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ON SOLVABILITY OF THE FINITE DIFFERENCE SCHEMES FOR A PARABOLIC EQUATIONS WITH NONLOCAL CONDITION

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ABSTRACT. The article deals with the problem of existence and uniqueness of the solution of parabolic equations subject to the nonlocal condition. First of all, we consider a one-dimensional parabolic equation with constant coefficients. Afterwards, the results of investigation are generalized to a two-dimensional case. The article ends with the discussion on possible ways of generalizing the results.

During the past two decades there was a significant increase in a number of research publications devoted to differential equations of various types subject to nonlocal conditions.

The reason of such an interest towards the equations mentioned above is unveiled by the fact of permanently emerging new areas of applications of these equations.

One of the earliest publications on the application of parabolic equations subject to nonlocal conditions describes a certain phenomenon in plasma physics [1]. This article deals with the one-dimensional heat equation subject to the conditions

$$u(0,t) = \mu_0, \quad \frac{\partial u(0,t)}{\partial x} = \frac{\partial u(1,t)}{\partial x} \tag{1}$$

Some other publications discuss the applications in the field of nonlinear diffusion in semiconductor devices [2]. One- and two-dimensional nonlinear diffusion equations subject to integral condition is discussed here. The monograph [3] considers various applications in biotechnology of the equation subject to different nonlocal conditions. A lot of nonlocal conditions are used in various problems of mathematical biology, as well [4]. The articles [5] – [6] deal with the quasistatic theory of thermoelasticity, while [7] considers atmosphere pollution problems. Publications [8] – [10] deal with one-dimensional parabolic equation subject to nonlocal conditions

$$\begin{aligned} u(0,t) &= \int_0^1 \alpha(x)u(x,t)dx + \mu_0(t), \\ u(1,t) &= \int_0^1 \beta(x)u(x,t)dx + \mu_1(t). \end{aligned} \tag{2}$$

The majority of authors in the field assume that functions $\alpha(x)$ and $\beta(x)$ satisfy a kind of "smallness" assumption, e.g.

$$|\alpha(x)| \leq 1, \quad |\beta(x)| \leq 1 \tag{3}$$

Key words and phrases. One- and two-dimensional parabolic equation, nonlocal condition, existence and uniqueness.

For a detailed discussion on this question see, e.g. [10]. A lot of theoretical problems and solution methods for parabolic equations subject to nonlocal condition are revealed in publications [11] – [21].

The author of this article substitutes assumption (3) for an assumption which is considerably more general, i.e. both functions $\alpha(x)$ and $\beta(x)$ are bounded. Throughout the paper, the difference approximation of the differential problem appears as the main research object rather than the differential problem itself. The main issue of the investigation is the conditions under which there exists a unique solution of approximating system of difference equations.

1. One-dimensional case. First we consider a linear parabolic equation with constant coefficients

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - qu - f(x,t), \tag{4}$$

$q \geq 0,\ 0 \leq x \leq 1,\ 0 \leq t \leq T$ subject to the initial condition

$$u(x,0) = \varphi(x) \tag{5}$$

and nonlocal conditions

$$u(0,t) = \int_0^1 \alpha(x) u(x,t) dx + \mu_0(t), \tag{6}$$

$$u(1,t) = \int_0^1 \beta(x) u(x,d) dx + \mu_1(t). \tag{7}$$

One of the most important points is the implications of nonlocal conditions to the existence and uniqueness of the solution of a differential problem. In the case of an ordinary differential equation this issue can have different answers. Let us consider an equation

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) - q(x)u = f(x) \tag{8}$$

subject to either single boundary condition and single nonlocal condition

$$u(0) = \mu_0, \quad u'(0) = u'(1) \tag{9}$$

or

$$u(0) = \mu_0, \quad u(1) = cu(\xi), \quad 0 < \xi < 1. \tag{10}$$

In both cases we rose the same question: Is there a number λ such that equation (8) subject to classic boundary value conditions

$$u(0) = \mu_0, \quad u(1) = \mu \tag{11}$$

have a solution, coincidental with the solution either of problem (8), (9) or (8), (10). For this purpose we put the general solution of equation (8) into the form

$$u(x) = c_1 w_1(x) + c_2 w_2(x) + w_0(x) \tag{12}$$

where $w_1(x), w_2(x)$ are fundamental solutions of the corresponding homogeneous equation, and w_0 is a partial solution of non-homogeneous equation. Let us define $w_0(x)$, $w_1(x)$, $w_2(x)$ in the following way:

$$\frac{d}{dx}\left(p(x)\frac{dw_1}{dx}\right) - q(x)w_1 = 0,\ w_1(0) = 1,\ w_1(1) = 0,$$
$$\frac{d}{dx}\left(p(x)\frac{dw_2}{dx}\right) - q(x)w_2 = 0,\ w_2(0) = 0,\ w_2(1) = 1,$$

$$\frac{d}{dx}\left(p(x)\frac{dw_0}{dx}\right) - q(x)w_0 = f(x),\ w_0(0) = w_0(1) = 0.$$

For solution (12), conditions (10) hold, if the following is true:

$$c_1 = \mu_0,$$
$$\lambda = c_2 = \frac{\mu_0 w_1'(1) + w_0'(1) - \mu_0 w_1'(0) - w_0'(0)}{w_2'(0) - w_2'(1)}.$$

Since $w_2(x)$ is a strictly increasing function in the interval [0, 1], we have $w_2'(0) < w_2'(1)$ and both c_1, c_2 exist. Thus, problem (8), (9) subject to nonlocal condition is always equivalent to the problem (8), (11). Now let us consider problem (8), (10). Solution (12) will be a solution of the problem (8), (10), if following holds:

$$c_1 = \mu_0,$$
$$\lambda = c_2 = \frac{c\mu_0 w_1(\xi) + cw_0(\xi)}{1 - cw_2(\xi)}.$$

Thus, the problem (8), (10) is equivalent to the classic boundary value problem (8), (11) if and only if

$$cw_2(\xi) \neq 1. \tag{13}$$

This is the necessary and sufficient condition for the existence of a unique solution of problem (8), (10). There exists a single value of c, $(c = 1/w_2(\xi) > 1)$, for which equation (8) subject to nonlocal condition has no unique solution. If we substitute the nonlocal condition $u(1) = cu(\xi)$ in conditions (10) with the expression

$$u(1) = c_1 u(\xi_1) + c_2 w_2(\xi_2), \tag{14}$$

then inequality (13) would be transformed into the following necessary and sufficient condition

$$c_1 w_2(\xi_1) + c_2 w_2(\xi_2) \neq 1. \tag{15}$$

In this case, equation (8) subject to both the boundary value condition $u(0) = \mu_k$ and nonlocal condition (14), has no unique solution for all values of c_1, c_2, if the point (c_1, c_2) lies on a straight line

$$w_2(\xi_1)x + w_2(\xi_2)y = 1.$$

Here we confront about natural question with a the influence of nonlocal conditions on the existence and uniqueness of the solution of parabolic differential equations. This article deals with the conditions of a existence of unique solution of the system of difference equations, which corresponds to differential problem (4) – (7).

Let us approximate differential problem (4) – (7) by the following difference problem of approximation order $O(h^2 + \tau)$:

$$\frac{u_i^j - u_i^{j-1}}{\tau} = \frac{u_{i-1}^j - 2u_i^j + u_{i+1}^j}{h^2} - qu_i^j - f_i^j, \tag{16}$$
$$i = \overline{1, N-1},\ j = \overline{1, M},$$

$$u_0^j = h\Big(\frac{\alpha_0 u_0^j + \alpha_N u_N^j}{2} + \sum_{i=1}^{N-1} \alpha_i u_i^j\Big) + \mu_0^j \equiv (\alpha, u^j) + \mu_0^j, \tag{17}$$

$$u_N^j = h\Big(\frac{\beta_0 u_0^j + \beta_N u_N^j}{2} + \sum_{i=1}^{N-1} \beta_i u_i^j\Big) + \mu_1^j \equiv (\beta, u^j) + \mu_1^j, \tag{18}$$
$$j = \overline{1, M},$$

$$u_i^0 = \varphi_i, \quad i = 0, 1, \dots, N, \tag{19}$$

where $h = 1/N,\ \tau = T/M$.

Let us consider the solution of equation (16) with a fixed value of j. This is the second order difference equation with respect to the discrete variable function $u_i^j, i = \overline{0,N}$. Therefore, its general solution has the form

$$u_i^j = c_1(w_1)_i + c_2(w_2)_i + (w_o)_i^j. \tag{20}$$

Let us define w_1, w_2, w_3 in the following way:

$$\begin{gathered}\frac{(w_1)_{i-1} - 2(w_1)_i + (w_1)_{i+1}}{h^2} - (q + \frac{1}{\tau})(w_1)_i = 0,\\ (w_1)_0 = 1, \quad (w_1)_N = 0;\end{gathered} \tag{21}$$

$$\begin{gathered}\frac{(w_2)_{i-1} - 2(w_2)_i + (w_2)_{i+1}}{h^2} - (q + \frac{1}{\tau})(w_2)_i = 0,\\ (w_2)_0 = 0, \quad (w_2)_N = 1;\end{gathered} \tag{22}$$

$$\begin{gathered}\frac{(w_0)_{i-1}^j - 2(w_0)_i^j + (w_0)_{i+1}^j}{h^2} - (q + \frac{1}{\tau})(w_0)_i^j = f_i^j - \frac{u_i^{j-1}}{\tau},\\ (w_0)_0 = 0, \quad (w_0)_N = 0.\end{gathered} \tag{23}$$

Lemma 1. *The system of equations (16) – (19) has a unique solution if and only if*

$$D = \begin{vmatrix} 1 - (\alpha, w_1) & -(\alpha, w_1) \\ -(\beta, w_1) & 1 - (\beta, w_2) \end{vmatrix} \neq 0. \tag{24}$$

Proof. Let us take a fixed value of $j, j = 1, 2, \dots, M$. Let us determine the conditions allowing to find unique values of c_1, c_2 such that solution (20) satisfies the nonlocal conditions (17) and (18). We put (20) into (17) and (18) and taking into account boundary value conditions of systems (21) – (23), we obtain

$$\begin{cases} (1 - (\alpha, w_1))c_1 - (\alpha, w_2)c_2 = (\alpha, w_0^j), \\ -(\beta, w_1)c_1 - (1 - (\beta, w_2))c_2 = (\beta, w_0^j). \end{cases} \tag{25}$$

Therefore, a unique solution of the system (25) there exists if and only if $D \neq 0$. Thus, the Lemma is proved.

On the ground that both w_1 and w_2 are defined in terms of difference equations with constant coefficients, let us put w_1 and w_2 into the explicit form:

$$(w_1)_i = \frac{e^{\sigma(N-i)h} - e^{-\sigma(N-i)h}}{e^{\sigma} - e^{-\sigma}} = \frac{\operatorname{sh}\sigma(N-i)h}{\operatorname{sh}\sigma},$$

$$(w_2)_i = \frac{e^{\sigma ih} - e^{-\sigma ih}}{e^{\sigma} - e^{-\sigma}} = \frac{\operatorname{sh}\sigma ih}{\operatorname{sh}\sigma},$$

where the number σ is defined by the formula

$$\operatorname{ch}\sigma h = 1 + \frac{1 + \tau q}{2\gamma}, \quad \gamma = \frac{\tau}{h^2}. \tag{26}$$

We will refer to the following statement.

Lemma 2. *If $\tau \to 0, h \to 0$, then independently of the value of $\gamma = \tau/h^2$,*

$$\lim_{\tau \to 0, h \to 0} (w_s)_i = 0, \quad i = \overline{1, N-1}, \ s = 1, 2.$$

For the function $(w_2)_i$, this lemma is proved in [21]. For the function $(w_1)_i$ the proof is analogous.

Lemma 3. *As $\tau \to 0$ and $h \to 0$, then*

$$(1, w_s) \to 0, s = 1, 2.$$

Proof. Since

$$\int_0^1 \frac{\operatorname{sh} \sigma x}{\operatorname{sh} \sigma} dx = \frac{1}{\sigma} \operatorname{th} \frac{\sigma}{2},$$

in accordance with formula (17), we get

$$(1, w_2) = \int_0^1 \frac{\operatorname{sh} \sigma x}{\operatorname{sh} \sigma} dx + O(h^2) = \frac{1}{\sigma} \operatorname{th} \frac{\sigma}{2} + O(h^2). \tag{27}$$

Let us find the value of the function

$$\Psi(\sigma) = \frac{1}{\sigma} \operatorname{th} \frac{\sigma}{2},$$

where σ is defined by equality (26), as $h \to 0$, and $\tau \to 0$. We consider the next three cases:

1) If $h \to 0$ and $\tau \to 0$ in the way that

$$0 < m_1 \le \frac{\tau}{h^2} \le M_1 < \infty$$

where constants m_1, M_1 do not depend on h, and τ then (26) implies, that σh is also bounded by the constants, independent of h, i.e., $\sigma = O(h^{-1})$, as $h \to 0$. Therefore,

$$\lim_{\tau \to 0, h \to 0} \Psi(\sigma) = 0.$$

2) If $\tau / h^2 \to 0$, as $h \to 0$ and $\tau \to 0$, then (26) implies that $\sigma h \to \infty$, i.e. $\sigma \to \infty$. Thus, again,

$$\lim_{\tau \to 0, h \to 0} \Psi(\sigma) = 0.$$

3) If $\tau / h^2 \to \infty$, as $h \to 0$ and $\tau \to 0$, then (26) implies that ch $\to 1$, i.e., $\sigma h \to 0$. Therefore,

$$\operatorname{ch} \sigma h = 1 + \frac{\sigma^2 h^2}{2} + O(h^4).$$

Equating this to (26), we get within the accuracy of $O(h^2)$ that

$$\sigma = O(\tau^{-1/2}).$$

Therefore, again,

$$\lim_{h \to 0, \tau \to 0} \Psi(\sigma) = 0.$$

In accordance with the expression of $(1, w_2)$ and formula (27), we obtain

$$\lim_{h \to \infty, \tau \to 0} (1, w_2) = 0.$$

The equality

$$\lim_{h \to \infty, \tau \to 0} (1, w_1) = 0.$$

is proved analogously.

Theorem 1. *Let functions $\alpha(x)$ and $\beta(x)$ be bounded in the interval $[0,1]$. Then there exist numbers $\tau_0 > 0$, $h_0 > 0$ such that for all $\tau \in (0, \tau_0]$ and $h \in (0, h_0]$, the system of equations (16) – (19) has a unique solution.*

Proof. According to Lemma 1, it is enough to prove, that $D \neq 0$. In accordance with Lemma 3, there exist τ_0 and h_0 such that $(1, w_s) < 1/2M, s = 1, 2$. For particular values of $\tau \le \tau_0$ and $h \le h_0$, we evaluate:

$$D = \big(1 - (\alpha, w_1)\big)\big(1 - (\beta, w_2)\big) - (\alpha, w_2)(\alpha, w_1) \ge$$
$$\ge \big(1 - M(1, w_1)\big)\big(1 - M(1, w_2)\big) - M(1 - w_2)M(1 - w_1) =$$
$$= 1 - 2M\frac{1}{\sigma}\text{th}\,\frac{\sigma}{2} > 0.$$

Thus, the theorem is proved.

Remark. The expressions of both $(1, w_1)$ and $(1, w_2)$ can be evaluated not only by means of Lemmas 2 and 3, but also directly by means of systems (21) and (22). Dropping the terms $(q + \frac{1}{\tau})(w_s)_i, s = 1, 2$, we have a trivial estimates

$$0 < (1, w_s) < 1/2.$$

Next, by evaluating the value of determinant D in the same way as in the theorem, we obtain

$$D \ge 1 - M > 0,$$

if $M < 1$. The same kind of sufficient conditions on the existence of the solution

$$\left|\alpha\right| < 1, \quad \left|\beta\right| < 1 \tag{28}$$

or very close to them were obtained in a series of articles (see, e.g. [5,6], [8] – [10]). Estimating w_1 and w_2, in accordance to Lemmas 2 and 3, we get conditions much weaker than (28): functions $\alpha(x)$ and $\beta(x)$ have to be bounded.

Example. We will apply the obtained theoretical results for an equation (4) subject to conditions of type (10):

$$u(0,t) = \mu_0(t),$$
$$u(1,t) = cu(t,\xi) + \mu_1(t), \quad 0 < \xi < 1.$$

Assuming that $\xi = kh$, where k is a integer, $1 \le k \le N - 1$, we put equation (16) into a difference form subject to the following conditions:

$$u_0^j = \mu_0^j, \tag{17a}$$

$$n_N^j = cu_k^j + \mu_1^j. \tag{18a}$$

For any fixed $j, (j = \overline{1, M})$, we get the following necessary and sufficient condition on the uniqueness of the solution of equation (16) subject to conditions (17a) and (18a):

$$c(w_2)_k \neq 1 \tag{29}$$

The sufficient condition of the existence of the solution is

$$-\infty < c < c^*; \tag{30}$$

where the value c^* is defined by the equality

$$c^* = \frac{1}{(w_2)_k}.$$

According to Lemma 2,

$$(w_2)_k \to 0, \quad \text{as } \tau \to 0, h \to 0.$$

Therefore,

$$c^* \to \infty, \quad \text{as } \tau \to 0, h \to 0.$$

In contrast to the ordinary differential equation, for the parabolic equation in the difference form the constraint (30) is insignificant, because $c* \to \infty$, as $\tau \to 0$ and $h \to 0$.

2. Two-dimensional parabolic equation. We will consider the ways of generalization of the foregoing results in a case of the two-dimensional parabolic equation. Let us consider the differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - qu - f(x,y,t) \tag{31}$$

over the domain $D = \{0 < x < 1,\ 0 < y < 1,\ 0 < t \leq T\}$ subject to boundary value, nonlocal and initial value conditions:

$$u(0,y,t) = \mu_0(y,t), \tag{32}$$

$$u(1,y,t) = cu(\xi,y,t) + \mu_1(y,t), \tag{33}$$

$$u(x,0,t) = g_0(x,t), \quad u(x,1,t) = g_1(x,t), \tag{34}$$

$$u(x,y,0) = \varphi(x,y). \tag{35}$$

Let us approximate this differential problem by a difference one, using the implicit scheme:

$$\frac{u_{ij}^k - u_{ij}^{k-1}}{\tau} = \left(u_{\bar{x}x}^k\right)_{ij} + \left(u_{\bar{y}y}^k\right)_{ij} - qu_{ij}^k - f_{ij}^k, \tag{36}$$

$$u_{0j}^k = (\mu_0)_j^k, \tag{37}$$

$$u_{Nj}^k = cu_{sj}^k + (\mu_1)_j^k, \tag{38}$$

$$u_{i0}^k = (g_0)_i^k, \quad u_{iM}^k = (g_1)_i^k, \tag{39}$$

$$u_{ij}^0 = \varphi_{ij}; \tag{40}$$

where $h = 1/N$, $\tau = T/K$, $i,j = \overline{1, N-1}$, $k = \overline{1,K}, \xi = sh$, s is a integer, $(1 \leq s \leq N-1)$,

$$\left(u_{\bar{x}x}^k\right)_{ij} = \frac{u_{i-1,j}^k - 2u_{ij}^k + u_{i+1,j}^k}{h^2},$$

$$\left(u_{\bar{y}y}^k\right)_{ij} = \frac{u_{i,j-1}^k - 2u_{ij}^k + u_{i,j+1}^k}{h^2}.$$

We put the difference scheme in the following form. Let us take a fixed number k and define:

$$v_{ij} = u_{ij}^k,$$

$$\bar{v}_i = \left(v_{i1}, v_{i2}, \ldots, v_{i,N-1}\right)',$$

$$(\bar{v}_{\bar{x}x})_i = \Big((v_{\bar{x}x})_{i1}, (v_{\bar{x}x})_{i2}, \ldots, (v_{\bar{x}x})_{iN-1}\Big)',$$

A is an $(N-1)$-dimensional tri-diagonal matrix, built by the pattern $\frac{1}{h^2}\Big(-1, 2+h^2(q+1/\tau), -1\Big)$. Now, for a fixed k, we put the difference scheme (36) – (40) into the form

$$(\bar{v}_{\bar{x}x})_i = A\bar{v}_i + \overline{F}_i, \quad i = \overline{1, N-1}, \tag{36a}$$

$$\bar{v}_0 = \bar{\mu}_0, \tag{37a}$$

$$\bar{v}_M = c\bar{v}_s + \bar{\mu}_1; \tag{38a}$$

where $\bar{\mu}_0, \bar{\mu}_1$ and are $(N-1)$-dimensional vectors, whose components are $(\mu_0)_j^k, (\mu_1)_j^k$, and $\bar{F}_i$ is $(N-1)$-dimensional vector, built of the values of the function, boundary values (39) and initial values (40).

Note, that A is symmetric positive definite matrix. Denote its eigenvalues as $\lambda_j (j = \overline{1, N-1})$. We refer to the following results [22].

Lemma 4. *Eigenvalues of matrix A are given by*

$$\lambda_j = \frac{4}{h^2}\sin^2\frac{j\pi h}{4} + \frac{1}{\tau} + q, \quad j = \overline{1, N-1}. \tag{41}$$

For a fixed number $j, (j = \overline{1, N-1})$ define the difference function $w_i^j, i = \overline{0, N}$ in terms of the solution of the following problem:

$$\frac{w_{i-1}^j - 2w_i^j + w_{i+1}^j}{h^2} - \lambda_j w_i^j = 0, \quad i = \overline{1, N-1}, \qquad w_0^j = 0, \quad w_N^j = 1. \tag{42}$$

Theorem 2. *The system of difference equations (36) – (40) has a unique solution if and only if*

$$cw_s^j \neq 1, \quad j = 1, 2, \ldots, N-1; \tag{43}$$

Proof. Let Q be a matrix whose columns are the eigenvestors of matrix A, Λ be a diagonal matrix whose diagonal elements are the eigenvalues of matrix A. Then

$$Q^{-1}AQ = \Lambda.$$

Put the system of difference equations (36) – (40) into the form of (36a) – (38a). Make the following substitution of the variables in the latter:

$$\bar{z}_i = Q^{-1}\bar{v}_i.$$

The substitution and (36a) – (38a) yield

$$\left(\bar{z}_{\bar{x}x}\right)_i = \Lambda\bar{z}_i + \tilde{F}_i, \quad i = \overline{1, N-1} \tag{36b}$$

$$\bar{z}_0 = \tilde{\mu}_0, \tag{37b}$$

$$\bar{z}_N = c\bar{z}_s + \tilde{\mu}_1; \tag{38b}$$

where $\tilde{F}_i = Q^{-1}f_i$, $\tilde{\mu}_0 = Q^{-1}\bar{\mu}_0$, $\tilde{\mu}_1 = Q^{-1}\bar{\mu}_1$. Since Λ is a diagonal matrix, (36b) is a system of $N-1$ mutually independent equations. Therefore, the system of equations (36b) – (38b) written down for vector $\bar{z}_i$ can be expressed in terms of $N-1$ separate systems, corresponding every to j-th component of vector $\bar{z}_i$. Thus, for any fixed $j = 1, 2, \ldots, N-1$, we obtain

$$\left(z_{\bar{x}x}\right)_{ij} - \lambda_j z_{ij} = \tilde{F}_{ij}, \quad i = \overline{1, N-1}, \tag{36c}$$

$$z_{0j} = \tilde{\mu}_{0j}, \tag{37c}$$

$$z_{Nj} = cz_{sj} + \tilde{\mu}_{1j}. \tag{38c}$$

To solve of this scalar system, the necessary and sufficient condition of existence and uniqueness is as follows: $cw_s^j \neq 1$. Thus, theorem is proved.

Corollary 1. There are exactly $N-1$ different values of c, $1 < c_1^* < c_2^* < \ldots < c_N^* < \infty$, for which the system (36) – (40) of difference equations has no unique solution.

Indeed, the values λ_j given by (41) all are different, and $0 < w_s^j < 1$, for $1 \le s \le N-1$.

Corollary 2. As $h \to 0$, $\tau \to 0$, then

$$c_j^* \to \infty, \quad i = 1, 2, \ldots, N-1.$$

Indeed, the solution w_s^j of system (42) tends to zero for all values of $j = \overline{1, N-1}$, $s = \overline{1, N-1}$ as $h \to 0$ $\tau \to 0$ (see Lemma 2).

3. Generalization and Remarks. Afore mentioned techniques for parabolic difference problems with a nonlocal condition can be applied far much wider than it is indicated in the article. First of all we note, that this approach can be used to investigate a great lot of various types of nonlocal conditions. It can be applied not only to examine the existence and uniqueness of the solution of difference problems with nonlocal conditions, but also to investigate the stability of difference equations, as well as convergence of the difference solution.

Both extension possibilities are based upon the fact, that solution (20) of the difference system $(w_1)_i$, $(w_2)_i$ contains terms $\tau \to 0, h \to 0$ convergent to zero, and the third term $(w_0)_i^j$ is solution of the classical problem.

The technique applies differential equations with variable coefficients, too. In this case, both functions $(w_1)_i$ and $(w_2)_i$ are the fundamental solutions of homogeneous differential equations with variable coefficients. Building up majorants of the solutions, it can be proved that

$$(w_s)_i \to 0, \quad s = 1, 2, \quad i = \overline{1, N-1}.$$

Next, the technique can be used to analyze nonlinear parabolic equations, if nonlinearities are subject to certain constraints. Similar topics were discussed in [15], [17].

In case of two spatial variables, this approach parabolic equations advances a problem of solving the system of difference equations (36) – (40). A possible way to deal with it is to apply the Tchebyshev iterative method. Such methods converge, if the eigenvalues of the main matrix are all positive even in case the matrix is non-symmetric. It is proved in [24], that this kind of situation is very likely in the difference problems with nonlocal conditions.

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