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NONSYMMETRIC GALERKIN FINITE ELEMENT METHOD WITH DYNAMIC MESH REFINEMENT FOR SINGULAR NONLINEAR PROBLEMS

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ABSTRACT. The paper is devoted to the numerical study of the model of heat structures in the radially symmetric case. The existence and the structural stability of its blow-up self-similar solutions beyond some critical exponents are investigated. Because the critical exponents appear in space dimensions $N \geq 3$, a special attention is given to the singularity at the origin. In order to deal successfully with this singularity we use the nonsymmetric Galerkin Finite element method. To proceed successfully with the single point blow-up, a special adaptive mesh refinement, consistent with the self similar law is made.

1. THE BLOWUP PROBLEM

The paper is devoted to the numerical study of the model of heat structures [17]

$$u_t = div\,(u^\sigma grad\, u) + u^\beta, \quad x \in I\!R^N, \quad t > 0, \quad \sigma > 0, \quad \beta > 1,$$
$$u(0,x) = u_0(x) \geq 0, \quad u_0 \not\equiv 0, \quad \sup_{x \in I\!R^N} u_0 < \infty, \tag{1}$$

in the radially symmetric case. This model was introduced about 30 years ago in [18] and it was increasingly investigated in the school of A.A. Samarskii. In spite of the fact, that the book [17] and many works were devoted to analyze the complexity and the unusual properties of the processes, described by this model, many problems remain open by now. The presence of two medium parameters σ and β, different space geometries and dimensions N, pose challenging questions and make the problem (1) interesting from mathematical and computational view points. It is worth mentioning, that in many cases the success was achieved by the combination of theoretical investigations and computational experiments.

In this work we consider the radially symmetric variant of problem (1):

$$u_t = r^{1-N}(r^{N-1}u^\sigma u_r)_r + u^\beta, \quad 0 < r < \infty, \ t > 0, \tag{2}$$

$$u(0,r) = u_0(r) \geq 0, \quad r \geq 0, \quad u_0 \not\equiv 0, \quad \mathop{sup}_{r \geq 0} u_0 < \infty. \tag{3}$$

It is well known [17], that for $\sigma > 0$, $\beta > 1$ equation (2) admits blowup self-similar solutions (s.-s.s.) of the form:

$$u_s(t,r) = (1 - t/T_0)^{-1/(\beta-1)}\theta(\xi), \quad \xi = r/(1 - t/T_0)^{m/(\beta-1)}, \quad m = (\beta - \sigma - 1)/2, \tag{4}$$

where $T_0 > 0$ is the blowup time.

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The function $\theta(\xi) \geq 0$ satisfies the nonlinear ordinary differential equation (self-similar equation)

$$-\frac{1}{\xi^{N-1}}\left(\xi^{N-1}\theta^\sigma\theta'\right)' + \frac{m}{(\beta-1)T_0}\xi\theta' + \frac{1}{(\beta-1)T_0}\theta - \theta^\beta = 0, \quad 0 < \xi < \infty \qquad (5)$$

and determines the space-time structure of the self-similar solution $u_s(t,r)$.

The invariant solutions, and in particular, the self-similar solutions, form an important class among all the possible solutions of partial differential equations (PDE). In many cases they are attractors of the solutions, which are determined from wide classes of initial data. Even more, the s.-s.s. describe the asymptotic behavior of these classes of solutions and the possible singularity formation, when the initial data are "forgotten". In the presence of arbitrary boundary conditions the invariant solutions determine the behavior of the noninvariant ones for times, which are away of the initial time, but smaller than the time, when the boundary conditions begin to act, so they are intermediate asymptotics [2].

Much efforts have been done to investigate the existence and the number of the different solutions of equation (5) under the conditions:

$$\theta'(0) = 0, \quad \theta(\xi) \to \infty, \quad \xi \to \infty. \qquad (6)$$

It occurred that the problem (5), (6) is very rich of different kind of solutions, depending on the parameters σ, β and N, determining thus different types of blow-up solutions [17]. Mentioning the cases $1 < \beta < \sigma+1$ (total blow-up) and $\beta = \sigma+1$ (regional blow-up), we will stress our attention to the case $\beta > \sigma+1$ (single point blow-up), where the following critical exponents have appeared by now (see [17], [11], [14] and the references therein):

$\beta_f = \sigma + 1 + 2/N$ (Fujita's exponent);

$\beta_{st} = (\sigma+1)N/(N-2),\ N \geq 3;\ \beta_{st} = \infty, \quad N = 1, 2;$

$\beta_s = (\sigma+1)(N+2)/(N-2),\ N \geq 3$ (Sobolev's exponent); $\beta_s = \infty,\ N = 1, 2;$

$\beta_u = (\sigma+1)(1 + 4/(N - 4 - 2\sqrt{N-1})),\ N \geq 11;\ \beta_u = \infty,\ N < 11;$

$$\beta_p = 1 + \frac{3(\sigma+1) + (\sigma^2(N-10)^2 + 2\sigma(5\sigma+1)(N-10) + 9(\sigma+1)^2)^{1/2}}{N-10}, \quad N \geq 11.$$

For $\beta \leq \beta_f$ and arbitrary initial data $u_0(r) \not\equiv 0$, the solution of problem (2), (3) blows up in finite time. For $\beta > \beta_f$ the problem (2), (3) has global and blowup solutions depending on the initial data $u_0(r)$.

For $\beta > \beta_{st}$ there exists singular stationary solution of equation (2): $U_{st}(r) = c_s r^{-1/m}$, and singular solution of equation (5):

$$\theta_s(\xi) = c_s\xi^{-1/m}, \quad c_s = [(N - 2 - (\sigma+1)/m)/m]^{1/2m}, \quad m = (\beta - \sigma - 1)/2. \qquad (7)$$

The critical Sobolev's exponent β_s determines the existence ($\beta \geq \beta_s$) and nonexistence ($\beta < \beta_s$) of strictly positive nonsingular stationary solutions of equations (1) and (2). It is connected with the possibility of nontrivial continuation of the blowup solutions for $t > T_0$ as well [11].

The exponents β_u and β_p are connected with some properties of the singular stationary solution $U_{st}(r)$ and with the existence of special kind of blowup solutions to problem (2),(3), which have a nontrivial continuation for $t > T_0$ [11].

As we know, the concept of continuation of the blowup solutions is posed first in the work [1] for a semilinear heat equation. If the continuation of the solution is trivial, that is, $u(t,r) \equiv \infty$ for $t > T_0$, the blowup is called complete; if $u(t,r) \not\equiv \infty$ for $t > T_0$, the blowup is called incomplete.

Let us return back to the self-similar problem (5),(6). It has been proved in the book [17], Chapter 4, that for $\sigma + 1 < \beta < \infty$, $N = 1, 2$, and $\sigma + 1 < \beta < \beta_s$, $N \geq 3$, the problem (5), (6) has strictly monotone solution $\theta(\xi) > 0$. Ten years later Galaktionov and Vazquez [11] extended this result to the range $\beta_s \le \beta < \beta_u$: there exists a positive solution of (5), (6), which is strictly monotone with asymptotics

$$\theta(\xi) = c_0 \xi^{-1/m}(1 + o(1)), \quad \xi \to \infty, \quad c_0 > c_s, \tag{8}$$

c_s being the constant in the singular solution $\theta_s(\xi)$ (7), and for which the s.-s.s. (4) blows-up completely. A hypothesis is made there ([11], p.56), that the corresponding to $\theta(\xi)$ self-similar solution $u_s(t, r)$ is asymptotically (structurally) stable and describes the behavior as $t \to {T_0}^{-}(u_0)$ of a wide class of solutions to (2) with initial data $u(0, r) = u_0(r) \ge 0$, that are eventually monotone and satisfy $u_t > 0$ in $\{u \gg 1\}$. This conjecture as well as the existence of solutions to (5),(6),(8) for $\beta \ge \beta_u$ are not theoretically proved by now. Some numerical results were briefly reported in [4].

The aim of this work is:

– to present numerical techniques, appropriate for the singularity at the origin of both - the self-similar and the parabolic equations, as well as for the singularity in time of the parabolic problem;

– to investigate numerically the accuracy of the method for solving the self-similar problem (5),(6),(8) in the case of strong singularity at the origin (big values of the space dimension N);

– to show the effectiveness and the reliability of this techniques when applied to investigate the structural stability of the blowup s.-s.s. $u_s(t, r)$, corresponding to the monotone solutions $\theta(\xi) > 0$ of (5),(6),(8) for $\beta_s \le \beta < \beta_u$ (which are proved to exist), as well as for the ranges $\beta_u \le \beta \le \beta_p$ and $\beta > \beta_p$.

2. Numerical methods

In order to deal successfully with the strong singularity at the origin (we are interested in big values of N) of both the self-similar equation (5) and the parabolic equation (2), we use the nonsymmetric Galerkin Finite Element Method (GFEM). It is proposed and theoretically investigated for linear singular at the origin problems in [9] and for semilinear ones – in [10].

The idea of the nonsymmetric method is to apply GFEM on a special nonsymmetric form of the original self-adjoint problem. For example, if the GFEM with continuous polynomial basis functions of degree $k - 1$ is applied on the linear radially symmetric problem, written in the form

$$-(x^{N-1}u')' + x^{N-1}qu = x^{N-1}f, \quad 0 < x < 1, \quad u'(0) = u(1) = 0,$$

then the approximate solution u_h satisfies the estimates:

$$||x^{(N-1)/2}(u_h - u)||_{L_2} \le Ch^k ||x^{(N-1)/2}u^{(k)}||_{L_2},$$

found in [19] by using a straightforward variational technique, and

$$||u_h - u||_{L_\infty} \le C \left(\ln \frac{1}{h}\right)^{\bar{k}} h^k ||u^{(k)}||_{L_\infty},$$

$\bar{k} = 1$ when $k = 2$ and $\bar{k} = 0$ when $k > 2$, found in [13] by a more refined analysis. It was mentioned in [9], that even in the case $k > 2$, when the theoretical order of convergence is optimal, the numerical experiments show a marked loss of accuracy near $x = 0$. For the GFEM, applied on the same equation, but written in the nonsymmetric form

$$-(xu')' - (N - 2)u' + xqu = xf, \quad 0 < x < 1, \quad u'(0) = u(1) = 0$$

for $N > 2$, an estimate of optimal order has been found for $k = 2$ as well in the paper [9]:

$$||u_h - u||_{L_\infty} \leq Ch^k ||u^{(k)}||_{L_\infty}, \quad k \geq 2. \tag{9}$$

The numerical tests, reported there, show good accuracy of the approximate solution just to the origin.

Realized and numerically tested on the nonlinear problem (5)-(6) for $N = 3$ in [5], the nonsymmetric GFEM occurred to give the optimal, second order of convergence just to the origin when using linear finite elements. Here we investigate the accuracy of the nonsymmetric GFEM for $N \gg 3$ and generalize it for the nonlinear parabolic equation (2).

2.1 Nonsymmetric GFEM for the self-similar problem.. For convenience and without loss of generality we first set $T_0 = 1/(\beta - 1)$ in equation (5) and then we write it in the following nonsymmetric form:

$$L(\theta) \equiv -(\xi^\gamma \theta^\sigma \theta')' - \gamma(N-2)\theta^\sigma \theta' + m\xi^{1+\gamma}\theta' + \xi^\gamma \theta(1 - \theta^{\beta-1}) = 0, \tag{10}$$

where $\gamma = 0$ for $N = 1$, $\gamma = 1$ for $N > 1$. From the known asymptotics (8) we derive a boundary condition of third kind, so in the case $\beta > \sigma + 1$ we solve the problem (10),(11):

$$\theta'(0) = 0, \quad \theta'(l) + (1/m)\theta(l)/l = 0, \quad l \gg 1. \tag{11}$$

In computations we choose the length l of the interval so, that the asymptotics (8) is fulfilled enough well: further increasing of l does not influence the numerical solution (in an appropriate range of accuracy). For completeness we give the boundary conditions for the case $1 < \beta \leq \sigma + 1$ as well:

$$\theta'(0) = 0, \quad \theta(l) = 0 \tag{12}$$

where we chose l so as to avoid the influence of the boundary condition on the solution (it is possible because of the finite support of the solution in this case). The further main steps of the method are given below.

According to the Continuous Analogue of the Newton's Method (CANM) [12] the stationary problem

$$L(\theta) = 0 \tag{13}$$

is reduced to the evolution one

$$L'(\theta)\frac{\partial \theta}{\partial t} = -L(\theta), \quad \theta(\xi, 0) = \theta_0(\xi), \tag{14}$$

by introducing a continuous parameter t, $0 < t < \infty$, on which the unknown solution depends: $\theta = \theta(\xi, t)$. By setting $v = \partial\theta/\partial t$ and applying the Euler's method to the Cauchy problem (14), one comes to the iteration scheme

$$L'(\theta_k)v_k = -L(\theta_k), \tag{15}$$

$$\theta_{k+1} = \theta_k + \tau_k v_k, \quad 0 < \tau_k \leq 1, \quad k = 0, 1, \ldots,$$

$$\theta_k = \theta_k(\xi) = \theta(\xi, t_k), \quad v_k = v_k(\xi) = v(\xi, t_k), \tag{16}$$

$$\theta_0(\xi) \text{ — initial approximation.}$$

For the nonlinear operator $L(\theta)$ from (10) the equation (15) takes the form:

$$
\begin{gathered}
-\left(\xi^{\gamma}\theta_k^{\sigma}v_k'\right)' - \gamma(N-2)\sigma\theta_k^{\sigma-1}\theta_k' v_k - \left(\xi^{\gamma}\sigma\theta_k^{\sigma-1}v_k\right)' - \\
-\gamma(N-2)\theta_k^{\sigma}v_k' + m\xi^{1+\gamma}v_k' + \xi^{\gamma}(1-\beta\theta_k^{\beta-1})v_k = \\
= -[-\left(\xi^{\gamma}\theta_k^{\sigma}\theta_k'\right)' - \gamma(N-2)\theta_k^{\sigma}\theta_k' + m\xi^{1+\gamma}\theta_k' + \xi^{\gamma}(1-\theta_k^{\beta-1})\theta_k].
\end{gathered} \tag{17}
$$

Let us mention, that for $N = 1, 2$ it is the same as for the symmetric equation (5). If $\theta_0(\xi)$ satisfies the boundary conditions (11) or (12), the iteration corrections $v_k(\xi)$ must satisfy:

$$
v_k'(0) = 0, \quad v_k'(l) + (1/m)v_k(l)/l = 0 \quad for \quad \beta > \sigma + 1, \tag{18}
$$

$$
v_k'(0) = 0, \quad v_k(l) = 0, \quad for \quad 1 < \beta \le \sigma + 1. \tag{19}
$$

The finite element discretization is made on the problems (17),(18) and (17),(19) in weak form:

Find a function $v_k(\xi) \in H^1(0,l)$, which satisfies the identity

$$
(L'(\theta_k)v_k, w) = -(L(\theta_k), w), \quad \forall w \in H^1_\alpha(0,l) \tag{20}
$$

and the boundary conditions (18) or (19) respectively. Here $(.,.)$ is the standard L^2 inner product, $\theta_k(\xi)$ is a given function, which satisfies the boundary conditions (11) or (12), $\theta_k \in D = \{\theta_k(\xi) : \theta_k^{\sigma+1}, d\theta_k^{\sigma+1}/d\xi \in L^2(0,l)\}$,

$$
H^1_\alpha(0,l) = \{w(\xi) : w, \xi^{\gamma/2}w' \in L^2(0,l), (1-\alpha)w(l) = 0\}.
$$

The value $\alpha = 1$ corresponds to the condition (18), and $\alpha = 0$ – to the condition (19).

The identity (20) is

$$
\begin{gathered}
\int_0^l \{\xi^{\gamma}a(\theta_k)v_k'w' + \left[\xi^{\gamma}q_1(\theta_k) - m\xi^{1+\gamma} + \gamma(N-2)a(\theta_k)\right]v_k w' + \\
+ \left[\xi^{\gamma}q_2(\theta_k) - (1+\gamma)m\xi^{\gamma}\right]v_k w\}\, d\xi + \gamma(N-2)a(\theta_k)v_k w|_{\xi=0} + \\
+\alpha\left[m\xi^{1+\gamma} + \left(((1+\sigma)/m)\xi^{\gamma-1} - \gamma(N-2)\right)a(\theta_k)\right]v_k w|_{\xi=l} = \\
= -\int_0^l \{\xi^{\gamma}a(\theta_k)\theta_k'w' + [-m\xi^{1+\gamma} + \gamma(N-2)a(\theta_k)]\theta_k w' + \\
+[\xi^{\gamma}q_3(\theta_k) - (1+\gamma)m\xi^{\gamma} + \gamma(N-2)q_1(\theta_k)]\theta_k w\}\, d\xi - \\
-\gamma(N-2)a(\theta_k)\theta_k w|_{\xi=0} - \alpha[m\xi^{1+\gamma} - (\gamma(N-2) - (1/m)\xi^{\gamma-1})a(\theta_k)]\theta_k w|_{\xi=l},
\end{gathered} \tag{21}
$$

where the functions a, q_1, q_2, q_3 are defined as follows:

$$
\begin{gathered}
a(\theta_k) = \theta_k^{\sigma}, \quad q_1(\theta_k) = \sigma\theta_k^{\sigma-1}\theta_k', \\
q_2(\theta_k) = 1 - \beta\theta_k^{\beta-1}, \quad q_3(\theta_k) = 1 - \theta_k^{\beta-1}.
\end{gathered} \tag{22}
$$

For discretization of (21), (22) linear finite elements on quasiuniform partitions $\{0 = \xi_1 < \xi_2 < \cdots < \xi_m = l, \ \ \xi_{i+1} - \xi_i \leq h\}$ of the interval $[0, l]$ are used. A system of linear equations is thus obtained for the vector $\bar{V}_k$ of nodal values of the iteration corrections $v_k = v(\xi, t_k)$:

$$A(\theta_k)\bar{V}_k = -B(\theta_k)\bar{\Theta}_k. \tag{23}$$

The matrices A and B are nonsymmetric band matrices, $\bar{\Theta}_k$ is the vector of nodal values of the function $\theta_k = \theta(\xi, t_k)$. To solve the system (23), LU decomposition of the matrix $A(\theta_k)$ is made at every iteration step.

The iteration parameter τ_k in (16) is determined by the extrapolation formula [16]:

$$\tau_k = \begin{cases} \min\left(1, \tau_{k-1}\dfrac{\delta_{k-1}}{\delta_k}\right), & \delta_k < \delta_{k-1}, \\ \max\left(\tau_0, \tau_{k-1}\dfrac{\delta_{k-1}}{\delta_k}\right), & \delta_k \geq \delta_{k-1}. \end{cases} \tag{24}$$

Here δ_k is some norm of the residual $L(\theta_k)$. In the computations the uniform norm of the discrete residual is used:

$$\delta_k = \max_{\eta \in \bar{\omega}_h} |B(\theta_k)\bar{\Theta}_k|. \tag{25}$$

The value of τ_0 was taken to be between 0.01 and 0.1. When δ_k decreases, the algorithm (24) ensures the convergence of τ_k to 1 ($\tau_k \to 1^-$), and the rate of convergence of the iteration process (15),(16) becomes quadratic. The stop criterion is $\delta_k < \delta$ for some small δ. When it is fulfilled we take $\theta_k = \theta(\xi, t_k)$ as approximate solution of problem (10),(11) (or (10),(12)) and set $\theta_h(\xi) = \theta(\xi, t_k)$.

The method of choosing initial approximations for the iteration process (15),(16) will not be presented here. It is the same as for the symmetric problem, described in [6]. For the case $\beta > \sigma + 1$ it relies on the linearization around the homogeneous solution of equation (5) and the sewing of its solution with the known asymptotics (8). Because we are interested here in the simplest monotone solutions of problem (5), (6), this choice is not so important, as it is for the multiple nonmonotone solutions. Of course, the rate of convergence of the iteration process depends essentially on this choice. In the examples below the method from [6] is used.

The numerous experiments made with the nonsymmetric GFEM and linear finite elements show its fast convergence (usually $15 - 60$ iterations are enough for $\delta = 10^{-7}$) and optimal, second order of accuracy at the nodal points just to the origin. In the table bellow the values of the approximate solution $\theta_h(\xi)$ of problem (10),(11) with parameters $\sigma = 2$, $\beta = 6.5$, $N = 10$ are given at some common points of 4 embedded meshes with steps $0.2, 0.1, 0.05$ and 0.025. The approximate order of accuracy α is computed by the method of Runge:

$$\alpha(\xi) = \ln \frac{\theta_h(\xi) - \theta_{h/2}(\xi)}{\theta_{h/2}(\xi) - \theta_{h/4}(\xi)} / \ln 2,$$

on the meshes with steps (0.2, 0.1, 0.05) and (0.1, 0.05, 0.025). It is shown in the columns "α" of the Table.

Table: Order of accuracy of $\theta_h(\xi)$ for $\sigma = 2, \ \beta = 6.5, \ N = 10$

h	$\xi = 0.0$	α	$\xi = 0.8$	α	$\xi = 1.6$	α	$\xi = 2.4$	α	$\xi = 3.2$	α
0.20	1.4482300		1.3243517		1.1036388		0.9257791		0.7997794	
0.10	1.4480463	2.04	1.3238326	2.00	1.1038749	2.02	0.9262001	1.98	0.8001835	1.98
0.05	1.4480015	1.98	1.3237025	1.99	1.1039333	2.01	0.9263064	2.00	0.8002862	2.00
0.025	1.4479902		1.3236697		1.1039478		0.9263331		0.8003120	
h	$\xi = 4.0$	α	$\xi = 6.0$	α	$\xi = 10.$	α	$\xi = 15.$	α	$\xi = 20.$	α
0.20	0.7090248		0.5654267		0.4230292		0.3356525		0.2847938	
0.10	0.7093822	1.97	0.5656965	1.97	0.4232200	1.97	0.3358005	1.96	0.2849181	1.96
0.05	0.7094733	1.99	0.5657655	1.99	0.4232688	1.99	0.3358384	1.99	0.2849500	1.99
0.025	0.7094961		0.5657828		0.4232811		0.3358479		0.2849580	

For the same problem, with the same initial approximations, the iteration process, based on the symmetric GFEM [6], does not converge at all. It is convergent for some smaller values of N (the range in N depends on the other parameters as well). For example, it gives an order of accuracy about 1.7 at the origin for $N = 3$, $\sigma = 2$, $\beta = 3$ [5].

2.2 Nonsymmetric GFEM for the parabolic problem.. We first use the Kirchhoff transformation of the nonlinear heat-conductivity coefficient:

$$G(u) = \int_0^u s^\sigma \, ds = u^{\sigma+1}/(\sigma+1).$$

This is essential for the further interpolation of the nonlinear coefficients and for the optimization of the computational process. As we know, the Kirchhoff transformation is first used for computational purposes in [3].

Then we write equation (2) in the following nonsymmetric form

$$r^\gamma u_t = (r^\gamma G_r)_r + \gamma(N-2)G_r + r^\gamma u^\beta, \quad \gamma = 0, \quad N = 1; \qquad \gamma = 1, \quad N > 1. \tag{26}$$

We solve equation (26) under the initial condition

$$u(0,r) = u_0(r) \geq 0, \quad 0 \leq r \leq R \tag{27}$$

and boundary conditions:

$$u_r(t,0) = 0, \; t > 0, \quad u(t,R) = 0 \;\; if \;\; messuppu_0(r) < \infty, \tag{28}$$

$$u_r(t,0) = 0, \; t > 0, \quad G_r(t,R) = 0 \;\; if \;\; u_0(r) = \theta(r). \tag{29}$$

The discretization is made on the problem (26)-(28) and (26),(27),(29) in weak form:

Find a function $u(t,r) \in D$, $D = \{u : r^{\gamma/2}u, r^{\gamma/2}\partial u^{\sigma+1}/\partial r \in L_2\}$, which satisfies the integral identity

$$(r^\gamma u_t, v) = A(t;u,v), \quad \forall v \in H^1_\alpha(0,R), \quad 0 < t < T_0, \tag{30}$$

the initial (27) and boundary conditions (28) or (29).

Here

$$(u,v) = \int_0^R u(r)v(r)dr,$$

$$A(t;u,v) = \int_0^R \left[r^\gamma \frac{\partial G(u)}{\partial r}\frac{\partial v}{\partial r} + \gamma(N-2)\frac{\partial G(u)}{\partial r} v + r^\gamma u^\beta v \right] dr,$$

$$H^1_\alpha(0,R) = \{v : r^{\gamma/2}v, r^{\gamma/2}v' \in L_2(0,R), \; (1-\alpha)v(R) = 0\}.$$

The value $\alpha = 0$ corresponds to the condition (28), and $\alpha = 1$ – to the condition (29).

The lumped mass finite element method [20] with interpolation of the nonlinear coefficients is used for discretization of (30). Let $\{0 = r_1 < r_2 < \cdots < r_m = R,\ \ r_{i+1} - r_i \leq h\}$ be a partition of the interval $[0, R]$ into elements $e_i = [r_i, r_{i+1}],\ \ i = 1, 2, , \ldots , m - 1$.

Let S_h be the space of the continuous functions on $[0, R]$, which are polynomials of degree $k - 1$ on e_i :

$$S_h = \{w(r) \in C([0, R]), w|_{e_i} \in P_{k-1};\ \ (1 - \alpha)w(R) = 0\}.$$

Let $\{\varphi_i\}_{i=1}^n$ be the standard Lagrangian nodal basis of S_h and $u_h(t, r)$ be the approximate solution in S_h for every fixed value of t. The semidiscrete problem is:

Find $u_h : [0, \tilde{T}_0] \to S_h$ such that

$$(r^\gamma u_{h,t}, w) = A(t; u_h, w)\ \ \forall w \in S_h, \tag{31}$$

$$u_h(0) = u_{0,h}. \tag{32}$$

The approximation $\tilde{T}_0$ of the blow-up time T_0 is found in the computations.

We use the finite element interpolants of the solution $u(t, r)$, of the nonlinear functions $G(u)$ and $q(u) = u^\beta$ and of the initial data $u_0(r)$:

$$u_h(t, r) = \sum_{i=1}^n u_i(t)\varphi_i(r), \quad u_0(r) = \sum_{i=1}^n u_0(r_i)\varphi_i(r), \tag{33}$$

$$G(u) \sim G_I = \sum_{i=1}^n G(u_i)\varphi_i(r), \quad q(u) \sim q_I = \sum_{i=1}^n q(u_i)\varphi_i(r). \tag{34}$$

Substituting (33) and (34) in (31), (32), we find a system of ordinary differential equations (ODE):

$$\dot{U} = \tilde{M}^{-1}(-KG(U) + \gamma(N - 2)BU) + q(U), \quad U(0) = U_0 \tag{35}$$

with respect to the vector $U(t)$ of the nodal values of the solution $u(t, r)$ at time t. Here

$$\tilde{M} = diag\{\tilde{m}_{ii}\}, \qquad \tilde{m}_{ii} = \sum_{j=1}^n m_{ij}, \qquad m_{ij} = \int_0^R r^\gamma \varphi_i \varphi_j dr,$$

$$K = \{k_{ij}\}, \quad k_{ij} = \int_0^R r^\gamma \varphi_i^{'} \varphi_j^{'} dr, \quad B = \{b_{ij}\}, \quad b_{ij} = \int_0^R r^\gamma \varphi_i^{'} \varphi_j dr,\ i, j = 1, 2, \ldots , n.$$

Let us mention, that because of the Kirchhoff transformation and the interpolation of the nonlinear coefficients only the two vectors $G(U)$ and $q(U)$ contain the nonlinearity of the problem, while the matrices K and B do not depend on the unknown solution.

To solve the system (35), a modification of the explicit Runge-Kutta method [15], which has second order of accuracy and extended region of stability is used. For convenience we write it down for a system of ODE in the form

$$y\prime = f(y), \qquad y(0) = y_0.$$

The value of the solution at time level $j + 1$ is then given by the formulas

$$y^{j+1} = y^j + p_1 k_1^j + p_2 k_2^j + p_3 k_3^j, \quad y^0 = y_0,$$

$$p_1 = 1/4, \quad p_2 = 15/32, \quad p_3 = 9/32,$$

$$k_1^j = \tau_j f(y^j), \quad k_2^j = \tau_j f(y^j + 2k_1^j/3), \quad k_3^j = \tau_j f(y^j + k_1^j/3 + k_2^j/3).$$

The step τ_j is chosen in order the two conditions (see [15] and references therein):

– for a given accuracy ϵ: $\max_{0 \leq i \leq n} |(k_2^j - k_1^j)_i| / (|(y^j)_i + E|) \leq 6.2\epsilon$,

– for stability: $\max_{0\le i\le n}|(3k_3^j-2k_2^j-k_1^j)_i|\le 5\max_{0\le i\le n}|(k_2^j-k_1^j)_i|$

to be fulfilled at every time level. For singular in time solutions the stop criterion $\tau_j<10^{-16}$ is used and then $\tilde{T_0}$ is the time reached in the computations.

2.3 Dynamic mesh adaptation.. To proceed successfully with the single point blow-up behavior of the solution, a special adaptive mesh refinement, consistent with the self similar law (4) is made.

The main idea of the dynamic mesh adaptation will be given on the case of a differential equation

$$u_t=Lu,\quad u=u(t,x),\quad x\in R^N,\quad t>0, \tag{36}$$

which admits self-similar solution of the kind

$$u_s(t,x)=\varphi(t)\theta(\xi),\qquad \xi=\frac{x}{\psi(t)} \tag{37}$$

The function $\varphi(t)$ determines the amplitude of the solution, while the function $\theta(\xi)$ determines its space structure (the geometry). In the new, similarity variable ξ, where the space and the time are connected in a special way, the function θ gives the "frozen" image of the nonstationary process, described by the equation (36). The function $\theta(\xi)$ satisfies a reduced order equation (ODE or PDE, depending on the operator L).

It is always possible to choose $\varphi(t)$ and $\psi(t)$ so, that $\varphi(0)=1$ and $\psi(0)=1$. In this case the invariant solution $u_s(t,x)$ corresponds to initial data

$$u_s(0,x)=\theta(x). \tag{38}$$

As it was mentioned earlier, the importance of the invariant solution $u_s(x,t)$ is that it is an attractor of the solutions of equation (36) for large classes of initial data, different from (38). So it is important to incorporate the structure (37) in the numerical method for solving equation (36). The relation between ξ and x (37) gives the idea how to adapt the mesh in space. Let $\Delta x^{(k)}$ is the step size in space at $t=t^k$. There are two possibilities:

- to choose $\Delta x^{(k)}=\psi(t^k)\Delta\xi^{(k)}$ so, that $\Delta\xi^{(k)}=h_0=\max_i\Delta x_i{}^{(0)}$ at every time step;
- to choose $\Delta x^{(k)}$ in such way, that $\Delta\xi^{(k)}$ be bounded from below and from above

$$h_0/\lambda\le\Delta\xi^{(k)}\le\lambda h_0$$

for appropriate λ (usually $\lambda=2$).

Further, by using the relation between $\psi(t)$ and $\varphi(t)$, it is possible to incorporate the structure (37) of the s.-s.s. in the adaptive procedure. We will show how it is done for equation (2). In this case

$$\xi=r(1-t/T_0)^{-\ (\beta-\sigma-1)/(2(\beta-1))} \tag{39}$$

$$u_s(t,r)=(1-t/T_0)^{-1/(\beta-1)}\ \theta(\xi). \tag{40}$$

From (40) we find

$$(1-t/T_0)^{-1/(\beta-1)}=\frac{u_s(t,r)}{\theta(\xi)}$$

and we define the function

$$\Gamma_s(t)\doteq\frac{u_s(t,r)}{\theta(\xi)}. \tag{41}$$

Then

$$(1-t/T_0)^{-(\beta-\sigma-1)/(2(\beta-1))}=\Gamma_s(t)^{(\beta-\sigma-1)/2}=\Gamma_s(t)^m$$

This, together with (39), gives the necessary connection between ξ and r, $\Delta\xi$ and Δr:

$$\xi=r\Gamma_s(t)^m,\qquad \Delta\xi=\Delta r\Gamma_s(t)^m,\qquad m=(\beta-\sigma-1)/2. \tag{42}$$

For arbitrary initial data

$$u(0, r) = u_0(r) \geq 0, \quad sup\, u_0 < \infty, \quad r \in R_+, \tag{43}$$

by analogy with (41) we introduce a new function

$$\Gamma(t) = \frac{\max_r u(t, r)}{\max_r u_0(r)}, \tag{44}$$

and then by analogy with (42)

$$\xi = r\Gamma(t)^m, \qquad \Delta\xi = \Delta r\Gamma(t)^m, \qquad m = (\beta - \sigma - 1)/2. \tag{45}$$

On the basis of relations (45) the following strategy is accepted.

Let $\Delta r^{(k)}$ be the step in space (the length of the finite element) at time $t = t^k$. In the case of single point blow-up, $m > 0$, we choose the step $\Delta r^{(k)}$ so that the step $\Delta\xi^{(k)}$ be bounded from above:

$$\Delta\xi^{(k)} = \Delta r^{(k)}\Gamma(t)^m \leq \lambda h_0. \tag{46}$$

This means, that when $\Gamma(t)$ increases, the mesh in r must be refined. When condition (46) is violated, the following procedure is made:

– every element in the region, where the solution is not established with a given accuracy δ_u (usually $\delta_u = 10^{-7}$), is divided into two equal elements and the values of the solution in the new mesh points are found by interpolation from the old values;

– the elements, where the solution is established with a given accuracy δ_u, are thrown off (the computations proceed in smaller interval).

We use the fact, that on the developed stage of the process ($t \to T_0$) the solution grows only in a neighborhood of the blow-up point, while it is established near the boundary of the localization region. Let us mention, that the condition (46) is checked at every time step, but the refinement is made only when this condition is violated. At that moment a checking for elements throwing-off is made.

In the computations we have used $\lambda = 2$.

3. Structural stability of the s.-s.s

It is clear, that the blowup solutions are not stable with respect to the initial data in the sense, that small changes of the initial data may produce small changes in the blowup time, but very big differences in the solution' values near the blowup time. For blowup solutions and more generally, for invariant solutions, a more important property is the preservation in time of some characteristics, as geometric form, rate of growth, localization in space. Such a property is called structural stability, and for the blowup solutions (4) it is introduced in [8]. It gives a possibility to investigate the asymptotic behavior of the blow-up solutions in a special "self-similar norm", consistent for every t with the geometric form of the solution. In order to introduce the notion of structural stability, we define the self-similar representation [8] $\Theta(t, \xi)$ of the solution $u(t, r)$ of problem (2), (3):

$$\Theta(t, \xi) = u(t, \xi\Gamma(t)^{-m})/\Gamma(t), \quad \Gamma(t) = \max_r u(t, r)/\max_r u_0(r). \tag{47}$$

The s-.s.s. $u_s(t, r)$ (4), corresponding to the solution $\theta(\xi)$ of the problem (5),(6), is called structurally stable [8], if there exists a class of initial data $u_0(r) \neq \theta(r)$, so that for the self-similar representations (47) $\Theta(t, \xi)$ of the corresponding solutions $u(t, r)$, it holds:

$$\|\Theta(t, \xi) - \bar{\theta}(\xi)\|_{C[0,\infty)} \to 0, \quad t \to T_0^-, \tag{48}$$
$$\bar{\theta}(\xi) = (\max_r u_0(r)/\max_\xi \theta(\xi))\theta(\xi).$$

It is clear, that if $u_0(r) \equiv \theta(r)$, then $\bar{\theta}(\xi) \equiv \theta(\xi)$ and $\Theta(t,\xi) \equiv \theta(\xi)$ for $\forall t$.

In some cases it is more convenient to introduce a little different self-similar representation:

$$\Theta(t,\xi) = u(t, \xi\bar{\Gamma}(t)^{-m})/\bar{\Gamma}(t), \quad \bar{\Gamma}(t) = \max_r u(t,r). \tag{49}$$

and another scaled solution of the problem (5),(6):

$$\bar{\theta}(\xi) = \theta(\xi \max_\xi \theta(\xi)^{-m})/\max_\xi \theta(\xi). \tag{50}$$

Then the s-.s.s. $u_s(t,r)$ is called structurally stable, if the convergence (48) takes place for (49) and (50). In the examples bellow the self-similar representation (49) is used, because we compare the evolution in time of self-similar and non self-similar initial data - in both cases the self-similar representations are scaled in such way, that $\Theta(t,0) = 1$ and the comparison is easier.

4. Numerical investigations

We constructed numerically solutions to (10),(11) for different values of $\sigma > 0$ and $\beta \geq \beta_s$. The comparison with the singular solution $\theta_s(\xi)$ showed that the constants c_0 in (8) for the computed solutions satisfy $c_0 > c_s$ for $\beta_s \leq \beta < \beta_u$, for $\beta_u \leq \beta < \beta_p$ and $\beta \geq \beta_p$.

Taking the computed $\theta(\xi)$ as initial data for the parabolic problem (26),(27),(29), we have compared both the exact blow-up time $T_0 = 1/(\beta - 1)$ with $\tilde{T}_0$ found in the computations and the self-similar representations $\Theta(t,\xi)$ with $\bar{\theta}(\xi)$. All of the experiments show: T_0 is close to $\tilde{T}_0$, (see the examples bellow), and $\Theta(t,\xi)$ are close to $\bar{\theta}(\xi)$ up to $\tilde{T}_0$ (their graphs coincide within the plotting resolution). These two observations show the good accuracy and the reliability of both methods - for the self-similar and for the parabolic problems.

Taking compactly supported initial data $u_0(r)$, but large enough [17] to produce blow-up (as it was mentioned, for $\beta > \beta_f$ and small initial data (2) the solution may be global), we show the validity of (48), i.e., the structural stability of the s.-s.s. (4), corresponding to the computed $\theta(\xi)$ for the same values of N, σ and $\beta \geq \beta_s$.

5. Examples

The evolution in time (on the left) and the self-similar representations (on the right) of self-similar (Fig. 1, 3, 5) and non-self-similar (Fig. 2, 4, 6) initial data for β beyond three critical exponents are shown.

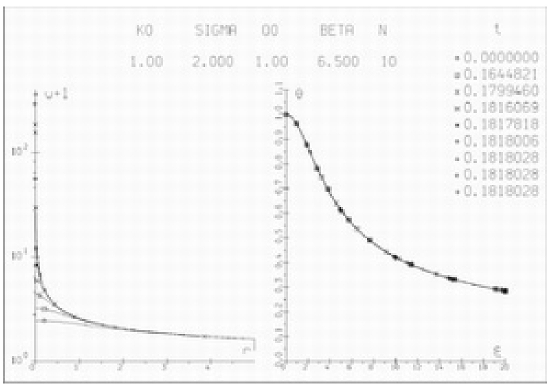

FIGURE 1. $\beta > \beta_s = 4.5$, self-similar initial data

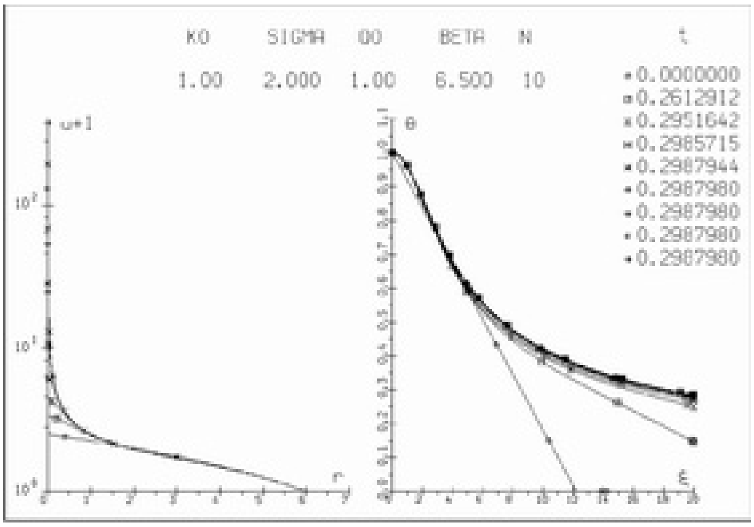

FIGURE 2. $\beta > \beta_s = 4.5$, non self-similar initial data

On Fig. 1 the evolution in time of the solution $\theta_h(\xi)$ for parameters $\sigma = 2$, $N = 10$, $\beta = 6.5 > \beta_s = 4.5$ is shown. The initial mesh $\{0(0.05)2(0.1)4(0.2)10(0.5)20\}$ contains 111 points with $h_0 = \min_i h_i{}^0 = 0.05$. The last mesh contains 3989 points, but 2058 of them are thrown-off, the smallest step is $h_l = h_0 * 2^{-13}$. The exact blowup time is $T_0 = 1/(\beta - 1) = 0.(18)$, the blow-up time $\tilde{T}_0$, found in computations, is 0.1818028, so $T_0 - \tilde{T}_0 \approx 0.000015$. The self-similar representations coincide with the scaled self-similar function $\bar{\theta}_h(\xi)$ within the plotting resolution.

On Fig. 2 the evolution in time of non self-similar initial data for the same parameters $\sigma = 2$, $N = 10$, $\beta = 6.5 > \beta_s = 4.5$ is shown. The initial mesh $\{0(0.1)7\}$ contains 71 points. The last mesh contains 4076 points, but 2182 of them are thrown-off, the smallest step is the same, as in the previous example: $h_l = h_0 * 2^{-13}$. The self-similar representations tends to the scaled self-similar function $\bar{\theta}_h(\xi)$.

On Fig. 3 and Fig. 4 the parameters are: $\sigma = 2$, $N = 15$, $\beta_u = 6.412 < \beta = 6.6 < \beta_p = 6.805$. For the self similar initial data we have $T_0 = 1/(\beta - 1) \approx 0.17857$, $T_0 - \tilde{T}_0 \approx 0.0037$.

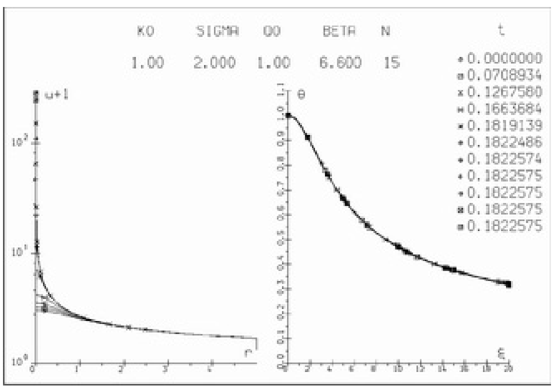

FIGURE 3. $\beta_u = 6.412 < \beta < \beta_p = 6.805$ self-similar initial data

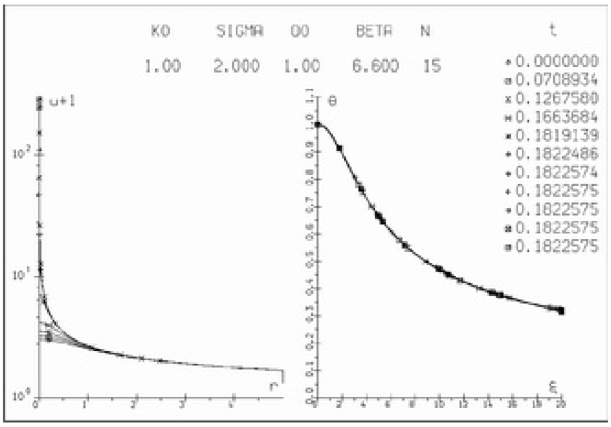

FIGURE 4. $\beta_u = 6.412 < \beta < \beta_p = 6.805$ non self-similar initial data

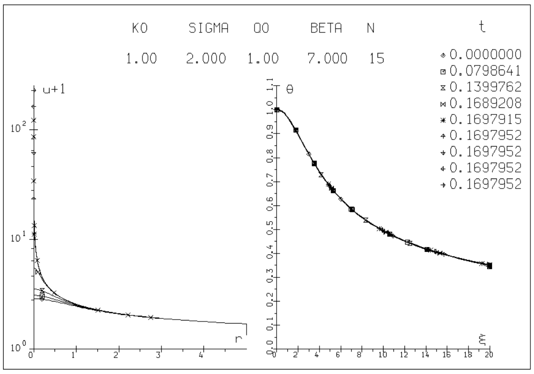

FIGURE 5. $\beta > \beta_p = 6.805$, self-similar initial data

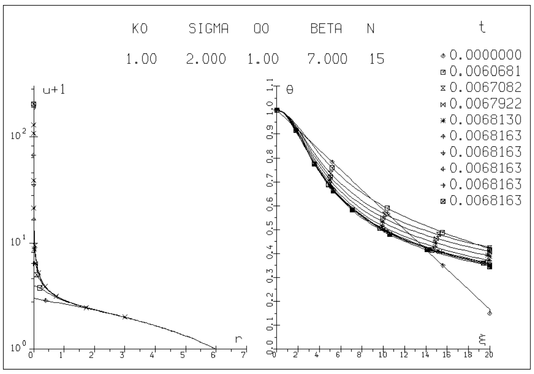

FIGURE 6. $\beta > \beta_p = 6.805$, non self-similar initial data

On Fig. 5 and Fig. 6 the parameters are: $\sigma = 2$, $N = 15$, $\beta = 7 > \beta_p$. For the self-similar initial data we have $T_0 = 1/(\beta - 1) = 0.1(6)$, $T_0 - \tilde{T}_0 \approx 0.0031$.

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