operator satisfies more strong condition than (2.5). More precisely, let us consider the sequence of the following problems: the unknown is $w_n \in L^2(0, T; V)$, $w_n' \in L^2(0, T; V')$, which in space $L^2(0, T; V')$ satisfies equation

$$\frac{dw_n}{dt} + Aw_n = f, \quad t \in (0, T), \tag{2.11}$$

and in space H initial condition

$$w_n(0) = Bw_{n-1} + u_0, \tag{2.12}$$

where $n \ge 1$, $w_0 \equiv 0$. As well known for any operator $B \in \mathfrak{L}(C^0([0, T]; H); H)$, $u_0 \in H$, $f \in L^2(0, T; V')$ the problem (2.11), (2.12) has a unique solution. In the following theorem we formulate the conditions on the operator B, which insure the convergence of the sequence $\{w_n\}_{n\ge 1}$ to the solution of the problem (2.2), (2.3).

Theorem 2.2. *Suppose that the operator $B \in \mathfrak{L}(C^0([0,T];H);H)$ satisfies condition* (2.4) *and there exists the positive constant $\gamma =$ const < 1 such as*

$$\left\|\widetilde{B}_n(e^{-A_n t})\right\|_S < \gamma, \quad \forall n \in \mathbb{N}. \tag{2.13}$$

If $u_0 \in H$, $f \in L^2(0,T;V')$, then the nonlocal in time problem (2.2), (2.3) *has a unique solution and the sequence of solutions to the problems* (2.11), (2.12) *converges to the solution of the problem* (2.2), (2.3) *in spaces $C^0([0,T];H)$ and $L^2(0,T;V)$.*

Proof. From condition (2.13) we infer that the matrix $I - \widetilde{B}_p(e^{-A_p t})$ is invertible and

$$\left\|\left(I - \widetilde{B}_p(e^{-A_p t})\right)^{-1}\right\| \leq \sum_{k=1}^{\infty}\left\|\widetilde{B}_p(e^{-A_p t})\right\|_S^k < \sum_{k=1}^{\infty}\gamma^k = \frac{1}{1-\gamma}, \quad \forall p \in \mathbb{N}.$$

Hence, from Theorem 2.1, we obtain the existence and uniqueness of the solution to the problem (2.2), (2.3).

Let us prove that the sequence of the solutions to the problems (2.11), (2.12) converges to the solution u of the problem (2.2), (2.3). Let $\theta_n(t) = u(t) - w_n(t)$, $n \geq 0$, and note that θ_n is a solution to the following problem

$$\frac{d\theta_n}{dt} + A\theta_n = 0, \qquad t \in (0,T), \tag{2.14}$$

$$\theta_n(0) = B\theta_{n-1}, \qquad n \geq 1. \tag{2.15}$$

The solution θ_n of the problem (2.14), (2.15) can be expressed by converging in $C^0([0,T];H)$ series

$$\theta_n(t) = \sum_{j=1}^{\infty} e^{-\lambda_j^2 t} c_j^n v_j, \quad n \geq 0. \tag{2.16}$$

From (2.16), by the condition (2.15), we obtain

$$c_j^n = \sum_{k=1}^{j} b_{jk}(e^{-\lambda_k^2 t}) c_k^{n-1}, \quad \forall j \in \mathbb{N},\ n \geq 1.$$

Hence, taking into account condition (2.13), we infer that

$$\|\bar{c}_j^n\|_j = \|\widetilde{B}_j(e^{-A_j t})\bar{c}_j^{n-1}\|_j = \ldots = \|[\widetilde{B}_j(e^{-A_j t})]^n \bar{c}_j^0\|_j \leq \gamma^n \|Bu + u_0\|_H, \tag{2.17}$$

where $\bar{c}_j^n$ is j component vector, whose i-th component is c_i^n, $1 \leq i \leq j$. Using inequality (2.17) we estimate the norm of $\theta_n(t)$,

$$\|\theta_n(t)\|_H^2 = \sum_{j=1}^{\infty}(e^{-\lambda_j^2 t} c_j^n)^2 = \lim_{j\to\infty}\|e^{-A_j t}\bar{c}_j^n\|_j^2 \leq \lim_{j\to\infty}\|e^{-A_j t}\|_S^2\|\bar{c}_j^n\|_j^2 \leq \widetilde{c}\gamma^n, \quad n \geq 1,$$

where $\widetilde{c} =$ const > 0 is independent of n. Thus, $\lim\limits_{n\to\infty}\|u - w_n\|_{C^0([0,T];H)} = 0$.

Note that, from the convergence in $C^0([0,T];H)$ follows the convergence in the space $L^2(0,T;V)$. Indeed, multiplying equation (2.14) by θ_n, integrating on $(0,T)$, and applying the formula for integration by parts, we obtain

$$\int_0^t a(\theta_n(\tau), \theta_n(\tau))\, d\tau = \frac{1}{2}\left(\|\theta_n(0)\|_H^2 - \|\theta_n(t)\|_H^2\right), \quad \forall t \in [0,T],$$

and consequently, by coerciveness of the bilinear form a, we deduce that $\|\theta_n\|_{L^2(0,T;V)} \to 0$, as $n \to \infty$.

It must be pointed out that the nonlocal in time problem (2.2), (2.3) can be investigated even in case of the operator A not satisfying spectral properties, which are assumed to be fulfilled in Theorem 2.1. More precisely, the following theorem is valid.

Theorem 2.3. *Let $A \in \mathfrak{L}(V;V')$ be coercive operator and $B \in \mathfrak{L}(C^0([0,T];H);H)$ is such that $\|B\|_{\mathfrak{L}} < 1$. If $u_0 \in H$, $f \in L^2(0,T;V')$, then the nonlocal in time problem (2.2), (2.3) has a unique solution u satisfying*

$$\max\left\{\|u\|_{C^0([0,T];H)}, \|u\|_{L^2(0,T;V)}\right\} \leq c\left(\|u_0\|_H + \|f\|_{L^2(0,T;V')}\right).$$

Proof. First, let us show that the nonlocal problem has at most one solution. Assume that $u_1(t)$ and $u_2(t)$ are solutions to the problem (2.2), (2.3). Then, their difference $\overline{u}(t) = u_1(t) - u_2(t)$ is a solution to the homogeneous equation (2.2) and it satisfies homogeneous nonlocal condition (2.3). Consequently, for $\overline{u}(t)$ we have the following identity:

$$\frac{1}{2}\|\overline{u}(t)\|_H^2 + \int_0^t a(\overline{u}(\tau), \overline{u}(\tau))\, d\tau = \frac{1}{2}\|\overline{u}(0)\|_H^2 = \frac{1}{2}\|B\overline{u}\|_H^2, \quad \forall t \in [0,T].$$

The latter identity, together with coerciveness of the form a, implies

$$\frac{1}{2}\|\overline{u}(t)\|_H^2 + \alpha\int_0^t \|\overline{u}(\tau)\|_V^2\, d\tau \leq \|B\|_{\mathfrak{L}}^2\left(\frac{1}{2}\|\overline{u}\|_{C^0([0,T];H)}^2 + \alpha\int_0^T \|\overline{u}(\tau)\|_V^2\, d\tau\right), \quad \forall t \in [0,T],$$

from which we have $\overline{u} \equiv 0$, since $\|B\|_{\mathfrak{L}} < 1$.

In order to prove existence of solution let us consider the sequence of problems (2.11), (2.12). According to the theorem conditions, for each $n \in \mathbb{N}$, the problem (2.11), (2.12) has a unique solution. Denote with $\chi_n(t) = w_n(t) - w_{n-1}(t)$, $n \geq 1$. $\chi_n(t)$ satisfies the homogeneous equation (2.2)

$$\frac{d}{dt}(\chi_n(.), v)_H + a(\chi_n(.), v) = 0, \quad \forall v \in V, \tag{2.18}$$

in the sense of distributions in $(0,T)$, and the following initial conditions

$$\chi_n(0) = B\chi_{n-1}, \quad n \geq 2. \tag{2.19}$$

From (2.18), taking into account (2.19), follows

$$\frac{1}{2}\|\chi_n(t)\|_H^2 + \int_0^t a(\chi_n(\tau), \chi_n(\tau))\, d\tau = \frac{1}{2}\|\chi_n(0)\|_H^2 = \frac{1}{2}\|B\chi_{n-1}\|_H^2, \quad n \geq 2,$$

and consequently

$$\frac{1}{2}\|\chi_n\|_{C^0([0,T];H)}^2 + \alpha\int_0^T \|\chi_n(\tau)\|_V^2\, d\tau \leq$$

$$\leq \|B\|_{\mathfrak{L}}^2\left(\frac{1}{2}\|\chi_{n-1}\|_{C^0([0,T];H)}^2 + \alpha\int_0^T \|\chi_{n-1}(\tau)\|_V^2\, d\tau\right).$$

The latter inequality implies

$$\|\chi_n\|^2_{C^0([0,T];H)} + \int_0^T \|\chi_n(\tau)\|^2_V \, d\tau \le \tilde{c}\|B\|^{n-1}_{\mathfrak{L}}, \quad \tilde{c} = \text{const} > 0, \; n \ge 1. \tag{2.20}$$

From (2.20), since $\|B\|_{\mathfrak{L}} < 1$, we obtain that the series $\sum_{n=1}^{\infty} \chi_n$ converges in the spaces $C^0([0,T];H)$, $L^2(0,T;V)$, with the limit designated as $u(t)$. By arguing as in the proof of Theorem 2.1 we can show that $u(t)$ is a solution to the nonlocal problem (2.2), (2.3).

Since u satisfies equation (2.2), the following identity is valid

$$\frac{1}{2}\|u(t)\|^2_H + \int_0^t a(u(\tau), u(\tau))\, d\tau = \int_0^t \langle f(\tau), u(\tau)\rangle \, d\tau + \frac{1}{2}\|Bu + u_0\|^2_H, \; \forall t \in [0,T],$$

and hence

$$\frac{1}{2}\|u(t)\|^2_H + \alpha \int_0^t \|u(\tau)\|^2_V \, d\tau \le \frac{\delta}{2}\int_0^t \|f(\tau)\|^2_{V'} \, d\tau + \frac{1}{2\delta}\int_0^t \|u(\tau)\|^2_V \, d\tau +$$

$$+\frac{1}{2}\|Bu\|^2_H + \frac{\delta_1}{2}\|Bu\|^2_H + \frac{\delta_1+1}{2\delta_1}\|u_0\|^2_H, \quad \forall t \in [0,T], \; \delta, \delta_1 > 0.$$

From the latter inequality, by selecting appropriate δ, δ_1, we obtain the estimate of the theorem.

3 Nonlocal in time problems for parabolic equations and systems

Now, we consider some applications of the theorems stated in the second section to concrete nonlocal in time problems and show that in some cases nonlocal in time problem can be considered as an approximation to classical initial-boundary value problem.

Let the spaces V and H, used in the statement of the problem (2.2), (2.3), to be featured in such a way, that the imbedding of V in H is compact. The linear operator $A \in \mathfrak{L}(V;V')$ is coercive and self-adjoint, i.e.,

$$|a(v,w)| \le c_a \|v\|_V \|w\|_V, \quad a(v,w) = a(w,v),$$
$$a(v,v) \ge \alpha\|v\|^2_V, \quad \forall v,w \in V, \quad c_a, \alpha = \text{const} > 0.$$

Then, there exists complete system $\{v_k\}_{k=1}^{\infty} \subset V$ of eigenvectors of the operator A, which is orthonormal in H ([26]). Moreover, if $\lambda_k^2 > 0$ is the eigenvalue corresponding to the eigenvector v_k, then $0 < \lambda_1^2 \le \lambda_2^2 \le \ldots$, $\lambda_j^2 \to \infty$, as $j \to \infty$.

Let us consider nonlocal in time problem for equation (2.2) with the following nonlocal initial condition

$$u(0) = \sum_{i=1}^{m} \alpha_i u(T_i) + \sum_{i,j=1}^{m_1} \int_{T_i^1}^{T_j^1} \rho_{ij}(\tau) u(\tau)\, d\tau + u_0, \tag{3.1}$$

where $\alpha_i \neq 0$, $0 < T_i, T_j^1 \le \tilde{T} \le T$, ρ_{jk} are real-valued bounded measurable functions, $1 \le i \le m$, $1 \le j,k \le m_1$, and integrals in (3.1) are Bochner's integrals ([27]). Denote with

$$\Delta(p) = \sum_{i=1}^{m} \alpha_i e^{-\lambda_p^2 T_i} + \sum_{i,j=1}^{m_1} \int_{T_i^1}^{T_j^1} \rho_{ij}(\tau) e^{-\lambda_p^2 \tau} d\tau$$

and notice that $\Delta(p) \to 0$, as $p \to \infty$. Hence, there exists minimal natural number $P \in \mathbb{N}$, such that $\Delta(k) < 1$, for $k \geq P$. From Theorem 2.1 we derive the theorem that follows.

Theorem 3.1. *Suppose that the eigenvalues $\{v_k\}_{k=1}^{\infty}$ of the operator A satisfy the condition $\Delta(k) \neq 1$, for $1 \leq k < P$. If $u_0 \in H$, $f \in L^2(0,T;V')$, then the problem* (2.2), (3.1) *has a unique solution $u \in L^2(0,T;V)$, $u' \in L^2(0,T;V')$ and the following estimate is valid*

$$\max\left\{\|u\|_{C^0([0,T];H)}, \|u\|_{L^2(0,T;V)}\right\} \leq c\left(\|u_0\|_H + \|f\|_{L^2(0,T;V')}\right).$$

Corollary 3.1. *If $\rho_{jk} \equiv 0$, when $T_j^1 > T_k^1$, $1 \leq j,k \leq m_1$, and coefficients in the nonlocal initial condition* (3.1) *satisfy the inequality*

$$1 - \sum_{i=1}^{m} \frac{\alpha_i + |\alpha_i|}{2} \geq \sum_{i,j=1}^{m_1} \int_{T_i^1}^{T_j^1} |\rho_{ij}(\tau)|\, d\tau, \tag{3.2}$$

then the problem (2.2), (3.1) *has a unique solution.*

Proof. According to Theorem 3.1 it suffices to show that $\Delta(k) \neq 1$, $\forall k \in \mathbb{N}$. Note that

$$1 - \sum_{i=1}^{m} \alpha_i e^{-\lambda_p^2 T_i} \geq 1 - \sum_{i=1}^{m} \frac{\alpha_i + |\alpha_i|}{2} e^{-\lambda_p^2 T_i} \geq 1 - \sum_{i=1}^{m} \frac{\alpha_i + |\alpha_i|}{2},$$

where one of the inequalities is strict. Moreover,

$$\left|\sum_{i,j=1}^{m_1} \int_{T_i^1}^{T_j^1} \rho_{ij}(\tau)\, e^{-\lambda_p^2 \tau} d\tau\right| \leq \sum_{i,j=1}^{m_1} \int_{T_i^1}^{T_j^1} |\rho_{ij}(\tau)|\, e^{-\lambda_p^2 \tau} d\tau \leq \sum_{i,j=1}^{m_1} \int_{T_i^1}^{T_j^1} |\rho_{ij}(\tau)|\, d\tau.$$

From the inequality (3.2) we obtain that $\Delta(k) \neq 1$, $\forall k \in \mathbb{N}$, and the proof is complete.

Notice that some nonlocal in time problems approximate classical initial boundary value problems. In particular, let us consider nonlocal in time problem for equation (2.2) with initial condition

$$\sum_{i=0}^{m} \beta_i u(T_i) + \sum_{i,j=1}^{m_1} \int_{T_i^1}^{T_j^1} \rho_{ij}(\tau)\, u(\tau)\, d\tau = u_0, \tag{3.3}$$

where $T_0 = 0$, $\beta_0 > 0$, $\sum_{i=0}^{m} \beta_i = 1$, $\beta_i > 0$, $0 < T_i, T_j^1 \leq \widetilde{T} \leq T$, ρ_{jk} are real-valued bounded measurable functions, $\rho_{jk} \equiv 0$, when $T_j^1 > T_k^1$, $1 \leq i \leq m$, $1 \leq j,k \leq m_1$. Passing to the limit in (3.3) as $\widetilde{T} \to 0$ formally we obtain classical initial condition

$$u(0) = u_0. \tag{3.4}$$

By the solution to the problem (2.2), (3.4) we mean vector-function $u \in L^2(0,T;V)$, $u' \in L^2(0,T;V')$, which satisfies equation (2.2) in the space $L^2(0,T;V')$ and condition (3.4) in H.

The following theorem is true.

Theorem 3.2. *If $u_0 \in H$, $f \in L^2(0,T;V')$, then for sufficiently small $\widetilde{T}$ the nonlocal problem* (2.2), (3.3) *has a unique solution u and $\lim\limits_{\widetilde{T}\to 0} \|u-\widehat{u}\|_{C^0([0,T];H)} = 0$, where $\widehat{u}$ is a solution to the problem* (2.2), (3.4).

Proof. Note that $\int_{T_i^1}^{T_j^1} |\rho_{ij}(\tau)|\, d\tau \to 0$, as $\widetilde{T} \to 0$, and consequently by Corollary 3.1, for sufficiently small $\widetilde{T}$ the nonlocal problem (2.2), (3.3) has a unique solution u, which is given by the following uniformly converging series

$$u(t) = \sum_{k=1}^{\infty} \left(e^{-\lambda_k^2 t} d_k + \int_0^t e^{-\lambda_k^2 (t-\tau)} f_k(\tau) d\tau \right) v_k,$$

where d_k we determine from the condition (3.3)

$$d_k = \left(\beta_0 + \sum_{i=1}^{m} \beta_i e^{-\lambda_k^2 T_i} + a_k \right)^{-1} \left(-b_k + (u_0, v_k)_H \right), \quad a_k = \sum_{i,j=1}^{m_1} \int_{T_i^1}^{T_j^1} \rho_{ij}(\tau)\, e^{-\lambda_k^2 \tau} d\tau,$$

$$b_k = \sum_{i,j=1}^{m_1} \int_{T_i^1}^{T_j^1} \int_0^{\tau} \rho_{ij}(\tau) e^{-\lambda_k^2(\tau - s)} f_k(s) ds d\tau + \sum_{i=1}^{m} \beta_i \int_0^{T_i} e^{-\lambda_k^2 (T_i - \tau)} f_k(\tau) d\tau, \ k = 1, 2, \ldots$$

The problem (2.2), (3.4) has a unique solution, which is given by

$$\widehat{u}(t) = \sum_{k=1}^{\infty} \left(e^{-\lambda_k^2 t} \widehat{d}_k + \int_0^t e^{-\lambda_k^2 (t-\tau)} f_k(\tau) d\tau \right) v_k, \quad \widehat{d}_k = (u_0, v_k)_H, \ k \in \mathbb{N}.$$

Hence, for any positive $0 < \varepsilon < 1$ there exists natural number $N(\varepsilon) \in \mathbb{N}$, such that

$$\sum_{k=N(\varepsilon)+1}^{\infty} e^{-2\lambda_k^2 t} (d_k - \widehat{d}_k)^2 \le 2 \sum_{k=N(\varepsilon)+1}^{\infty} \left(|d_k|^2 + |\widehat{d}_k|^2 \right) \le \frac{\varepsilon^2}{2}, \quad \forall t \in [0, T].$$

Also, for $N(\varepsilon)$ we can find $T_\varepsilon > 0$, such that for any $\widetilde{T} < T_\varepsilon$ we have

$$|a_k| < \frac{\varepsilon(1-\varepsilon)}{3\sqrt{2N(\varepsilon)} \left(|\widehat{d}_k| + 1 \right)}, \quad |b_k| < \frac{\varepsilon(1-\varepsilon)}{3\sqrt{2N(\varepsilon)}},$$

$$1 - e^{-\lambda_k^2 T_i} < \frac{\varepsilon(1-\varepsilon)}{3m\sqrt{2N(\varepsilon)} \left(\max\limits_{1\le i \le m} |\beta_i| + 1 \right) \left(|\widehat{d}_k| + 1 \right)}, \quad 1 \le k \le N(\varepsilon).$$

From the latter inequalities we infer that $\sum\limits_{k=1}^{N(\varepsilon)} \left(d_k - \widehat{d}_k \right)^2 < \frac{\varepsilon^2}{2}$, and consequently for any $\widetilde{T} < T_\varepsilon$,

$$\|u(t) - \widehat{u}(t)\|_H^2 = \sum_{k=1}^{\infty} e^{-2\lambda_k^2 t} \left(d_k - \widehat{d}_k \right)^2 < \varepsilon^2, \quad 0 \le t \le T.$$

It must be pointed out, that theorems stated above allow investigating nonlocal in time problems for parabolic equations and systems. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with regular boundary. We designate the usual Sobolev space $W^{l,2}(\Omega)$ of an $l \in \mathbb{N}$ order as $H^l(\Omega)$ and denote with $H_0^l(\Omega)$ the closure of $\mathfrak{D}(\Omega)$ ($\mathfrak{D}(\Omega)$ is a set of infinitely differentiable functions with compact support in Ω) in the space $H^l(\Omega)$. The dual space of $H_0^l(\Omega)$ is denoted with $H^{-l}(\Omega)$. Let $V = \mathfrak{H}_0^l(\Omega) = [H_0^l(\Omega)]^N$, $H = \mathfrak{L}^2(\Omega) = [L^2(\Omega)]^N$, and A to be an elliptic operator of order $2l$:

$$A = \sum_{k=0}^{l} (-1)^k \sum \frac{\partial^k}{\partial x_{i_1}...\partial x_{i_k}} \left(A_{j_1...j_k}^{i_1...i_k}(x) \frac{\partial^k}{\partial x_{j_1}...\partial x_{j_k}} \right),$$

where in the inner sum each index $i_1, ..., i_k$, $j_1, ..., j_k$ independently range over the set $\{1, ..., n\}$, u is N-component vector-function, $A_{j_1...j_k}^{i_1...i_k}(x)$ is the square matrix of N order, which does not change for any transposition of upper or lower indices and turns into conjugate matrix if all upper indices are transposed with all lower indices. Assume that the elements of the matrices belong to $L^\infty(\Omega)$ and for almost all $x \in \Omega$,

$$\begin{gathered}\left(\sum A_{j_1...j_l}^{i_1...i_l}(x) t_{j_1...j_l}, t_{i_1...i_l}\right)_N \geq \alpha \sum \|t_{i_1...i_l}\|_N^2, \\ \left(\sum A_{j_1...j_k}^{i_1...i_k}(x) t_{j_1...j_k}, t_{i_1...i_k}\right)_N \geq 0, \quad k = 0, 1, ..., l-1.\end{gathered} \tag{3.5}$$

In the latter inequalities $t_{i_1...i_k}$ is N-component vector, which does not vary for any transposition of indices $i_1, .., i_k$ $(k = \overline{1, l})$.

Let us consider nonlocal in time problem for parabolic system

$$\frac{\partial u}{\partial t} + Au = f, \quad (x,t) \in \Omega \times (0,T), \tag{3.6}$$

with the following nonlocal initial and homogeneous boundary conditions

$$u(x,0) = \sum_{i=1}^{m} \alpha_i u(x, T_i) + \sum_{i,j=1}^{m_1} \int_{T_i^1}^{T_j^1} \rho_{ij}(\tau) u(x,\tau) d\tau + u_0(x), \quad x \in \Omega, \tag{3.7}$$

$$u = \frac{\partial u}{\partial \nu} = ... = \frac{\partial^{l-1} u}{\partial \nu^{l-1}} = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{3.8}$$

where ν is the unit outward normal to $\partial\Omega$, $\alpha_i \neq 0$, $0 < T_i, T_j^1 \leq \widetilde{T}$, ρ_{jk} are bounded measurable functions $(1 \leq i \leq m,\ 1 \leq j,k \leq m_1)$. According to the conditions (3.5) A is coercive and self-adjoint operator. Since $\mathfrak{H}_0^m(\Omega)$ is dense in $\mathfrak{L}^2(\Omega)$ and continuously, compactly imbedded in it, the set of eigenfunctions of the operator $A : \mathfrak{H}_0^m(\Omega) \to \mathfrak{H}^{-m}(\Omega)$ is complete in $\mathfrak{H}_0^m(\Omega)$ and corresponding eigenvalues $0 < \lambda_1^2 \leq \lambda_2^2 \leq ...$, $\lambda_k^2 \to \infty$, as $k \to \infty$.

Applying Theorem 3.1 to the formulated nonlocal in time problem we obtain the following theorem.

Theorem 3.3. *If* $u_0 \in \mathfrak{L}^2(\Omega)$, $f \in L^2(0,T; \mathfrak{H}^{-1}(\Omega))$ *and eigenvalues of the operator* A *satisfy the condition*

$$\begin{gathered}\Delta(k) \neq 1, \quad 1 \leq k < P, \\ P = \min \Big\{ q \,\Big|\, \lambda_q^2 > \\ > \max_{\substack{s=\overline{1,m},\\ i,j=\overline{1,m_1}}} \left(\ln(2m|\alpha_s|)/T_s, 2m_1^2 \sup_{[T_i^1, T_j^1]} |\rho_{ij}| e^{-\lambda_q^2 \min\{T_i^1, T_j^1\}} \left(1 - e^{-\lambda_q^2 |T_i^1 - T_j^1|}\right)\right) \Big\},\end{gathered}$$

then the nonlocal problem (3.6)-(3.8) *has a unique solution* $u \in L^2(0,T; \mathfrak{H}^m(\Omega))$, $u' \in L^2(0,T; \mathfrak{H}^{-m}(\Omega))$.

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