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UNIFORM EXPONENTIALLY CONVERGENT METHOD FOR THE FIRST ORDER EVOLUTION EQUATION WITH UNBOUNDED OPERATOR COEFFICIENT

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ABSTRACT. A new algorithm is proposed for differential equations of the first order in a Hilbert space with an unbounded operator coefficient. A solution of differential equation is represented as a Dunford-Cauchy integral along a curve in the right half of the complex plane, then transform it into real integral over $(-\infty, \infty)$, and finally apply an exponentially convergent Sinc-quadrature formula to this integral. Algorithm provides possibility to perform computations in parallel.

Let us consider the initial value problem

$$\begin{aligned} &u'(t) + Au(t) = 0, \quad t \in (0, T], \\ &u(0) = u_0, \end{aligned} \tag{1}$$

where $u : R_+ \to H-$ is a vector-valued function, $A-$ self-adjoint, positive densely defined operator in Hilbert space H. $A = A^* \geq \gamma_0 I$, $\gamma_0 > 0$, $\overline{D(A)} = H$. Using the improper Dunford-Cauchy integral we can represent the solution of (1) in the form (see [1] for details)

$$u(t) = -\frac{1}{2\pi i} \int\limits_{\Gamma} e^{-zt} R_A(z) u_0 dz, \tag{2}$$

where $\Gamma-$ is a curve in the plain $\mathbb{C}$, that envelops the spectrum of the operator A. For the approximate solution of the problem (1) different numerical integration formulas are used. So, using the Sinc-approximation [2] and trapezoidal quadrature rule there was built an algorithm in the work [3] for the numerical solution of the problem (1) when A is a strongly $P-$positive densely defined closed operator in a Banach space. The main advantages of this method are the exponential rate of convergence of algorithm and natural possibility to perform computations in parallel. In the work [5] Sinc-approximation and trapezoidal quadrature rule was used for the integral (2) in assumption that the spectrum of the operator A is enveloped by a curve $\Gamma = \{z = \xi + i\eta : \ \xi = \gamma_0 \cosh(a\xi)\}$.

We need to notice the slow rate of convergence of proposed algorithms at the point $t = 0$ as disadvantage of these methods. The rate of convergence is polynomial at the point $t = 0$ in contrast to the case when $t > 0$ where the rate of convergence is exponential. So, it is $O(N^{-1/3})$ for the strongly $P-$positive operator.

Another way for the numerical solution of the problem (1) was proposed in the work [4]. It was built a method with an exponential rate of convergence in Hilbert space by means of expanding of exponent to the Fourier-Chebyshov series. Difficulties in computation of the series terms and absence of parallel implementation are disadvantages of this method. Besides at the point $t = 0$ the rate of convergence is polynomial as for the

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above mentioned methods. We have to remark that all these methods don't require the smoothness of the initial data u_0.

Let us make assumption that $u_0 \in D(A^\sigma)$, $\sigma \in \mathbb{N}$. Then $\exists\, u^* : \; u^* = A^\sigma u_0$, and the integral (2) one can write down in the form

$$u(t) = -\frac{1}{2\pi i}\int_\Gamma e^{-zt} R_A(z) A^{-\sigma} u^* dz =$$

$$= -\frac{1}{2\pi i}\int_\Gamma \frac{e^{-zt}}{z^\sigma} R_A(z) u^* dz. \tag{3}$$

We chose an integration curve in the form

$$\begin{aligned}\Gamma = \{z = \xi(s) + i\eta(s) :\; &\xi = \cosh(bs) + a - 1,\; \eta = -s\cosh(bs),\\ &a < \gamma_0,\, b > 0,\; s \in (-\infty, \infty)\}.\end{aligned} \tag{4}$$

Then (3) we can write as follows

$$u(t) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \exp\{-z(s)t\}\,(z(s) - A)^{-1}\,\frac{b\sinh(bs) - i\,(\cosh(bs) + bs\sinh(bs))}{z^\sigma(s)} u^* ds =$$

$$= -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \exp\{-z(s)t\}\,\frac{\psi(s)}{z^\sigma(s)}\,(z(s) - A)^{-1} u^* ds = \int_{-\infty}^{\infty} F(s,t) ds. \tag{5}$$

Let us estimate the function $F(s,t)$. First of all we shall see the resolvent $R_A(z),\; z \in \Gamma$. Due to the fact that A is self-adjoint , we further get the estimate

$$\|R_A(z)\| \leqslant \frac{1}{d},$$

where $d-$ is a distance from z to the spectrum of the operator A (see [1]). Taking into account that Γ is situated in the right half-plain we have the estimate

$$\|R_A(z)\| \leqslant \frac{1}{|\mathrm{Im}\; z|}, \quad for\, z : \,\mathrm{Re}\; z > \gamma_0.$$

So, we get

$$\|R_A(z)\|_{z\in\Gamma} \leqslant \frac{1}{|s\cosh(bs)|} \leqslant \frac{C_1}{(|s|+1)\cosh(bs)}, \quad s \geqslant \frac{1}{b}\,\mathrm{ar}\cosh(\gamma_0 + 1 - a).$$

Then we can write

$$\|\psi(s) R_A(z)\|_{z\in\Gamma} \leqslant C_1 \frac{\psi(s)}{(|s|+1)\cosh(bs)} \leqslant C_2 b,$$

where

$$C_2 = C_1 \max_s \frac{b\sinh(bs) - i\,(\cosh(bs) + bs\sinh(bs))}{(|s|+1)\cosh(bs)}.$$

For $z^{-\sigma}$ we use the simple estimate

$$\left|\frac{1}{z^\sigma}\right|_{z\in\Gamma} = \frac{1}{\cosh^\sigma(bs)\left|1 + (a-1)\cosh^{-1}(bs) - is\right|^\sigma} \leqslant \frac{e^{-b\sigma|s|}2^\sigma}{|1 - is|^\sigma} = \frac{e^{-b\sigma|s|}2^\sigma}{(1+s^2)^{\sigma/2}}.$$

Here we have used the estimate $(\cosh(bs))^{-1} \leqslant 2e^{-b|s|}$.
Taking into account that

$$|\exp\{-z(s)t\}| \leqslant \exp\{-(\cosh(bs) + a - 1)t\},$$

we have

$$\|F(s,t)\| \leqslant C_2 b \frac{\exp\{-(\cosh(bs)-1+a)t - b\sigma|s|\}}{(1+s^2)^{\sigma/2}} \|u^*\| \tag{6}$$

It follows from the estimate (6) that the integral (5) converges for all t and $\sigma > 0$. Let us consider $u'(t)$

$$u'(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \exp\{-z(s)t\} \frac{\psi(s)}{z^{\sigma-1}(s)} (z(s)-A)^{-1} u^* ds = \int_{-\infty}^{\infty} F_1(s,t) ds \tag{7}$$

Using the estimate of $F(s,t)$ we have

$$\|F_1(s,t)\| \leqslant C_2 b \frac{\exp\{-(\cosh(bs)-1+a)t - b(\sigma-1)|s|\}}{(1+s^2)^{(\sigma-1)/2}} \|u^*\|.$$

These estimates provide the convergence of the integral (5) for $\sigma > 0$, $t \geqslant 0$ and the integral (7) for $\sigma \geqslant 1$, $t > 0$. So we have proved the following result

Theorem 1. *Let for the initial value problem (1) $A-$ self-adjoint, positive densely defined operator in Hilbert space H, $u_0 \in D(A^\sigma)$, $\sigma > 0$. Then the solution of the problem (1) is represented by the integral (5).*

Further we shall construct quadrature formula for the integral (5) as it was done in works [3], [5]. For this purpose let us introduce the family $H^p(D_d)$ of all vector-valued functions, which are analytic in the infinite strip D_d,

$$D_d = \{z \in \mathbb{C} : -\infty < \operatorname{Re} z < \infty, |\operatorname{Im} z| < d\},$$

with the norm

$$\|F\|_{H^p(D_d)} = \begin{cases} \lim\limits_{\varepsilon \to 0} \left(\int\limits_{\partial D_d(\varepsilon)} \|F(z)\|^p |dz| \right)^{1/p}, & 1 \leqslant p < \infty, \\ \lim\limits_{\varepsilon \to 0} \sup\limits_{z \in D_d(\varepsilon)} \|F(z)\|, & p = \infty, \end{cases}$$

where

$$D_d(\varepsilon) = \left\{ z \in \mathbb{C} : |\operatorname{Re} z| < \frac{1}{\varepsilon}, |\operatorname{Im} z| < d(1-\varepsilon) \right\}.$$

We have to find the width d of the strip D_d, where we can analytically extend the integrand $F(s,t)$ because we construct the quadrature rule in the space $H^p(D_d)$. Let us consider a parametric family of curves $\Gamma(\nu)$ which we obtain by substitution of $s + i\nu$ instead of s in Γ. Analyticity of the integrand can be violated if the set $\Gamma(\nu)$ intersects the part of real axis $\eta > \gamma_0$ where the spectrum of A is situated (in this case the resolvent is unbounded) or when the set $\Gamma(\nu)$ includes the point $(0,0)$ (in this case we obtain 0 in denominator). So we have

$$\begin{aligned} \Gamma(\nu) &= \{\cosh(b(s+i\nu)) + a - 1 - i(s+i\nu)\cosh(b(s+i\nu))\} = \\ &= \{\cosh(bs)\cos(b\nu) + i\sinh(bs)\sin(b\nu) + a - 1 + \\ &+ i(\nu - is)(\cosh(bs)\cos(b\nu) + i\sinh(bs)\sin(b\nu))\} = \\ &= \{\cosh(bs)\cos(b\nu)(1+\nu) + a - 1 + s\sinh(bs)\sin(b\nu) + \\ &+ i(\sinh(bs)\sin(b\nu)(1+\nu) - \cosh(bs)\cos(b\nu))\}. \end{aligned}$$

$\Gamma(\nu)$ intersects the real axes ($\operatorname{Im} z = 0$) when $s = 0$. Then the width of the strip D_d is defined by inequality

$$0 < \operatorname{Re}\Gamma(\nu)|_{s=0} < \gamma_0,$$

$$0 < a - 1 + (1+d)\cos(d) < \gamma_0. \tag{8}$$

Let

$$S(k,h)(x) = \frac{\sin\left[\pi(x-kh)/h\right]}{\pi(x-kh)/h}, \qquad k \in \mathbb{Z},\ h > 0,\ x \in \mathbb{R}$$

be the k–th Sinc function [2] with step size h, evaluated at x. Given $f \in H^p(D_d)$, $h > 0$, and $N \in \mathbb{N}$, we use the notations

$$I(f) = \int_{\mathbb{R}} f(\xi)d\xi,$$

$$T(f,h) = h\sum_{k=-\infty}^{\infty} f(kh), \qquad T_N(f,h) = h\sum_{k=-N}^{N} f(kh),$$

$$\eta(f,h) = I(f) - T(f,h), \qquad \eta_N(f,h) = I(f) - T_N(f,h).$$

Lemma 1. *For any operator valued function $f \in H^1(D_d)$, that satisfies on $\mathbb{R}$ the condition*

$$\|f(x)\| \leqslant c\frac{e^{-\alpha|x|}}{(1+x^2)^\beta}, \quad c = const, \quad \alpha, \beta > 0, \tag{9}$$

the following estimate is true

$$\|\eta_N(f,h)\| \leqslant \frac{2c}{\alpha}\left[\frac{e^{-\pi d/h}}{\sinh(\pi d/h)} + \frac{e^{-\alpha hN}}{\left(1+(hN)^2\right)^\beta}\right]. \tag{10}$$

Proof. As it has been shown in [3], [5] $\forall f \in H^1(D_d)$ the following estimate is true

$$\|\eta(f,h)\| \leqslant \frac{e^{-\pi d/h}}{2\sinh(\pi d/h)}\|f\|_{H^1(D_d)}. \tag{11}$$

Taking into account the condition (9) we obtain

$$\|f\|_{H^1(D_d)} = 2\int_{-\infty}^{\infty}\|f(x)\|\,dx \leqslant 2c\int_{-\infty}^{\infty}\frac{e^{-\alpha|x|}}{(1+x^2)^\beta}dx \leqslant 2c\int_{-\infty}^{\infty}e^{-\alpha|x|}dx = \frac{4c}{\alpha}.$$

Then from (11)

$$\|\eta(f,h)\| \leqslant \frac{2ce^{-\pi d/h}}{\alpha\sinh(\pi d/h)}. \tag{12}$$

For $\eta_N(f,h)$ we have

$$\|\eta_N(f,h)\| \leqslant \|\eta(f,h)\| + h\sum_{|k|>N}\|f(kh)\|. \tag{13}$$

For the last sum assumption (9) leads to

$$h\sum_{|k|>N}\|f(kh)\| \leqslant hc\sum_{|k|>N}\frac{e^{-\alpha|kh|}}{(1+(kh)^2)^\beta} = 2hc\sum_{k=N+1}^{\infty}\frac{e^{-\alpha|kh|}}{(1+(kh)^2)^\beta} \leqslant$$

$$\leqslant 2hc\int_N^{\infty}\frac{e^{-\alpha hx}}{(1+(xh)^2)^\beta}dx \leqslant \frac{2hc}{(1+(Nh)^2)^\beta}\int_N^{\infty}e^{-\alpha hx}dx =$$

$$= \frac{2hc}{(1+(hN)^2)^\beta}\frac{1}{\alpha h}e^{-\alpha hN} = \frac{2c}{\alpha\left(1+(hN)^2\right)^\beta}e^{-\alpha hN}. \tag{14}$$

Combining (12), (13), (14) we obtain the estimate (10) which completes the proof.

$F(s,t)$ can be analytically extended into the strip D_d as it has been shown above. So we can apply quadrature rule T_N for approximating of the integral (5). Taking into account the estimate (6) we can use the lemma 1 substituting c for $bC_2e^{-at}2^{\sigma}$, α for $b\sigma$, β for $\sigma/2$. Equalizing the exponent by setting $h = \frac{1}{\sqrt{b\sigma N}}$, we obtain

$$\|\eta_N(F,h)\| \leqslant \frac{C_2 e^{-at}2^{\sigma+1}}{\sigma}\left[\frac{2\exp\left(-2\pi d\sqrt{b\sigma N}\right)}{1-\exp\left(-2\pi d\sqrt{b\sigma N}\right)} + \frac{\exp\left(-\sqrt{b\sigma N}\right)}{\left(1+\frac{N}{b\sigma}\right)^{\sigma/2}}\right]. \tag{15}$$

Algorithm 1.

1. Given γ_0, chose $a < \gamma_0$, $b > 0$, $\sigma \geq 1$, N.
2. For $k = \overline{-N,N}$ compute $h = \frac{1}{\sqrt{b\sigma N}}$,
   $z_k = \cosh(bkh) + a - 1 - i(kh)\cosh(bkh)$,
   $\alpha_k = \frac{\psi(kh)}{z_k^{\sigma}}$.
3. Solve the equations $(z_k - A)\,\hat{u} = u^*$, $k = \overline{-N,N}$, where $u^* = A^{\sigma}u_0$.
4. Find the approximation u_N for the solution of (1) in the form

$$u_N(t) = h\sum_{k=-N}^{N} \alpha_k \exp\{-z_k t\}\,\hat{u}(z_k).$$

Remark. The above algorithm possesses two sequential levels of parallelism: first, we can compute all $\hat{u}(z_k)$ at Step 3 in parallel and, second, each operator exponent at different time values $(t_1, t_2, \ldots, t_M)$.

Now we can formulate the main result of the work.

Theorem 2. *Let the assumption of the theorem 1 is valid. The approximate solution is computed using the algorithm 1. Then the estimate (15) is true.*

Example. Let us consider the following problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial t^2}, \quad u(x,0) = \sin(\pi x),$$
$$u(0,t) = u(1,t) = 0,$$

with the exact solution $u(x,t) = e^{-\pi^2 t}\sin(\pi x)$.

Operator

$$A = -\frac{d^2}{dx^2},$$

defined on

$$D(A) = H^2(0,1) \cap H_0^1(0,1),$$

is self-adjoint, positive definite and $\gamma_0 = \pi^2$. The numerical solution was computed accordance with the Sinc-algorithm ($a = 1$, $\sigma = 1$, $b = 1$) where the step 3 was performed using explicit formulas. The error $\varepsilon_N = u(x,t) - u_N(x,t)$ for $x = 0.5$ as a function of N is given by Table 1 and Table 2.

The estimate (15) shows that $\varepsilon_N \approx c\exp(-\sqrt{N})$. If we set $\delta_N = \frac{\varepsilon_{N/2}}{\varepsilon_N}$ then $Ln(\delta_N) \approx \sqrt{N}\left(1 - \frac{1}{\sqrt{2}}\right)$. In Table 3 comparison of $Ln(\delta_N)$ obtained from the Table 1, 2 at $t = 0$ and $\sqrt{N}\left(1 - \frac{1}{\sqrt{2}}\right)$ is shown in Table 3.

Table 3 shows that ε_N is in a well agreement with the estimate (15).

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TABLE 1

t	$\varepsilon 16$	$\varepsilon 32$	$\varepsilon 64$	$\varepsilon 128$
0	0.0235251	0.0033945	0.0002416	$6.4074683 * 10^{-6}$
0.2	0.0000513	0.0000103	$2.7622945 * 10^{-7}$	$6.2785482 * 10^{-10}$
0.4	0.0000862	$9.6637569 * 10^{-6}$	$1.5685791 * 10^{-7}$	$3.1343728 * 10^{-9}$
0.6	0.0000714	$6.7490393 * 10^{-6}$	$9.0428640 * 10^{-8}$	$4.9384672 * 10^{-9}$
0.8	0.0001015	$4.7352868 * 10^{-6}$	$2.2676695 * 10^{-7}$	$3.8738405 * 10^{-9}$
1.0	0.0000264	$4.8744014 * 10^{-6}$	$2.1572527 * 10^{-7}$	$2.6352864 * 10^{-9}$

TABLE 2

t	$\varepsilon 256$	$\varepsilon 512$	$\varepsilon 1024$
0	$4.2575771 * 10^{-8}$	$4.0334624 * 10^{-11}$	$2.3314683 * 10^{-15}$
0.2	$1.5860607 * 10^{-10}$	$9.5479180 * 10^{-14}$	$8.3266726 * 10^{-17}$
0.4	$3.3733536 * 10^{-12}$	$4.4002995 * 10^{-14}$	$2.7755575 * 10^{-17}$
0.6	$2.6156579 * 10^{-11}$	$2.1379599 * 10^{-14}$	$4.8138576 * 10^{-17}$
0.8	$1.2210844 * 10^{-12}$	$9.5989839 * 10^{-16}$	$9.9746599 * 10^{-18}$
1.0	$6.1291827 * 10^{-12}$	$2.7885950 * 10^{-15}$	$4.3977950 * 10^{-18}$

TABLE 3

N	$Ln(\delta_N)$	$\sqrt{N}\left(1-\frac{1}{\sqrt{2}}\right)$
32	1.9358	1.65685
64	2.6422	2.34315
128	3.6302	3.31371
256	5.0139	4.68629
512	6.9618	6.62742
1024	9.7584	9.37258

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