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ON UNIQUENESS AND INVARIANCE OF OPERATOR INTERPOLATIONAL POLYNOMIALS IN BANACH SPACES

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ABSTRACT. It's obtained new constructions of interpolation operator polynomials: for abstract Banach space with countable sequence of nodes and for space $C[0,\infty)$ with finite sequence of continual nodes. These interpolation polynomials are unique and retain operator polynomials of the same degree.

Introduction

The publications [10-14] were the first papers devoted to polynomial operator interpolation questions in abstract spaces. Bibliography of corresponding works and current state of theoretical developments on 2000 year in the line of investigation presented in the monograph [2]. One of the important properties of interpolational polynomials in finite-dimensional space is their uniqueness and invariance relative to the multinomials of the same power at some correlation between the number of the nodes m and the power of interpolainal polynomial n. In case of real one-variable functions such correlation is the equality $m = n + 1$. In infinite dimensional in general case the interpolational operator polynomials has not such properties at any correlation between m and n. It was showed in the papers [1-4], [15] that in infinite dimensional spaces the uniqueness and invariance can be obtained by the nodes selection, which depend from the real parameters (in following calls such nodes as continual). At that constructed interpolational formulas has either the integrals, which have derivatives, Gato differentials of integrated operator, or Stilties integrals of operator of scalar variable and continual nodes contained functions of Heviside type "jump" -function. The interpolational polynomial of such type we will refer to the class D. In the paper, based on traditional for that problems technique ([5], [6], [2]), we obtained two fundamentally new constructions of interpolational operator polynomials. The first construction (section 1) obtained for abstract Banach space with corresponding using the countable sequence of the nodes, concerned to space basis. At that constructed interpolational polynomials are not belong to the class D, but at that time have the properties of uniqueness and invariance. The second construction (section 2) obtained for specific Banach space $C[0,\infty)$ with using the finite sequence of continual nodes. Here also constructed interpolational polynomials (of integral form) are not belong to the class D and also have the property of uniqueness and invariance relative to all polynomials of integral form of the same degree.

1. Polynomial interpolation in Banach space with the basis

Let at first X, Y are linear spaces, π_n - the set of polynomials $P_n : X \to Y$ of the n-th power of the form:

$$\pi_n = \{P_n : P_n(x) = L_0 + L_1 x + ... + L_n x^n\} \tag{1}$$

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where $L_0 \in Y, L_k x^k = L_k(x, ..., x) : X \to Y, k = 1, 2, ..., n$ are k-th operator powers, which obtained from the symmetric k-linear operator forms $L_k(v_1, v_2, ..., v_k) : X^k \to Y$ at $v_1 = v_2 = ... = v_k = x$. Let further $x_1, x_2, ..., x_m (m \geq n)$ are liner dependent elements from X. Let us construct the set $Z(m) = \{z_i\}_{i=0}^{N}, N = N(m)$ in such way: $z_0 = 0, ..., z_i = x_i, i = 1, 2, ..., m$ and the rest $z_{m+1}, z_{m+2}, ..., z_N$ as every possible sums (repetition included) of the elements $x_1, x_2, ..., x_m$ with two, three and so on to n items in each. It is not hard to show, that $N = \sum_{k=1}^{n} C_{m+k-1}^{k}$. Let $F : X \to Y$ is an (in general case nonlinear) operator which defined by its values $F(z_i), i = 0, 1, 2, ..., N$. Then based on [5-6] we have the following result.

Theorem 1. *Let the polynomial $P_{m,n}^I(F; x) \in \pi_n$, the values of k-linear operator forms $L_k^I(x_{i_1}, x_{i_2}, ..., x_{i_k})$ of which, defined by the formulas* [5-6]

$$L_0^I = F(0),$$

$$\begin{aligned} L_n^I(x_{i_1}, ..., x_{i_n}) = \frac{1}{n!}\{\varphi(x_{i_1} + x_{i_2} + ... + x_{i_n}) - [\varphi(x_{i_1} + x_{i_2} + ... + x_{i_{n-1}}) + \\ + \varphi(x_{i_1} + x_{i_2} + ... + x_{i_{n-2}} + x_{i_n}) + ... + \varphi(x_{i_2} + x_{i_3} + ... + x_{i_n})] + \\ + [\varphi(x_{i_1} + x_{i_2} + ... + x_{i_{n-2}}) + \varphi(x_{i_1} + x_{i_2} + ... + x_{i_{n-3}} + \\ + x_{i_n}) + ... + \varphi(x_{i_3} + x_{i_4} + ... + x_{i_n})] + (-1)^{n-1}[\varphi(x_{i_1}) + \varphi(x_{i_2}) + ... + \varphi(x_{i_n})]\}, \end{aligned} \tag{2}$$

where $\varphi(x) = F(x) - F(0)$ and to find $L_{n-1}^I(x_{i_1}, x_{i_2}, ..., x_{i_{n-2}})$ we need to replace in (2) n for $n-1$, $\varphi(x)$ for $\varphi(x) - L_n^I x^n$, to find $L_{n-2}^I(x_{i_1}, x_{i_2}, ..., x_{i_{n-2}})$ replace n for $n-2$, $\varphi(x)$ for $\varphi(x) - L_{n-1}^I x^{n-1} - L_n^I x^n$ and so on. Then $P_{m,n}^I$ is interpolational for F on the node set $Z(m)$, i.e. the conditions hold true

$$P_{m,n}^I(F; z_i) = F(z_i), i = 0, 1, 2, ..., N. \tag{3}$$

Note, that the interpolant $P_{m,n}^I(F : x)$ has not the property of uniqueness and invariance on the set t_n. This fact explanation, and the construction of polynomial $P_{m,n}^I(F; x)$ itself will be demonstrated on the example $m = n = 2$. We have $Z(2) = \{z_i\}_{i=0}^{5} (N = 5), z_0 = 0, z_1 = x_1, z_2 = x_2$ (x_1, x_2 are linear depend), $z_3 = 2x_1, z_n = 2x_2, z_5 = x_1 + x_2$. From the conditions (3) we obtain

$$P_{2,2}^I(F; z_0) = L_0^I = F(0)$$

$$P_{2,2}^I(F; z_1) = L_1^I x_1 + L_2^I x_1^2 = \varphi(x_1),$$

$$P_{2,2}^I(F; z_2) = L_1^I x_2 + L_2^I x_2^2 = \varphi(x_2),$$

$$P_{2,2}^I(F; z_3) = 2L_1^I x_1 + 4L_2^I x_1^2 = \varphi(2x_1),$$

$$P_{2,2}^I(F; z_4) = 2L_1^I x_2 + 4L_2^I x_2^2 = \varphi(2x_2),$$

$$P_{2,2}^I(F; z_5) = L_1^I x_1 + L_1^I x_2 + L_2^I x_1^2 + 2L_2^I(x_1, x_2) + L_2^I x_2^2 = \varphi(x_1 + x_2).$$

From the system of linear equations we obtain

$$
\begin{aligned}
L_1^I(x_1,x_2) &= \frac{1}{2}\{\varphi(x_1+x_2)-\varphi(x_1)-\varphi(x_2)\},\\
L_2^I x_1^2 &= \frac{1}{2}\{\varphi(2x_1)-2\varphi(x_1)\},\\
L_2^I x_2^2 &= \frac{1}{2}\{\varphi(2x_2)-2\varphi(x_2)\},\\
L_1^I x_1 &= \frac{1}{2}\{4\varphi(x_1)-\varphi(2x_1)\},\\
L_1^I x_2 &= \frac{1}{2}\{4\varphi(x_2)-\varphi(2x_2)\},\\
L_0^I &= F(0),
\end{aligned} \tag{4}
$$

which corresponding to the formulas (2). As mentioned above, the interpolant $P_{2,2}^I(F;x)$ with found values of n-linear forms is not unique and invariant on the set of multinomials of the second power. Indeed, consider for example

$$L_0^I = K_0^I \in R_1, L_1^I x = \int_0^1 K_1(t)x(t)dt, L_2^I x^2 = \int_0^1\int_0^1 K_2(t_1,t_2)x(t_1)x(t_2)dt_1 t_2.$$

Then, according to (4) we have

$$
\begin{aligned}
K_0^I &= F(0),\\
\int_0^1 K_1(t)x_i(t)dt &= \frac{1}{2}\{4\varphi(x_i)-\varphi(2x_1)\},\\
\int_0^1\int_0^1 K_2^I(t_1,t_2)x_i(t_1)x_j(t_2)dt_1dt_2 &= \frac{1}{2}\{\varphi(x_i+x_j)-\varphi(x_i)-\varphi(x_j)\}, i,j=1,2
\end{aligned} \tag{5}
$$

We obtain the system of linear integral equations (5) for definition of kernels K_1^I, K_2^I. It is clearly, that the system has the set of solutions and, therefore, the interpolational polynomial $P_{2,2}^I(F;x)$ is not unique. Let us now $\bar{P}_2(x)$ is some fixed polynomial of the form

$$\bar{P}_2(x) = \bar{K}_0 + \int_0^1 \bar{K}_1(t)x(t)dt + \int_0^1\int_0^1 \bar{K}_2(t_1,t_2)x(t_1)x(t_2)dt_1dt_2$$

Then to define $P_{2,2}^I(\bar{P}_2;x)$ we use the formula (2) and obtain

$$K_0^I = \bar{K}_0$$

$$\int_0^1 K_1^I x_i(t)dt = \int_0^1 \bar{K}_1 x_1(t)dt$$

$$\int_0^1\int_0^1 K_2^I(t_1,t_2)x_i(t_1)x_j(t_2)dt_1dt_2 = \int_0^1\int_0^1 \bar{K}_2(t_1,t_2)x_i(t_1)x_j(t_2)dt_1dt_2$$

$$i,j=1,2$$

From that equalities in general case are not follows the equalities $K_i^I = \bar{K}_i, i=1,2,$ that in one's part is not guaranteed the invariance of the interpolant $P_{2,2}^I(F;x)$ relative to polynomials of integral form of the second power.

Let as above $x_1, x_2, ..., x_m$ are linear independent elements from X. Let us construct corresponding to these elements the set $Z(m) = \{z_i\}_{i=0}^N, N = N(m)$. It is known [7] that in the space X' conjugated to X there exists a system of linear functionals $l_i(x)$, $i = 1, 2, ..., m$ biorthogonal (conjugated) to the system $x_i, i = 1, 2, ..., m$, i.e. $l_i(x_j) = \delta_{ij}$, $i, j = 1, 2, ..., m, \delta_{ij}$ is Kronecker symbol. Consider a set of operator polynomials of the n-th power of form

$$\pi_{m,n} = \{P_{m,n} : P_{m,n}(x) = \sum_{k=0}^{n} \sum_{i_1,...,i_k=1}^{m} a_{i_1...i_k} l_{i_1}(x)...l_{i_k}(x)\}, \tag{6}$$

where $a_{i_1...i_k}$ symmetrical elements from Y concerning of its indexes. There exists the following

Theorem 2. *Let $P^I_{m,n} \in \pi_{m,n}$ is an operator polynomial, which $a^I_{i_1...i_k}$ defined by the formulas (2) with replacement $L^I_k x^k$ by*

$$\sum_{i_1,...,i_k}^{m} a^I_{i_1...i_k} l_{i_1}(x)...l_{i_k}(x),$$

and $L^I_k(x_{i_1}, ..., x_{i_k})$ by $a^I_{i_1...i_k}$. Then this polynomial is interpolational for F on the set of the nodes $Z(m)$ with the interpolational conditions (3). At that the interpolant $P^I_{m,n}(F; x)$ is uniqueness and invariance relative to polynomials of the n-th power on the set $\pi_{m,n}$.

Proof. Inasmuch as $a^I_{i_1...i_k}$ defined by the formula (2) are the solutions of linear system of equations, which equivalent to the equation (3), then the interpolationnes of the polynomial $P^I_{m,n}(F; x)$ is obvious. Further, since that solution is uniqueness, then the corresponding interpolant $P^I_{m,n}(F; x)$ on the set $\pi_{m,n}$ is also unique. Let now $F(x) = \bar{P}_{m,n}(\bar{x}) \in \pi_{m,n}$. Then from the formulas (2) taking into account the algebraic identities

$$\begin{gathered}(x_1 + x_2 + ... + x_n)^m - [(x_1 + x_2 + ... + x_{n-1})^m + ... + (x_2 + x_3 + ... + x_n)^m]+ \\ +[(x_1 + x_2 + ... + x_{n-2})^m + ... + (x_3 + x_4 + ... + x_n)^m] + ...+ \\ +(-1)^{m-1}[x_1^m + x_2^m + ... + x_n^m] = 0\end{gathered} \tag{7}$$

$\forall m = 1, 2, ..., n-1$, applied to operator polynomial $\bar{P}_{m,n}(x)$ we obtain

$$a^I_{i_1...i_k} = \bar{a}_{i_1...i_k}, k = 0, 1, ..., n, 1 \leq i_j \leq m \tag{8}$$

where $\bar{a}_{i_1...i_k}$ are an elements from Y, which corresponds to the polynomial $\bar{P}_{m,n}(x)$. The equalities (8) are refer to invariantness of the interpolant $P^I_{m,n}(F; x)$ concerning to all multinomials of the set $\pi_{m,n}$: $P^I_{m,n}(\bar{P}_{m,n}; x) = \bar{P}_{m,n}(x)$. The theorem is proved completely.

Remark 1. If X is a pre-Hilbert space with an inner product $(.,.)$, then according to the results [2] $l_i(x)$ can be written in the form

$$l_i(x) = \sum_{j=1}^{m} \alpha_{ij}(x, x_j), i = 1, 2, ..., m,$$

where α_{ij} are the elements of the matrix inverse to the Gramm matrix, constructed on system of linear independent elements $x_1, x_2, ..., x_m$

Remark 2. It follows from the paper [2], that in pre-Hilbert space X for invariant solvability interpolational problem with the conditions (3) (i.e. for existence of the operator polynomial interpolant on the set of the nodes $Z(m)$ at any values $F(z_i), i = 0, 1, ..., N$)

it is necessary and sufficiently fulfillment the condition $A = 0$, where A is a matrix which rows are the coordinates of orthonormal eigenvalues of the matrix

$$\Gamma = \| \sum_{k=0}^{n} (z_i, z_j)^k \|_{i,j=0}^{N}.$$

It is obvious that

$$A = 0 \Leftrightarrow \operatorname{rank} \Gamma = N + 1.$$

But inasmuch as interpolational problem on the set of the nodes $Z(m)$ (i.e. with interpolational conditions (3)) according to the correlations (2) is invariantly solvable, then

$$\operatorname{rank} \| \sum_{k=0}^{n} (z_i, z_j)^k \|_{i,j=0}^{N} = N + 1.$$

2. Polynomial interpolation in $C[0, \infty)$

Let us now X, Y are Banach spaces, π_n is a set of continuous polynomials of form (1). Let us set the following problem: it is necessary to define such sequence of the nodes and the interpolational polynomial of the n-th power for the operator $F : X \to Y$ on that sequence, which has the properties of uniqueness and invariance concerning to multinomials of the same power on the all set π_n Let $\{x_i\}_{i=1}^{\infty}$ is a basis of the space X. Construct the node set $Z(m) = \{z_i\}_{i=0}^{N} N = N(m)$, where $z_0 = 0, z_i = x_i, i = 1, 2, ..., m$ are the first of the basis elements. Grounded on [8] in the space X', conjugated to X, there exists a system of linear continuous functionals $\{l_i(x)\}_{i=1}^{\infty}$, which is biorthogonal to the element system $\{x_i\}_{i=1}^{\infty}$, i.e. $l_i(x_j) = \delta_{ij}, i, j = 1, 2, ...$ Consider a sequence (by the index m) of interpolational polynomial $P_{m,n}^{I}(F; x)$ on the node set $Z(m)$ of the following form

$$P_{m,n}^{I}(F; x) = \sum_{k=0}^{n} \sum_{i_1, \ldots, i_k=1}^{m} a_{i_1 \ldots i_k}^{I} l_{i_1}(x) \ldots l_{i_k}(x), \tag{9}$$

where $a_{i_1 \ldots i_k}^{I}$ are symmetrical relative to their indicies elements from Y, defined by the formulas (2). Without decrease a commonness let us consider, that $\|x\| = 1, \|l_i\| = 1, i = 1, 2, ...$ The following statement holds true.

Theorem 3. *Let*

$$\sum_{i_1, \ldots, i_k=1}^{\infty} \|a_{i_1 \ldots i_k}^{I}\| \le M_k = \text{const}, \quad k = 0, 1, ..., n. \tag{10}$$

Then limit polynomial

$$P_{\infty,n}^{I}(F; x) = \lim_{m \to \infty} P_{m,n}^{I}(F; x) \in \pi_n,$$

exists $\forall x \in X$, is an interpolational polynomial for the operator F on the countable sequence of the nodes $Z(m) = \{z_i\}_{i=0}^{N}, N = N(m), m = n, n+1, ...,$ uniqueness on the set π_n of continuous polynomials of the form (1), and invariant relative to all multinomials of the n-th power from π_n.

Proof. The limit existence. It follows from the condition (10) of the theorem, that

$$\| \sum_{i_1, \ldots, i_k=1}^{\infty} a_{i_1 \ldots i_k}^{I} l_{i_1}(x) \ldots l_{i_k}(x) \| \le \sum_{i_1, \ldots, i_k=1}^{\infty} \|a_{i_1 \ldots i_k}^{I}\| \|x\|^k \le M_k \|x\|^k, k = 0, 1, ..., n.$$

The last is knowing, that the serieses in polynomial $P_{\infty,n}^{I}(F; x)$ are converged, and the k-th operator powers of the polynomial are continuous. Thus $P_{\infty,n}^{I}(F; x)$ exists, continuous, and belongs to π_n.

The interpolationness. Inasmuch as at any fixed $m, (m \geq n)$ the polynomial $P^I_{\infty,n}(F;x)$ is an interpolational for F on the set of the nodes $Z(m)$, and there exists inclusion $Z(n) \subset Z(n+1) \subset ...$, then limit polynomial will be interpolational on the countable sequence of the nodes $Z(\infty)$. The uniqueness of the polynomial $P^I_{\infty,n}(F;x)$ is obvious, because every element of the sequence $P^I_{m,n}(F;x)$ defined by the formulas (2) by unique way. The invariance. Let $F(x) \equiv P_n(x) = L_0 + L_1 x + ... + L_n x^n \in \pi_n$. With account the algebraical identities (7), applied to the polynomial $P_n(x)$, grounded by the formulas (2) we have

$$P^I_{m,n}(P_n;x) = \sum_{k=0}^{n} \sum_{i_1,...,i_k=1}^{m} L_k(x_{i_1}, ..., x_{i_k}) l_{i_1}(x)...l_{i_k}(x).$$

Then

$$P^I_{\infty,n}(P_n;x) = \lim_{m\to\infty} P^I_{m,n}(P_n;x) = \lim_{m\to\infty} \sum_{k=0}^{n} \sum_{i_1,...,i_k=1}^{m} L_k(x_{i_1}, ..., x_{i_k}) l_{i_1}(x)...l_{i_k}(x) =$$

$$= \sum_{k=0}^{n} \lim_{m\to\infty} L_k(\sum_{i_1=1}^{m} l_{i_1}(x)x_{i_1}, ..., \sum_{i_k=1}^{m} l_{i_k}(x)x_{i_k}) =$$

$$\sum_{k=0}^{n} L_k(\lim_{m\to\infty} \sum_{i_1=1}^{m} l_{i_1}(x)x_{i_1}, ..., \lim_{m\to\infty} \sum_{i_k=1}^{m} l_{i_k}(x)x_{i_k}) =$$

$$= \sum_{k=0}^{n} L_k(x, ..., x) = \sum_{k=0}^{n} L_k x^k = P_n(x) \in \pi_n.$$

It proves the invariance. Here we used a representation of the element $x \in X$ in the form

$$x = \sum_{i=1}^{\infty} l_i(x) x_i,$$

and so a possibility of limit transition, inasmuch as the k-linear operator forms of the polynomial $P_n(x)$ are continuous. The theorem has been proved.

Consider now a set of polynomials of integral form, defined in the spase $C[0,\infty)$ of continuous functions with the value area on the real line. Here we construct a new type of interpolational polynomial with continual nodes, which has a uniqueness and an invariance properties. At that the interpolational formula is not belong to D class, i.e. is not contain neither differential operation, nor Stilties integrals, and do not need to implementation the "substitution rule" [9], as described below. It was proved in [9], that there exists an interpolational functional polynomial of form:

$$P_n(x) = K_0 + \int_0^1 K_1(t)(x(t) - \varphi_0(t))dt +$$

$$+ \int_0^1 \int_{t_1}^1 K_2(t_1,t_2)(x(t_1) - \varphi_0(t_1))(x(t_2) - \varphi_1(t_2))dt_2 dt_1 + \tag{11}$$

$$+ \int_0^1 \int_{t_1}^1 ... \int_{t_{n-1}}^1 K_n(t_1,t_2,...,t_n)(x(t_1) - \varphi_0(t_1))(x(t_2) - \varphi_1(t_2))... \times$$

$$\times (x(t_n) - \varphi_{n-1}(t_n))dt_n dt_{n-1}...dt_1$$

with the symmetrical kernels $K_i \in Q(\Omega_i)$, $\varphi_i(t) \in Q[0,1] i = 0,..,n-1$, where $\Omega_i = \{(\xi_1, \xi_2, ..., \xi_i) : 0 \le \xi_1 \le \xi_2 \le ... \le \xi_i \le 1\}$, $Q(\Omega_i)$ is a set of piece linear functions on every variable with finite number of points of discontinuity of the first type. At that the interpolational continual nodes are the functions

$$x^k(t, \vec{\xi}^{(k)}) = \varphi_0(t) + \sum_{i=1}^{k} H(t-\xi_i)(\varphi_i(t) - \varphi_{i-1}(t)), \vec{\xi}^{(k)} \in \Omega_k, k = 1, 2, ..., n$$

($H(t)$ is Heviside function) i.e. holding true the following equalities

$$P_n(x^k(\cdot, \vec{\xi}^{(k)})) = F(x^k(\cdot, \vec{\xi}^{(k)})), \forall \vec{\xi}^{(k)} \in \Omega_k. \tag{12}$$

The following statements are the necessary and sufficiently conditions of existence and uniqueness such interpolant with equalities (12):

1.

$$\frac{\partial^k}{\partial \xi_1 ... \partial \xi_k} F(x^k(\cdot, \vec{\xi}^{(k)})) \in Q(\Omega_k), \tag{13}$$

2. The "substitution rule"

$$\{\frac{\partial}{\partial z_k}[\frac{\partial^{k-1}}{\partial z_1 \partial z_2 ... \partial z_{k-1}} F(x^{k-1}(\cdot, \vec{\xi}^{k-1}) + H(\cdot - z_k)(\phi_k(\cdot) - \phi_{k-1}(\cdot)) +$$
$$+ H(\cdot - z_{k+1})(\phi_{k+1}(\cdot) - \phi_k(\cdot)))]\}_{z_{k+1}=z_k} =$$
$$= \frac{\phi_k(z_k) - \phi_{k-1}(z_k)}{\phi_{k+1}(z_k) - \phi_{k-1}(z_k)} \frac{\partial}{\partial_k}[\frac{\partial^{k-1}}{\partial z_1 \partial z_2 ... \partial z_{k-1}} F(x^{k-1}(\cdot, \vec{\xi}^{k-1}) +$$
$$+ H(\cdot - z_k)(\phi_{k+1}(\cdot) - \phi_{k-1}(\cdot)))], \quad k = 1, 2, ..., n-1.$$

At that the kernel K_i defined by the formulas

$$K_i(\xi_1, \xi_2, ..., \xi_i) = \frac{\partial^i}{\partial \xi_1 \partial \xi_2 ... \partial \xi_i} F(x^i(\cdot, \vec{\xi}^i)),$$

Now we pass to a setting problem of construction of interpolational functional polynomial of integral type with defined above properties of uniqueness and invariance, which is not belong to the class D. Define as π_n the set of functional polynomials of the n-th power of form

$$\pi_n = \{P_n : P_n(x) = K_0 + \int_0^\infty K_1(t)x(t)dt + \int_0^\infty \int_0^\infty K_2(t_1, t_2)x(t_1)x(t_2)dt_1 dt_2 + ... +$$
$$+ \int_0^\infty \cdots \int_0^\infty K_n(t_1, t_2, ..., t_n)x(t_1)x(t_2)...x(t_n)dt_1 dt_2 ... dt_n\} P_n : C[0,\infty) \to R^1, \tag{15}$$

where K_i are an symmetrical functions of its variables, $K_i \in L_1(\Omega_i), \Omega_i = [0,\infty) \times ..(i).. \times [0,\infty), x \in C[0,\infty)$. Introduce the system of linear independent (at different ξ_i) functions

$$x_i(t, \xi_i) = \psi(t) sint\xi_i, i = 1, 2, ..., n, \psi(t) \in C[0,\infty). \tag{16}$$

Construct by that system, as it showed above, the set of continual nodes $Z(n) = \{z_i\}_{i=0}^N, N = N(n)$, which depend from real parameters ξ_i. We need to find such polynomial of the n-th power $P_n^I(F;x) \in \pi_n$ with the kernels K_i^I, which satisfy the interpolational conditions

$$P_n^I(F; z_i(\cdot, \xi)) = F(z_i(\cdot, \xi)), i = 0, 1, 2, ..., N, \forall \xi \in \Omega_n. \tag{17}$$

Notice, that instead the conditions (17) we can substitute one interpolational condition on the continual node

$$z^n(t,\xi) = \psi(t) = \sum_{i=1}^{n} sin(t\xi_i),$$

i.e. instead (17) we demand holding true the following identity

$$P_n^I(F; z^n(\cdot,\xi)) \equiv F(z^n(\cdot,\xi)), \forall \xi \in \Omega_n,$$

from which at suitable values ξ_i as partial case, follows the conditions (17). The continual interpolation nodes choice as the set $Z(n)$, based on (16) is determined by convenience for the following statements. Thus, we have the following problem: to define the kernels $K_i^I, i = 0,1,...,n$ in such way, that the correspondent polynomial $P_n^I(F;x)$ will be interpolational on continual node set $z_i(z,\xi)$, which depends from continuous vector-parameter $\xi \in \Omega_n$ with interpolational conditions (17). According to (15) the p-th operator power of polynomial $P_n^I(F;x)$ has the form

$$L_p^I x^p = \int_0^\infty ... \int_0^\infty K_p^I(t_1,t_2,...,t_p)x(t_1)x(t_2)...x(t_p)dt_1 dt_2 ... dt_p.$$

From the conditions (17) based on the formulas (2) we can find a values of the p-linear operator forms $L_p^I(z_1(\cdot,\xi_1), z_2(\cdot,\xi_2),...,z_p(\cdot,\xi_p))$ as the right parts of equalities (2). On the other hand

$$\begin{gathered} L_p^I(z_1(\cdot,\xi_1), z_2(\cdot,\xi_2),...,z_p(\cdot,\xi_p)) = \\ = \int_0^\infty ... \int_0^\infty K_p^I(t_1,t_2,...,t_p) z_1(t_1,\xi_1) z_2(t_2,\xi_2)...z_p(t_p,\xi_1)dt_1 dt_2 ... dt_p = \\ = \int_0^\infty ... \int_0^\infty K_p^I(t_1,t_2,...,t_p) \prod_{i=1}^{p} \psi(t_i) sin(t_i\xi_i)dt_i, \end{gathered} \tag{18}$$

Further, using the inverse sinus transformation, from the formula (18) we obtain

$$\begin{gathered} K_p^I(\xi_1,\xi_2,...,\xi_p) = \\ = \left(\frac{2}{\pi}\right)^p [\prod_{i=1}^{p} \psi(\xi_i)]^{-1} \int_0^\infty ... \int_0^\infty L_p^I(z_1(\cdot,t_1), z_2(\cdot,t_2),...,z_p(\cdot,t_p)) \times \\ \times \prod_{i=1}^{p} sin(t_i\xi_i)dt_i, \quad p = 1,2,...,n \end{gathered} \tag{19}$$

Thus, if the conditions (17) holds true, then the kernels K_p^I of the polynomial P_n^I can be defined by the formulas (19). Inversely, if the kernels calculated by the formulas (19), then sinus transformation define the values of the p-linear operator forms

$$L_p^I(z_{i_1}(\cdot,\xi_1), z_{i_2}(\cdot,\xi_2),...,z_{i_p}(\cdot,\xi_{i_p})), p = 1,2,...,n,$$

and $1 \leq i_j \leq n$, which in its turn are the solutions of the system of linear equations, which equivalent to the interpolational conditions (17). So we proved the following

Theorem 4. *Let*

$$L_p^I(z_1(\cdot,t_1),..,z_p(\cdot,t_p)) \in L_1(\Omega_p), p = 1,2,...,n.$$

Then in order that operator polynomial $P_n^I(F;x)$ is interpolational on continual node set $Z(n) = \{z_i\}_{i=0}^N$, it is necessary and sufficient to that kernels defines by the formulas (19).

Notice, that interpolant $P_n^I(F;x)$ on continual node set is unique. This fact is obvious as the corollary of the previous theorem. Besides there holds true the following statement.

Theorem 5. *The interpolational polynomial $P_n^I(F;x)$ on continual node set $Z(n)$ is invarianted relative to all multinomials of the form (15).*

Proof. Let us the interpolated operator F present by itself a functional polynomial $P_n \in \pi_n$ of the form (15), i.e. $F \equiv P_n$. Then based on algebraic identities (7), applied to the polynomial P_n, an so the conditions

$$P_n^I(F;z_i) = \varphi(z_i), \varphi(x) = P_n(x) - P_n(0), i = 1,2,...,N$$

we obtain by the formulas (2) the values

$$L_p^I(z_1(\cdot,\xi_1), z_2(\cdot,\xi_2),...,z_p(\cdot,\xi_p)) =$$

$$= \int_0^\infty ... \int_0^\infty K_p(t_1,t_2,...,t_p) z_1(t_1,\xi_1) z_2(t_2,\xi_2)...z_p(t_p,\xi_p) dt_1 dt_2...dt_p, \qquad (20)$$

$$p = 1,2,...,n$$

where K_p are a kernels of the polynomial $P_n(x)$. From (20) by the inverse sinus transformation we obtain

$$K_p(\xi_1,\xi_2,...,\xi_p) = \left(\frac{2}{\pi}\right)^p [\prod_{i=1}^p \psi(\xi_i)]^{-1} \int_0^\infty ... \int_0^\infty L_p^I(z_1(\cdot,t_1), z_2(\cdot,t_2),...,z_p(\cdot,t_p)) \times$$

$$\times \prod_{i=1}^p sin(t_i\xi_i)dt_i = K_p^I(\xi_1,\xi_2,...,\xi_p), \quad p = 1,2,...,n.$$

The last means the polynomial P_n^I and P_n coincide, i.e. $P_n^I(P_n;x) = P_n$. The theorem has proved.

Remark 3. Instead the sinus transformation we can use any other integral transformations, for which we know transformation formulas (cosine, Hankel, Kantorovich-Lebedev etc.). Such approach to construction of interpolational functional polynomials used by L.A.Yanovich with the followers [2], but in that paper authors used Stilties integrals on scalar argument operator, so their constructions are not interpolational on continual nodes.

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