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SOME NEW STABILITY RESULTS FOR DIFFERENTIAL AND DIFFERENCE EQUATIONS WITH UNBOUNDED OPERATOR COEFFICIENTS IN HILBERT AND BANACH SPACES

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Dedicated to the 85-th anniversary of A. A. Samarskij

ABSTRACT. New stability results for the first order differential equations with an unbounded strongly positive operator coefficient in Banach and Hilbert spaces are presented. The framework of Hilbert spaces allows one to get stability estimates in somewhat stronger norms as it is the case for Banach spaces. An exact two-level difference approximation is derived using the evolution operator. Considering an arbitrary two-level approximation as a perturbation of the exact difference scheme the stability and accuracy theorems in a strong norm are proven. The stability results for three-level difference schemes with unbounded strongly P-positive operator coefficients are discussed.

1. INTRODUCTION

Many initial value problems for parabolic partial differential equations in abstract setting have the form

$$\frac{\partial u(t)}{\partial t} + Au(t) = f(t), \quad u(0) = u_0, \tag{1.1}$$

where $u(t),\ u:\ \mathbb{R}_+ \to X$ is an unknown, $f(t)$ is a given vector-valued function with values in a Hilbert or Banach space X and A is a closed linear operator in X with a domain $D(A)$. The simplest finite difference approximations to (1.1) on the equidistant grid $\omega_\tau = \{t_i = i\tau :\ i = 0, 1, \dots\}$ is the explicit scheme

$$\frac{y_{n+1} - y_n}{\tau} + Ay_n = f_n, \quad n = 0, 1, \dots \quad y_0 = u_0, \tag{1.2}$$

where $y_n = y(t_n)$ is an approximation to $u_n = u(t_n)$. An other approximation is the implicit scheme

$$\frac{y_{n+1} - y_n}{\tau} + Ay_{n+1} = f_n, \quad n = 0, 1, \dots \quad y_0 = u_0. \tag{1.3}$$

Both these schemes belong to the class of two-level schemes of the kind

$$By_{n+1} = Cy_n + F_n, \quad n = 0, 1, \dots \quad y_0 = u_0. \tag{1.4}$$

where $B = I, C = I + \tau A, F_n = \tau f_n$ for the first scheme and $B = I + \tau A, C = I, F_n = \tau f_n$ for the second one.

There are various definitions of stability for two-level difference schemes [11, 17, 13, 12, 16, 21] which mean, roughly speaking, that the error remains bounded or increases in accordance with some apriori defined low when $n \to \infty$. For example, the following two definitions can be found in [13,12,16].

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Definition 1.1. A two-level difference scheme is called stable with respect to the initial data, if

$$\|y_n\|_{(1)} \le M_1 \|y_0\|_{(1)}, \quad n = 0, 1, \dots \tag{1.5}$$

in some norm $\|\cdot\|_{(1)}$ with a positive constant M_1 independent of n, τ provided that $F_n = 0$.

Definition 1.2. A two-level difference scheme is called stable with respect to the right-hand side, if

$$\|y_n\|_{(1)} \le M_2 \max_{0 \le k < n} \|F_k\|_{(2)}, \quad n = 0, 1, \dots \tag{1.6}$$

with some norms $\|\cdot\|_{(1)}$, $\|\cdot\|_{(2)}$ and with a positive constant M_1 independent of n, τ provided that $u_0 = 0$.

The both stabilities (with respect to the right-hand side and with respect to the initial data) imply the stability of the difference scheme with respect to the initial data in the sense

$$\|y_n\|_{(1)} \le M_1 \|y_0\|_{(1)} + M_2 \max_{0 \le k < n} \|F_k\|_{(2)}, \quad n = 0, 1, \dots \tag{1.7}$$

It is possible to study the stability of a two-level difference scheme in the form (1.4). For example, one get immediately that the necessary and sufficient condition of the stability with respect to the initial data is

$$\|\rho(B^{-1}C)\| \le 1 \tag{1.8}$$

where $\rho(B^{-1}C)$ is the spectral radius of the operator $B^{-1}C$. But the complexity of the problem of estimating the spectral radius is often equivalent to the initial one. For this reason A.A.Samarskij has introduced the so called canonical form of a two-level difference scheme

$$B\frac{y_{n+1} - y_n}{\tau} + Ay = \phi_n, n = 0, 1, \dots \tag{1.9}$$

with a given y_0. One of the most beautiful results of the Samarskij's stability theory is the following [13,12,16,17]: Let $A = A^*$ and the operator B^{-1} exists then difference scheme (1.9) is stable in the norm $\|y\|_A = \sqrt{(Ay, y)}$ iff $B \ge \frac{\tau}{2} A$. The use of the canonical form has allowed to develop almost closed theory of stability of two-level difference schemes in [13,12,16,17]. Only the case of unbounded operator coefficients especially in a Banach space remains not complete studied despite the fact that there are many interesting results in [17]. A canonical form of three-level difference schemes approximating the second order differential equations was also introduced by A.A.Samarskij and has allowed to obtain a number of important results [13,12,16,17].

The stability theory together with the regularization principle [14] provide a powerful tool to obtain stable difference schemes. The main idea of regularization is to start from any simple scheme (even unstable) and by perturbing its coefficients (while taking into consideration the stability conditions) obtain a stable difference scheme or a scheme with other desired properties. All the main classes of difference schemes for the problems of mathematical physics have been designed and analyzed on the basis of this approach in [13,12,16,17].

An important kind of stability is the so called coefficient stability of differential equations and difference schemes. Before we define this kind of stability let us consider the following two Cauchy problems:

$$\frac{du}{dt} + Au = f(t), \quad u(0) = u_0 \tag{1.10}$$

and

$$\frac{dv}{dt} + Bv = g(t), \quad v(0) = v_0. \tag{1.11}$$

Definition 1.3. Problem (1.10) is called stable with respect to the initial data if in some norms

$$\|u - v\| \leq M\|u_0 - v_0\| \tag{1.12}$$

provided that $B = A,\ g = f$, stable with respect to the right-hand side if

$$\|u - v\| \leq M\|f - g\| \tag{1.13}$$

provided that $B = A,\ v_0 = u_0$, and stable with respect to the operator coefficient A, if there exists an operator C such that $(A - B)C^{-1}$ is bounded and

$$\|u - \tilde{u}\| \leq M\|(A - B)C^{-1}\| \tag{1.14}$$

provided that $g = f,\ v_0 = u_0$ where M is a positive constant.

The problem (1.10) is called strongly stable if it is stable with respect to the initial data, to the right-hand side and with respect to the operator coefficient in the same time.

Analogously one can define the stability with respect to the initial data, with respect to the right-hand side, with respect to the operator coefficient and the strong stability of the difference schemes (cf. [17]).

In this paper we present some new results for the case of unbounded operator coefficients which develop results from [13,12,16,17] and were partly obtained in [5,15].

The paper is organized as follows. In Section 2 we derive the strong stability of the first order differential equations with an unbounded operator coefficient in Hilbert and Banach spaces. The framework of Hilbert spaces allows one to get the stability estimate in a somewhat stronger norm as it is the case for Banach spaces. Section 3 is devoted to the two-level grid approximations of the first order differential equations with strongly positive unbounded operator coefficients. We derive an exact two-level difference scheme and consider an arbitrary two-level approximation as a perturbation of this scheme. This allows us to get new stability and convergence results in some strong norm in a Banach space. Section 4 deals with three-level difference schemes which involve unbounded strongly P-positive operator coefficients. In this section we introduce the ρ-stability of three-level difference schemes, discuss the strong P-positivity and represent the solution of the difference scheme as a function of the operator coefficient. This solution is represented then in Section 5 by an improper Dunford-Cauchy integral being the basis for various stability results where the strong P-positivity is the crucial sufficient condition of the ρ-stability. Analytical and numerical examples are given to confirm and clarify theoretical results.

Note that common results about differential and difference equations with unbounded operator coefficients is of great importance also for finite-difference and finite-element approximations of non-stationary partial differential equations. These approximations possesses although matrix coefficients, i.e. are formally bounded, but their norms depend on the discretization parameter h and tend to infinity as $h \to 0$.

2. Stability of the first order differential equations

In this section we consider the following two initial value problems

$$\frac{du}{dt} + A(t)u = f(t), \quad u(0) = u_0 \tag{2.1}$$

and

$$\frac{dv}{dt} + B(t)v = g(t), \quad v(0) = v_0 \tag{2.2}$$

with densely defined, closed operators $A(t), B(t)$ having a common domain $D(A) = D(B) = D(A(t)) = D(B(t))$ independent of t.

First of all we remind some facts about the evolution operator. The evolution operator (family of evolution operators) $U(t,s)$ for the equation (2.1) satisfies the equations

$$\frac{\partial U(t,s)}{\partial t} + A(t)U(t,s) = 0, \; U(s,s) = I, \tag{2.3}$$

where I is the unit operator. Given the evolution operator $U(t,s)$ the solution of the equation (2.1) can be represented by

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\xi)f(\xi)d\xi \tag{2.4}$$

It is known (see i.e [3]) that the evolution operator can be written down in the form

$$U(t,s) = e^{-(t-s)A(s)} + \int_s^t e^{-(t-r)A(s)}R(r,s)dr, \tag{2.5}$$

where $R(t,s)$ is the solution of the integral equation of the Volterra type

$$R(t,s) - \int_s^t R_1(t,r)R(r,s)dr = R_1(t,s) \tag{2.5}$$

with

$$R_1(t,s) = -(A(t) - A(s))e^{-(t-s)A(s)}. \tag{2.7}$$

The difference $z(t) = u(t) - v(t)$ is the solution of the initial value problem

$$\begin{gathered} \frac{dz}{dt} + A(t)z = -[A(t) - B(t)]v(t) + f(t) - g(t), \\ z(0) = u_0 - v_0. \end{gathered} \tag{2.8}$$

We investigate the estimates for $z(t)$ in Banach and Hilbert spaces under various assumptions and in various norms.

2.1 Stability estimates in Banach space.

Here we make the following assumptions:

(B1) The operators $A\,(t)$, $B\,(t)$ are densely defined in a Banach space X and possess domains $D\,(A) = D\,(B) = D\,(C)$ independent of t. There exist the bounded inverses $A^{-1}(t)$, $B^{-1}(t)$ and for the resolvents $R_{A(t)}(z) = (z - A(t))^{-1}, R_{B(t)}(z) = (z - B(t))^{-1}$ holds

$$\|R_{A(t)}(z)\| \le \frac{1}{1+|z|}, \; \|R_{B(t)}(z)\| \le \frac{1}{1+|z|} \qquad (\theta + \epsilon \le |\arg z| \le \pi) \tag{2.9}$$

with $\theta \in (0, \pi/2), \epsilon > 0$ uniformly in $t \in [0,T]$.

(B2) The operators $A(t), B(t)$ are strongly differentiable on $D\,(A)$.

(B3) There exists a constant M such that

$$\left\|A^{\beta}(s)B^{-\beta}(s)\right\| \le M. \tag{2.10}$$

(B4) For the evolution operators there holds

$$\|A^{\beta}(t)U_A(t,s)\| \le \frac{C_\beta}{|t-s|^{\beta}}, \qquad \|B^{\beta}(t)U_B(t,s)\| \le \frac{C_\beta}{|t-s|^{\beta}}, \qquad \beta \in [0,1]. \tag{2.11}$$

(B5) It holds

$$\left\|A^{\rho}(t)A^{-\rho}(s) - I\right\| \le C|t-s|^{\alpha}, \qquad \rho \in [0,1], \quad \alpha \ge 0. \tag{2.12}$$

Remark 2.1. Let $\Omega \subset \mathbb{R}^2$ be a polygon and

$$\mathcal{L}(x,t,D) = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} a_{i,j}(x,t) \frac{\partial}{\partial x_j} + \sum_{j=1}^{2} b_j(x,t) \frac{\partial}{\partial x_j} + c(x,t) \tag{2.13}$$

be a second order elliptic operator with time-dependent real smooth coefficients satisfying the uniform ellipticity condition

$$\sum_{i,j=1}^{2} a_{ij}(x,t)\xi_i\xi_j \geq \delta_1|\xi|^2 \qquad (\xi = (\xi_1,\xi_2) \in \mathbb{R}_2) \tag{2.14}$$

with a positive constant δ_1. Taking $X = L^2(\Omega)$, and $V = H_0^1(\Omega)$ or $V = H^1(\Omega)$ accordingly to the boundary condition

$$u = 0 \quad on \quad \partial\Omega \times (0,T) \tag{2.15}$$

or

$$\frac{\partial u}{\partial \nu_{\mathcal{L}}} + \sigma u = 0 \quad on \quad \partial\Omega \times (0,T), \tag{2.16}$$

we set

$$\begin{aligned} \mathcal{A}_t(u,v) = \sum_{i,j=1}^{2} \int_\Omega a_{i,j}(x,t)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx + \sum_{j=1}^{2}\int_\Omega b_j(x,t)\frac{\partial u}{\partial x_j}v\,dx \\ + \int_\Omega c(x,t)uv\,dx + \int_{\partial\Omega}\sigma(x,t)uv\,dS \end{aligned} \tag{2.17}$$

for $u, v \in V$. An m-sectorial operator $A(t)$ in X can be defined through the relation

$$\mathcal{A}_t(u,v) = \big(A(t)u, v\big), \tag{2.18}$$

where $u \in D(A(t)) \subset V$ and $v \in V$. The relation

$$D(A(t)) = H^2(\Omega) \cap H_0^1(\Omega) \tag{2.19}$$

follows for $V = H_0^1(\Omega)$ and

$$D\big(A(t)\big) = \left\{ v \in H^2(\Omega) \mid \frac{\partial v}{\partial \nu_{\mathcal{L}}} \text{ on } \partial\Omega \right\} \tag{2.20}$$

for $V = H^1(\Omega)$, if $\partial\Omega$ is smooth for instance.

It was proved in [3][pp. 95–101], that all the assumptions above hold for such an operator $A(t)$.

Assumptions **(B1)**- **(B2)** yield (see e.g. [8][Th. 3.11, pp. 255, 240], [9]) that the homogeneous problems (2.1), (2.2) are uniformly correct, and the evolution operators $U_A(t,s), U_B(t,s)$ map the domain $D(A)$ into itself, the operators $V_A(t,s) = A(t)U_A(t,s)A^{-1}(s), V_B(t,s) = B(t)U_B(t,s)B^{-1}(s)$ are bounded and strongly continuous in the triangle $T_\Delta = \{(s,t) : 0 \leq s \leq t \leq T\}$. The solution of (2.8) can be represented by

$$z(t) = U_A(t,0)z(0) + \int_0^t U_A(t,s)\big\{ - [A(s) - B(s)]v(s) + f(s) - g(s)\big\}\,ds, \tag{2.21}$$

where $U_A(t,s)$ is the evolution operator. This equality yields

$$\begin{aligned} A^\beta(t)z(t) = A^\beta(t)U_A(t,0)A^{-\beta}(0)A^\beta(0)z(0) \\ + \int_0^t A^\beta(t)U_A(t,s)\big[B(s) - A(s)\big]A^{-\beta}(s)A^\beta(s)v(s)\,ds \\ + \int_0^t A^\beta(t)U_A(t,s)A^{-1}(s)A(s)[f(s) - g(s)]\,ds, \end{aligned} \tag{2.22}$$

Let us show that the following estimate holds:

$$\begin{gathered} \big\|A^\beta(t)\,U_A(t,s)\,A^{-\beta}(s)\big\| \leq M, \\ 0 \leq s \leq t \leq T. \end{gathered}$$

To this end we use the formula [8][see (4.14), p. 262]:

$$U_A(t,s) = U_{A(s)}(t,s) + \int_s^t U_A(t,\tau)\left[A(\tau) - A(s)\right] U_{A(s)}(\tau - s)\, d\tau,$$

where

$$U_{A(s)}(\tau - s) = \exp\left(-A(s)(t-s)\right).$$

Then we get

$$\left\|A^{\beta}(t)\, U_A(t,s)\, A^{-\beta}(s)\right\| = \Big\|A^{\beta}(t)\, A^{-\beta}(s)\, A^{\beta}(s)\, U_{A(s)}(t,s)\, A^{-\beta}(s) +$$

$$+\int_s^t A^{\beta}(t)\, U(t,\tau)\left[A(\tau) - A(s)\right] A^{-1}(s)\, A(s)\, U_{A(s)}(\tau - s)\, A^{-\beta}(s)\, d\tau\Big\|$$

$$\le (1 + CT^{\alpha})\, M + C_{\beta} C M \int_s^t \frac{d\tau}{(t-\tau)^{\beta}(\tau - s)^{1-\alpha-\beta}} =$$

$$= (1 + CT^{\alpha})\, M + C_{\beta} C M (t-s)^{\alpha} B(1-\beta, 1-\beta-\alpha) \le M_1,$$

where $B(x,y)$ is the beta-function. Now, equality (2.22) implies

$$\begin{aligned} \|A^{\beta}(t)z(t)\| &\le M\|A^{\beta}(0)z(0)\| \\ &+ C_{\beta}\max_{s}\left\|[B(s)-A(s)]A^{-\beta}(s)\right\| \int_0^t \frac{1}{(t-s)^{\beta}}\|A^{\beta}(s)v(s)\|\,ds \\ &+ M\int_0^t \left\|A^{\beta}(s)\left[f(s)-g(s)\right]\right\| ds. \end{aligned} \tag{2.23}$$

Assumption **(B3)** allows us to replace the first integral on the right-hand side of (2.23) by the integral

$$M\int_0^t \frac{1}{(t-s)^{\beta}}\left\|B(s)v(s)\right\| ds. \tag{2.24}$$

For this integral we get from (2.2)

$$\|B(t)v(t)\| \le M\|B(0)v(0)\| + M\int_0^t \|B(s)g(s)\|\,ds. \tag{2.25}$$

Substituting this estimate into (2.23) we get the following stability estimate

$$\begin{aligned} \|A^{\beta}(t)z(t)\| &\le M\|A^{\beta}(0)z(0)\| + c_{\beta}M\max_{0\le s\le T}\left\|[B(s)-A(s)]A^{-1}(s)\right\| \\ &\times \frac{t^{1-\beta}}{1-\beta}\left\{\|B(0)v(0)\| + \int_0^t \|B(s)g(s)\|\,ds\right\} \\ &+M\int_0^t \|A^{\beta}(s)[f(s)-g(s)]\|\,ds,\ \beta\in[0,1). \end{aligned} \tag{2.26}$$

Thus, we have proven the following assertion.

Theorem 2.2. *Problem (2.1) is strongly stable in the Banach space X with the stability estimate (2.26) provided that conditions* **(B1)** - **(B5)** *hold true.*

Note that stability estimate (2.26) has sense for $0 \le \beta < 1$ only, i.e. it is not valid for the norm $\|Az(t)\|$. The strong stability in a similar strong norm will be shown in the next section in the case of a Hilbert space.

2.2 Stability estimates in Hilbert space.

In this section we assume that there exists an operator $C = C^* \geq c_0 I$ such that

(H1)

$$\big\|[A(s) - B(s)]C^{-1}\big\| \leq \delta < \infty, \tag{2.27}$$

(H2)

$$\begin{aligned} \big(A(s)y, Cy\big) &\geq c_0\|Cy\|^2, \\ \big(B(s)y, Cy\big) &\geq c_0\|Cy\|^2 \quad \forall s \in [0,T], c_0 > 0. \end{aligned} \tag{2.28}$$

The equality (2.8) yields

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|z(t)\|_C^2 + \big(A(t)z(t), Cz(t)\big) &= -\big([A(t) - B(t)]v(t), Cz(t)\big) \\ &+ \big(f(t) - g(t), Cz(t)\big) \end{aligned} \tag{2.29}$$

and further

$$\begin{aligned} &\frac{1}{2}(Cz(t), z(t)) + \int_0^t (A(s)z(s), Cz(s))ds \\ &= \int_0^t \big([B(s) - A(s)]C^{-1}Cv(s), Cz(s)\big)\, ds \\ &+ \int_0^t \big(f(s) - g(s), Cz(s)\big)\, ds + \frac{1}{2}\big(C(u_0 - v_0), (u_0 - v_0)\big). \end{aligned} \tag{2.30}$$

Using the assumptions **(H1)**, **(H2)** we get

$$\begin{aligned} &\frac{1}{2}\big(Cz(t), z(t)\big) + c_0\int_0^t \|Cz(s)\|^2 ds \\ &\leq \epsilon\int_0^t \|Cz(s)\|^2 ds + \frac{\delta^2}{2\epsilon}\int_0^t \|Cv(s)\|^2 ds \\ &+ \epsilon_1\int_0^t \|Cz(s)\|^2 ds + \frac{1}{2\epsilon_1}\int_0^t \|f(s) - g(s)\|^2 ds \\ &+ \frac{1}{2}\big(C(u_0 - v_0), (u_0 - v_0)\big), \end{aligned} \tag{2.31}$$

where ϵ, ϵ_1 are arbitrary positive numbers. Choosing $\epsilon + \epsilon_1 < c_0$ we get

$$\begin{aligned} &\frac{1}{2}\big(Cz(t), z(t)\big) + (c_0 - \epsilon - \epsilon_1)\int_0^t \|Cz(s)\|^2 ds \leq \\ &\leq \frac{\delta^2}{2\epsilon}\int_0^t \|Cv(s)\|^2 ds + \\ &+ \frac{1}{2\epsilon_1}\int_0^t \|f(s) - g(s)\|^2 ds + \frac{1}{2}\big(C(u_0 - v_0), (u_0 - v_0)\big). \end{aligned} \tag{2.32}$$

In order to estimate the first term on the right-hand side we use the equation (2.2), from where we get

$$\frac{1}{2}\frac{d}{dt}\big(Cv(t), v(t)\big) + \big(B(t)v(t), Cv(t)\big) = \big(g(t), Cv(t)\big). \tag{2.33}$$

This implies the inequality

$$\begin{aligned} &\frac{1}{2}(Cv(t), v(t)) + c_0\int_0^t \|Cv(s)\|^2 ds \leq \\ &\leq \epsilon_2\int_0^t \|Cv(s)\|^2 ds + \frac{1}{4\epsilon_2}\int_0^t \|g(s)\|^2 ds + \frac{1}{2}(Cv_0, v_0) \end{aligned} \tag{2.34}$$

or

$$\int_0^t \|Cv(s)\|^2 ds \le (c_0-\epsilon_2)^{-1}\Big[\frac{1}{4\epsilon_2}\int_0^t \|g(s)\|^2 ds + \frac{1}{2}(Cv_0, v_0)\Big]. \tag{2.35}$$

After substitution of (2.35) into (2.32) we get the following stability estimate

$$\begin{gathered}\frac{1}{2}(Cz(t), z(t)) + (c_0-\epsilon-\epsilon_1)\int_0^t \|Cz(s)\|^2 ds \le \\ \le \max_{0\le s\le T}\|[A(s)-B(s)]C^{-1}\|^2\frac{(c_0-\epsilon_2)^{-1}}{2\epsilon}\Bigg[\frac{1}{4\epsilon_2}\int_0^t \|g(s)\|^2 ds + \\ +\frac{1}{2}(Cv_0, v_0)\Bigg] + \frac{1}{2\epsilon_1}\int_0^t \|f(s)-g(s)\|^2 ds + \frac{1}{2}\big(C(u_0-v_0), u_0-v_0\big),\end{gathered} \tag{2.36}$$

which means the strong stability, i.e. the stability with respect to the right-hand side, to the initial condition and the coefficient stability. Thus, we have proven the following assertion.

Theorem 2.3. *Let $A(t)$, $B(t)$ be densely defined operators in a Hilbert space H and there exists a self-adjoint, positive definite operator C with a domain $D(C) = D(A(t)) = D(B(t))$ independent of t, then the conditions* **(H1)**, **(H2)** *provide the strong stability of the problems (2.1), (2.2).*

Note, that an analogous estimate in the case of an finite dimensional Hilbert spaces and of a constant operator A was proved in [17][p. 62]. In some cases it is more convenient to use other sufficient stability conditions for (2.1), (2.2) using properties of the operators $A(t)$, $B(t)$ only (without the assumption about C). These properties are discussed below.

Let us suppose that the operators $A(t)$, $B(t)$ are self-adjoint, positive definite in some Hilbert space H with the domains $D(A) = D(B)$ independent of t and satisfy

(H3) The operators $A(t)$, $B(t)$ are strongly continuous and differentiable on $D(A)$.

Starting from (2.8) one gets the equality

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}\|z(t)\| - \frac{1}{2}(A'(t)z(t), z(t)) + \|A(t)z(t)\|^2 = \\ = -([A(t)-B(t)]V(t), A(t)z(t)) + (f(t)-g(t), A(t)z(t)),\end{gathered} \tag{2.37}$$

which implies after integration

$$\begin{gathered}\frac{1}{2}\|z(t)\|^2_{A(t)} + \int_0^t \|A(s)z(s)\|^2 ds = \int_0^t (A'(s)z(s), z(s))^2 ds - \\ -\int_0^t ([A(s)-B(s)]v(s), A(s)z(s))ds + \\ +\int_0^t (f(s)-g(s), A(s)z(s))ds + \frac{1}{2}\|z(0)\|^2_{A(0)} \\ = \frac{1}{2}I_1 + I_2 + I_3 + \frac{1}{2}\|z(0)\|^2_{A(0)}.\end{gathered} \tag{2.38}$$

Next, we estimate each of the integrals on the right-hand side of (2.38). We get

$$\begin{aligned} I_1 &= \int_0^t \left(A^{-1/2}(s)\, A'(s)\, A^{-1/2}(s)\, A^{-1/2}(s)\, z(s)\,, A^{-1/2}(s)\, z(s)\right) ds \\ &\leq c_1 \int_0^t \left\|A^{-1/2}(s)\, z(s)\right\|^2 ds = c_1 \int_0^t \|z(s)\|_{A(s)}^2\, ds, \end{aligned} \tag{2.39}$$

where Lemma 1.9, p. 229 from [8] (see also [9]) was used. Further we have

$$\begin{aligned} |I_2| &= \left| \int_0^t \left([A(s) - B(s)]\, A^{-1}(s)\, A(s)\, V(s)\,, A(s)\, Z(s)\right) ds \right| \\ &\leq \frac{1}{4\varepsilon} \max_{0\leq s\leq T} \left\|[A(s) - B(s)]\, A^{-1}(s)\right\| \int_0^t \|A(s)\, V(s)\|^2\, ds \\ &\quad + \varepsilon \int_0^t \|A(s)\|^2\, ds, \end{aligned} \tag{2.40}$$

and, finally,

$$I_3 = \varepsilon_1 \int_0^t \|A(s)\, z(s)\|^2\, ds + \frac{1}{4\varepsilon} \int_0^t \|f(s) - g(s)\|^2\, ds. \tag{2.41}$$

Using inequalities (2.39)-(2.41) with $\varepsilon = \varepsilon_1 = 1/4$ equality (2.38) implies

$$\begin{aligned} &\frac{1}{2} \|z(t)\|_{A(t)}^2 + \frac{1}{2} \int_0^t \|A(s)\, z(s)\|^2\, ds \leq \frac{c_1}{2} \int_0^t \|z(s)\|_{A(s)}^2\, ds \\ &\quad + \max_{0\leq s\leq T} \left\|[A(s) - B(s)]\, A^{-1}(s)\right\| \int_0^t \|A(s)\, V(s)\|^2\, ds \\ &\quad + \int_0^t \|f(s) - g(s)\|^2\, ds + \frac{1}{2} \|z(0)\|_{A(0)}^2\, . \end{aligned} \tag{2.42}$$

Using the Gronwall's lemma we get from (2.42)

$$\begin{aligned} &\|z(t)\|_{A(t)}^2 + \int_0^t \|A(s)\, z(s)\|^2\, ds \\ &\leq \left\{ 2 \max_{0\leq s\leq T} \left\|[A(s) - B(s)]\, A^{-1}(s)\right\| \int_0^t \|A(s)\, V(s)\|^2\, ds \right. \\ &\quad \left. + \int_0^T \|f(s) - g(s)\|^2\, ds + \|z(0)\|_{A(0)}^2 \right\} e^{c_1 t}. \end{aligned} \tag{2.43}$$

The right-hand side of (2.43) contains an integral, where the integrand involves $V(t)$. Let us estimate this integral through the input data. Analogously as above we get for the solution of problem (2.2) the following equality

$$\frac{1}{2}\|V(t)\|_{B(t)}^{2}+\int_{0}^{t}\|B(s)V(s)\|^{2}ds$$
$$=\frac{1}{2}\int_{0}^{t}(B'(s)V(s),V(s))\,ds+\int_{0}^{t}(g(s),B(s)V(s))\,ds$$
$$+\frac{1}{2}\|V(0)\|_{B(0)}^{2},$$

which yields the estimate

$$\begin{gathered}\frac{1}{2}\|V(t)\|_{B(t)}^{2}+\int_{0}^{t}\|B(s)V(s)\|^{2}ds\\ \leq c_{1}\int_{0}^{t}\|V(s)\|_{B(s)}^{2}ds+\int_{0}^{t}\|g(s)\|^{2}ds+\|V_{0}\|_{B(0)}^{2}.\end{gathered}\tag{2.44}$$

This estimate and the Gronwall's lemma imply

$$\begin{gathered}\frac{1}{2}\|V(t)\|_{B(t)}^{2}+\int_{0}^{t}\|B(s)V(s)\|^{2}ds\leq\\ =\left[\int_{0}^{T}\|g(s)\|^{2}ds+\|V_{0}\|_{B(0)}^{2}\right]e^{c_{1}t},\end{gathered}\tag{2.45}$$

where c_1 is a constant, which bounds the norm of the operator $B^{-1/2}(s)\,B'(s)\,B^{-1/2}(s)$ (see Lemma 1.9, p. 229 from [8,9]).

Since the operator $A(t)\,B^{-1}(t)$ is bounded, i.e. there exists a positive constant M such that

$$\left\|A(t)\,B^{-1}(t)\right\|\leq M,\quad t\in[0,T],$$

then (2.45) implies

$$\begin{gathered}\int_{0}^{t}\|A(s)V(s)\|^{2}ds=\int_{0}^{t}\left\|A(s)B^{-1}(s)B(s)V(s)\right\|^{2}ds\leq\\ \leq M^{2}\left[\int_{0}^{T}\|g(s)\|^{2}ds+\|V_{0}\|_{B(0)}^{2}\right]e^{c_{1}t}.\end{gathered}\tag{2.46}$$

Substituting estimate (2.46) into the right-hand side of (2.43), we get the following assertion.

Theorem 2.4. *Let the operators $A(t)$,$B(t)$ be self-adjoint, positive definite, densely defined with domains $D(A(t))=D(B(t))=D(A)$ independent of t and satisfy* **(H3)**, *then the Cauchy problem (2.1) is strongly stable with the stability estimate*

$$\|z(t)\|^2_{A(t)} + \int_0^t \|A(s)\, z(s)\|^2\, ds \leq$$

$$\leq \Bigg\{ 2 \max_{0\leq s\leq T} \left\| [A(s) - B(s)]\, A^{-1}(s) \right\| M^2 \left[\int_0^T \|g(s)\|^2\, ds + \right. \tag{2.47}$$

$$\left. + \|V_0\|^2_{B(0)} \right] e^{c_1 T} + 2 \int_0^T \|f(s) - g(s)\|^2\, ds + \|z(0)\|^2_{A(0)} \Bigg\} e^{c_1 t}.$$

3. Strong stability of two-level difference schemes in Banach spaces

Let us consider the Cauchy problem

$$\begin{gathered} u'(t) + A(t)\, u(t) = f(t), \qquad t \in (0,1] \\ u(0) = u_0 \end{gathered} \tag{3.1}$$

in a Banach space X. In this section we assume the operator $A(t)$ for each fixed t to be densely defined, strongly positive operator, i.e. its spectrum is situated inside the domain

$$\Omega_\Sigma = \{ z = \rho e^{i\theta} : \rho_0 \leq \rho < \infty,\ -\varphi \leq \theta \leq \varphi, \varphi \in (0, \pi/2) \}$$

and on the boundary and outside of Ω_Σ the resolvent satisfies

$$\|(zI - A(t))^{-1}\| \leq \frac{M}{1 + |z|}$$

with some positive constant M.

Besides we make the following assumptions.

(BB1) The operator $A(t)$ possesses a domain $D(A(t)) = D(A(0))$ independent of t and satisfies the following Hölder condition

$$\left\| [A(t) - A(s)]\, A^{-1}(\tau) \right\|_{X\to X} \leq M\, |t - s|^{\varepsilon} \tag{3.2}$$

for arbitrary $t, s, \tau \in [0,1]$ with some positive constant M and $\varepsilon \in (0,1]$.

It is well known [18] that problem (3.1) possesses the unique continuously differentiable solution $u(t)$ for $t \in [0,1]$ and it holds

$$u(t) = U(t,0)\, u_0 + \int_0^t U(t,s)\, f(s)\, ds, \tag{3.3}$$

provided that $u_0 \in D(A(0))$ and $f(t)$ is continuously differentiable.

Here $U(t,s)$ is the evolution operator of problem (3.1) defined as the continuous solution of the integral equation

$$\begin{gathered} U(t,s) = \exp\{-(t-s)\, A(s)\} \\ + \int_s^t U(t,\eta)\, [A(s) - A(\eta)]\, \exp\{-(\eta - s)\, A(s)\} d\eta. \end{gathered} \tag{3.4}$$

The operator $U(t,s)$ satisfies the semi-group identity

$$\begin{gathered} U(t,s) = U(t,\eta)\, U(\eta,s), \\ 0 \leq s \leq \eta \leq t \leq 1. \end{gathered} \tag{3.5}$$

Using representation (3.3) and identity (3.5) we get the following exact two-level difference scheme

$$u_{t,k} + B_k u\left(t_k\right) = \varphi_k, \quad k = 0, 1, \cdots, N-1, \tag{3.6}$$

where $t_k = \tau k$, $u_{t,k} = \tau^{-1}(u(t_{k+1}) - u(t_k))$, $\quad k = 0, 1, \cdots, N, \quad \tau N = 1$,

$$B_k = \tau^{-1}\left(I - U\left(t_{k+1}, t_k\right)\right), \tag{3.7}$$

$$\varphi_k = \tau^{-1} \int\limits_{t_k}^{t_{k+1}} U\left(t_{k+1}, s\right) f\left(s\right) ds \tag{3.8}$$

The evolution operator $U(t,s)$ and the vectors $\varphi_k \in X$ of the exact difference scheme (3.6) can be found exactly in some very special cases only. That is why an other difference scheme

$$\begin{gathered} y_{t,k} + \tilde{B_k} y_k = \tilde{\varphi_k}, \quad k = 0, 1, \cdots, N-1, \\ y_0 = \tilde{u_0} \end{gathered} \tag{3.9}$$

is usually used which approximate scheme (3.6) in some sense.

For the error

$$z_k = u\left(t_k\right) - y_k$$

we get from (3.6), (3.9)

$$\begin{gathered} z_{t,k} + B_k z_k = \psi_k, \\ k = 0, 1, \cdots, N-1, \\ z_0 = u_0 - \tilde{u_0}, \end{gathered} \tag{3.10}$$

where

$$\psi_k = \left(\tilde{B_k} - B_k\right) y_k + \varphi_k - \tilde{\varphi_k}. \tag{3.11}$$

The solution of problem (3.10) can be written down in the form

$$\begin{gathered} z_k = U\left(t_k, 0\right) z_0 + \sum_{j=1}^{k} \tau U\left(t_k, t_j\right) \psi_j, \\ k = 0, 1, \cdots, N. \end{gathered} \tag{3.12}$$

It was shown in [18] that

$$\left\|U\left(t_k, t_n\right)\right\|_{X \to X} \leq M, \tag{3.13}$$

$$\left\|A\left(t_k\right) U\left(t_k, t_n\right) A^{-1}\left(t_n\right)\right\|_{X \to X} \leq M, \tag{3.14}$$

$$\left\|\tau\left(k-n\right) A\left(t_k\right) U\left(t_k, t_n\right)\right\|_{X \to X} \leq M, \tag{3.15}$$

$$0 \leq n \leq k \leq N,$$

provided that condition (3.2) holds. Using (3.13) – (3.15) we get from (3.12)

$$\begin{gathered} \left\|A\left(t_k\right) z_k\right\|_{C_\tau(X)} \leq M\left[\left\|A\left(0\right) z_0\right\|_X + \left\|\psi_k\right\|_{C_\tau(X)}\left(1 + \sum_{j=1}^{k-1} \frac{\tau}{\left(k-j\right)\tau}\right)\right] \leq \\ \leq M_1\left[\left\|A\left(0\right) z_0\right\|_X + \right. \\ + \ln\frac{1}{\tau}\left(\max_{0 \leq k \leq N-1}\left\|\left(\tilde{B_k} - B_k\right)\tilde{B_k}^{-1}\right\|_{X \to X}\left\|\tilde{B_k} y_k\right\|_{C_\tau(X)}\right. \\ \left.\left. + \left\|\varphi_k - \tilde{\varphi_k}\right\|_{C_\tau(X)}\right)\right] \end{gathered} \tag{3.16}$$

where

$$\left\|V_k\right\|_{C_\tau(X)} = \max_{0 \leq k \leq N}\left\|V_k\right\|_X$$

Let us estimate the term $\left\|\tilde{B}_k y_k\right\|_{C_\tau(X)}$ on the right-hand side of (3.16) through the initial data of problem (3.9). To this end we consider the operator

$$\tilde{U}\left(t_{k+1}, t_n\right) = \left(I - \tau\tilde{B}_k\right)\left(I - \tau\tilde{B}_{k-1}\right)\cdots\left(I - \tau\tilde{B}_{n+1}\right) \tag{3.17}$$

which satisfies the equations

$$\begin{gathered}\tilde{U}\left(t_k, t_s\right) = \left(I - \tau\tilde{B}_s\right)^{k-s} \\ + \sum_{p=s+1}^{k} \tilde{U}\left(t_k, t_p\right)\left[\tilde{B}_s - \tilde{B}_{p-1}\right]\left(I - \tau\tilde{B}_s\right)^{p-s-1},\end{gathered} \tag{3.18}$$

$$\begin{gathered}\tilde{U}\left(t_k, t_s\right) = \left(I - \tau\tilde{B}_{k-1}\right)^{k-s} \\ - \sum_{p=s}^{k-2} \tau\left(I - \tau\tilde{B}_{k-1}\right)^{k-p-1}\left[\tilde{B}_{k-1} - \tilde{B}_p\right]\tilde{U}\left(t_p, t_s\right),\end{gathered} \tag{3.19}$$

Let us further assume that

(BB2) The operators $\tilde{B}_k$, $k = 0, 1, \cdots, N$ are densely defined in X with the domains

$$D\left(\tilde{B}_k\right) = D\left(A\left(0\right)\right), \qquad k = 0, 1, \cdots, N \tag{3.20}$$

and for each fixed k the operators $\tilde{B}_k$ are strongly positive.

(BB3) The operators $\tilde{B}_k$, $k = 0, 1, \cdots, N$ satisfy the Lipschitz condition

$$\left\|\tilde{B}_n\left(\tilde{B}_k - \tilde{B}_n\right)\tilde{B}_k^{-1}\right\|_{X \to X} \leq M\left|t_k - t_n\right|. \tag{3.21}$$

(BB4) The inequalities

$$\begin{gathered}\left\|\left(I - \tau\tilde{B}_k\right)^n\right\|_{X \to X} \leq M, \qquad \left\|\tilde{B}_k\left(I - \tau\tilde{B}_k\right)^n\right\|_{X \to X} \leq \frac{M}{n\tau}, \\ \left\|\tau\tilde{B}_k\right\|_{X \to X} \leq M, \qquad k = 0, 1, \cdots, N-1, \quad n \geq 1\end{gathered} \tag{3.22}$$

with some positive constant M hold true.

Using (3.21), (3.22) we get from equation (3.18)

$$\left\|\tilde{U}\left(t_k, t_s\right)\right\|_{X \to X} \leq M\left\{1 + \sum_{p=s+1}^{k} \tau\left\|\tilde{U}\left(t_k, t_p\right)\right\|_{X \to X}\right\}. \tag{3.23}$$

The solutions of this inequality can be majorized by the solution of the recurrence system of equations

$$\begin{gathered}V\left(k, s\right) = M\left\{1 + \sum_{p=s+1}^{k} \tau V\left(k, p\right)\right\} \\ s = k-1, k-2, \cdots, 0, \\ V\left(k, k\right) = M.\end{gathered} \tag{3.24}$$

This solution is

$$V\left(k, s\right) = \left(1 + M\tau\right)^{k-s} M$$

and we arrive at the estimate

$$\begin{gathered}\left\|\tilde{U}\left(t_k, t_s\right)\right\| \leq \left(1 + M\tau\right)^{k-s} M \leq \left(1 + M\tau\right)^{N} M = \\ = \left[\left(1 + M\tau\right)^{\frac{1}{M\tau}}\right]^{M} M \leq e^{M} M,\end{gathered} \tag{3.25}$$

which is analogous to estimate (3.13).

Let us prove an analogon of estimate (3.15). Using inequalities (3.21), (3.22) we obtain from (3.19)

$$\left\|\tilde{B}_{k-1}\tilde{U}\left(t_k,t_s\right)\tilde{B}_s^{-1}\right\|_{X\to X} \le \left\|\left(I-\tau\tilde{B}_{k-1}\right)^{k-s}\right\|_{X\to X}\left\|\tilde{B}_{k-1}\tilde{B}_s^{-1}\right\|_{X\to X}$$
$$+\sum_{p=s}^{k-2}\tau\left\|\tilde{B}_{k-1}\left(I-\tau\tilde{B}_{k-1}\right)^{k-p-1}\right\|_{X\to X}\left\|\left[\tilde{B}_{k-1}-\tilde{B}_p\right]\tilde{B}_{p-1}^{-1}\right\|_{X\to X}\times$$
$$\times\left\|\tilde{B}_{p-1}\tilde{U}\left(t_p,t_s\right)\tilde{B}_s^{-1}\right\|_{X\to X} \le M\left(1+\sum_{p=s}^{k-2}\tau\left\|\tilde{B}_{p-1}\tilde{U}\left(t_p,t_s\right)\tilde{B}_s^{-1}\right\|_{X\to X}\right),$$

which looks similar to (3.23). Thus, in accordance with (3.25) we get for its solution

$$\left\|\tilde{B}_{k-1}\tilde{U}\left(t_k,t_s\right)\tilde{B}_s^{-1}\right\|_{X\to X} \le Me^M. \tag{3.26}$$

It remains to find an analogon of estimate (3.15). Starting with (3.19) and taking into account (3.21), (3.22) we arrive at

$$\left\|\tilde{B}_{k-1}\tilde{U}\left(t_k,t_s\right)\right\|_{X\to X} \le \left\|\tilde{B}_{k-1}\left(I-\tau\tilde{B}_{k-1}\right)^{k-s}\right\|_{X\to X}$$
$$+\sum_{p=s}^{k-2}\tau\left\|\tilde{B}_{k-1}\left(I-\tau\tilde{B}_{k-1}\right)^{k-p-1}\right\|_{X\to X}\left\|\left[\tilde{B}_{k-1}-\tilde{B}_p\right]\tilde{B}_{p-1}^{-1}\right\|_{X\to X}\times \tag{3.27}$$
$$\times\left\|\tilde{B}_{p-1}\tilde{U}\left(t_p,t_s\right)\right\|_{X\to X} \le \frac{M}{(k-s)\tau}+M\sum_{p=s}^{k-1}\tau\left\|\tilde{B}_{p-1}\tilde{U}\left(t_p,t_s\right)\right\|_{X\to X}$$

Solutions of inequality (3.27) are majorized by the solution of the system of equations

$$\begin{gathered} V(k,s)=\frac{M}{(k-s)\tau}+M\sum_{p=s}^{k-1}\tau V(p,s), \\ k=s+1,s+2,\cdots,N, \\ \tau V(s,s)=M, \end{gathered} \tag{3.28}$$

which yields

$$\begin{gathered} V(k+1,s)=\alpha V(k,s)-\frac{M}{\tau(k+1-s)(k-s)} \\ k=s+1,s+2,\cdots,N, \\ V(s+1,s)=\frac{m}{\tau}+M^2, \quad \alpha=(1+\tau M). \end{gathered} \tag{3.29}$$

From (3.28), (3.29) we get the estimate

$$V(k,s)=\alpha^{k-s-1}V(s+1,s)-\frac{M}{\tau}\left(\frac{\alpha^{k-s-2}}{2\cdot 1}+\frac{\alpha^{k-s-3}}{3\cdot 2}+\cdots\right.$$
$$\left.+\frac{1}{(k-s)(k-s-1)}\right)$$
$$\le \alpha^{k-s-1}\left(\frac{M}{\tau}+M^2\right)-\frac{M}{\tau}\left(\frac{1}{2\cdot 1}+\frac{1}{3\cdot 2}+\cdots+\frac{1}{(k-s)(k-s-1)}\right)$$
$$=\alpha^{k-s-1}\left(\frac{M}{\tau}+M^2\right)-\frac{M}{\tau}\left(1-\frac{1}{k-s}\right)$$

$$\leq \left(e^{M}-1\right)\frac{M}{\tau}+e^{M}M^{2}+\frac{M}{\tau\left(k-s\right)}\leq\frac{M_{1}}{\tau\left(k-s\right)},$$
$$k=s+1,s+2,\cdots,N,$$

which implies

$$\left\|\tilde{B}_{k-1}\tilde{U}\left(t_{k},t_{s}\right)\right\|_{X\to X}\leq\frac{M_{1}}{\left(k-s\right)\tau}. \tag{3.30}$$

The estimates (3.25), (3.26), (3.30) lead to the following assertion.

Theorem 3.1. *Let the operators $\tilde{B}_k$, $k=0,1,\cdots,N-1$ satisfy assumptions* **(BB2)**-**(BB4)**, *then for the solution of problem (3.9) the following estimate holds true*

$$\left\|\tilde{B}_{k}y_{k}\right\|_{C_{\tau}(X)}\leq\left(\left\|\tilde{B}_{0}y_{0}\right\|_{X}+\left\|\tilde{\varphi}_{j}\right\|_{C_{\tau}(X)}\ln\frac{1}{\tau}\right). \tag{3.31}$$

Proof. The solution of problem (3.9) can be written down in the form

$$y_{k}=\tilde{U}\left(t_{k},0\right)y_{0}+\sum_{j=1}^{k}\tau\tilde{U}\left(t_{k},t_{i}\right)\tilde{\varphi}_{j}.$$

Together with (3.26), (3.30) this representation implies

$$\begin{gathered}\left\|\tilde{B}_{k}y_{k}\right\|_{C_{\tau}(X)}\leq\\ \leq\left\|\tilde{B}_{k}\tilde{B}_{k-1}^{-1}\right\|_{X\to X}\left\{\left\|\tilde{B}_{k-1}\tilde{U}\left(t_{k},0\right)\tilde{B}_{0}^{-1}\right\|_{X\to X}\left\|\tilde{B}_{0}y_{0}\right\|_{X}+\right.\\ \left.+\sum_{j=1}^{k}\tau\left\|\tilde{B}_{k-1}\tilde{U}\left(t_{k},t_{i}\right)\right\|_{X\to X}\left\|\tilde{\varphi}_{j}\right\|_{X}\right\}\leq M\left(\left\|\tilde{B}_{0}y_{0}\right\|_{X}+\right.\\ \left.+\left\|\tilde{\varphi}_{j}\right\|_{C_{\tau}(X)}\left(\sum_{j=1}^{k-1}\frac{1}{k-j}+1\right)\right).\end{gathered} \tag{3.32}$$

Let us prove the auxiliary estimate

$$\sum_{j=1}^{k-1}\frac{1}{k-j}\leq\frac{1}{\ln 2}\ln k. \tag{3.33}$$

Actually, the function

$$f\left(k\right)=\sum_{j=1}^{k-1}\frac{1}{k-j}-a\ln k,\qquad k=2,3,\cdots,\quad f\left(1\right)=0,$$

satisfies

$$f\left(k+1\right)-f\left(k\right)=g\left(k\right)$$

with

$$g\left(k\right)=\frac{1}{k}-a\ln\frac{k+1}{k}.$$

Since

$$g'\left(k\right)=\frac{-k-1+ak}{k^{2}\left(k+1\right)},$$

the derivative $g'\left(k\right)$ is non-positive for all $k=2,3,\cdots$ provided hat

$$a\leq\frac{3}{2}. \tag{3.34}$$

The function $g(x)$ is non-increasing for these values of a which means that

$$g(k) = f(k+1) - f(k) \le g(1) = 1 - a\ln 2$$

and further, choosing

$$a = 1/\ln 2, \tag{3.35}$$

we get

$$f(k) \le f(k-1) \le \cdots \le f(2) = 1 - a\ln 2 = 0.$$

Thus, condition (3.35) provides that all $f(k)$, $k = 1, 2, 3, \cdots$ are non-positive and we arrive at (3.33).

Inequality (3.32) together with (3.33) imply (3.31) which proves the theorem.

Now, we are in the position to prove the main result of this section.

Theorem 3.2. *Under assumptions* **(BB1)**- **(BB4)** *the two-level exact difference scheme (3.6) is strongly stable, i.e. the following stability estimate holds true*

$$\begin{aligned} \|A(t_k) z_k\|_{C_\tau(X)} \le M \Bigg\{ \|A(0) z_0\|_X + \ln\frac{1}{\tau}\Bigg[\max_{0\le k\le N-1} \left\| \left(\tilde{B}_k - B_k\right)\tilde{B}_k^{-1} \right\|_{X\to X} \cdot \\ \cdot \left(\left\|\tilde{B}_0 y_0\right\|_X + \ln\frac{1}{\tau}\|\tilde{\varphi}_j\|_{C_\tau(X)} \right) + \|\varphi_k - \tilde{\varphi}_k\|_{C_\tau(X)} \Bigg]\Bigg\}. \end{aligned} \tag{3.36}$$

Proof. The proof is due to (3.16) and Theorem 3.1.

Let us consider the following example of implicit difference scheme (3.9)

$$\begin{aligned} \frac{y_k - y_{k-1}}{\tau} + A(t_k) y_k = f(t_k), \qquad k = 1, 2, \cdots, N, \\ y_0 = u_0, \end{aligned} \tag{3.37}$$

which approximates the exact difference scheme (3.6). This scheme can be easily transformed to the form (3.9) with

$$\begin{aligned} \tilde{B}_k = \frac{1}{\tau}\left[I - (I + \tau A(t_{k-1}))^{-1}\right] = A(t_{k-1})\left[I + \tau A(t_{k-1})\right]^{-1}, \\ \tilde{\varphi}_k = (I + \tau A(t_{k-1}))^{-1} f(t_{k-1}), \qquad k = 1, 2, \cdots N. \end{aligned} \tag{3.38}$$

Let us check the conditions (3.21), (3.22). We have

$$\begin{aligned} &\left\|\tilde{B}_n\left(\tilde{B}_k - \tilde{B}_n\right)\tilde{B}_k^{-1}\right\|_{X\to X} \\ &= \left\|(I + \tau A(t_{k-1}))^{-1}\left[A(t_{k-1}) - A(t_{n-1})\right]A^{-1}(t_{k-1})\right\|_{X\to X} \\ &\le M\left\|\left[A(t_{k-1}) - A(t_{n-1})\right]A^{-1}(t_{k-1})\right\|_{X\to X} \le M|t_k - t_n|^{\varepsilon} \end{aligned}$$

and one can recognize that due to (3.2) condition (3.21) is fulfilled with $\varepsilon = 1$.

The following statement was proved in [19]: an operator A with a dense domain $D(A)$ in a Banach space X is strongly positive generator of an analytic semigroup $(\exp -tA)$ which satisfies the estimates

$$\|\exp(-tA)\|_{X\to X} \le M, \qquad \|A\exp(-tA)\|_{X\to X} \le \frac{M}{t}$$

iff the estimates

$$\left\|(I + \tau A)^{-k}\right\|_{X\to X}, \qquad \left\|A(I + \tau A)^{-k}\right\|_{X\to X} \le \frac{M}{k\tau} \qquad k > 1 \tag{3.39}$$

hold with M independent of τ.

This theorem implies

$$\left\|\left(I-\tau\tilde{B}_k\right)^n\right\|_{X\to X}=\left\|\left(I+\tau A\left(t_{k-1}\right)\right)^{-n}\right\|_{X\to X}\leq M \tag{3.40}$$

and, further,

$$\begin{gathered}\left\|\tilde{B}_k\left(I-\tau\tilde{B}_k\right)^n\right\|_{X\to X}=\left\|A\left(t_{k-1}\right)\left[I+\tau A\left(t_{k-1}\right)\right]^{-n-1}\right\|_{X\to X}\leq\\ \leq\frac{M}{(n+1)\,\tau}<\frac{M}{n\tau},\\ \tau\left\|\tilde{B}_k\right\|_{X\to X}=\left\|I-\left(I+\tau A\left(t_{k-1}\right)\right)^{-1}\right\|_{X\to X}\leq 1+M.\end{gathered}$$

Thus, the operators $\tilde{B}_k$, $k=0,1,\cdot,N-1$ satisfy the assumptions of Theorem 3.2 which asserts that the exact two-level difference scheme (3.6) is strongly stable as compared with the scheme (3.9), (3.38) and estimate (3.36) holds true.

In order to get an accuracy estimate for the difference scheme (3.9), (3.37) one has to estimate the terms

$$\varphi_k-\tilde{\varphi}_k,\quad \left[\tilde{B}_k-B_k\right]\tilde{B}_k^{-1}.$$

For this end we represent

$$\begin{gathered}\left\|\varphi_k-\tilde{\varphi}_k\right\|_X=\left\|\frac{1}{\tau}1\left[U\left(t_{k+1},s\right)-I\right]f\left(s\right)ds+\frac{1}{\tau}\int\limits_{t_k}^{t_{k+1}}\left[f\left(s\right)-\right.\right.\\ \left.\left.-f\left(t_k\right)\right]ds+\left[I-\left(I+\tau A\left(t_k\right)\right)^{-1}\right]f\left(t_k\right)\right\|_X\end{gathered} \tag{3.41}$$

and estimate successively each summand under the norm.

Using integral equation (3.4) we get

$$\begin{gathered}\left[U\left(t_{k+1},s\right)-I\right]f\left(s\right)=\left[\exp\left\{-\left(t_k-s\right)A\left(s\right)\right\}-I\right]f\left(s\right)+\\ +\int\limits_{t_k}^{t_{k+1}}U\left(t_{k+1},\eta\right)\left[A\left(s\right)-A\left(\eta\right)\right]\exp\left\{-\left(\eta-s\right)A\left(s\right)\right\}f\left(s\right)d\eta.\end{gathered} \tag{3.42}$$

We assume that $f\left(s\right)\in D\left(A\left(0\right)\right)$ and $f\left(s\right)$ is continuously differentiable, then we get

$$\begin{gathered}\left\|\left[\exp\left\{-\left(t_{k+1}-s\right)A\left(s\right)\right\}-I\right]f\left(s\right)\right\|_X=\\ =\left\|-\int\limits_{s}^{t_{k+1}}\exp\left\{-\left(t_{k+1}-s\right)A\left(s\right)\right\}A\left(s\right)f\left(s\right)d\eta\right\|_X\leq\\ \leq M\tau\left\|A\left(0\right)f\left(s\right)\right\|_X\leq M\tau\left\|A\left(0\right)f\left(s\right)\right\|_{C(X)}\end{gathered} \tag{3.43}$$

and

$$\begin{gathered}\left\|\frac{1}{\tau}\int\limits_{t_k}^{t_{k+1}}\left[f\left(s\right)-f\left(t_k\right)\right]ds\right\|_X=\left\|\frac{1}{\tau}\int\limits_{t_k}^{t_{k+1}}\int\limits_{t_k}^{s}f'\left(\eta\right)d\eta ds_X\right\|\leq\\ \leq\frac{\tau}{2}\left\|f'\left(t\right)\right\|_{C(X)}.\end{gathered} \tag{3.44}$$

Finally, we have

$$\left\|\left[I-\left(I+\tau A\left(t_k\right)\right)^{-1}\right]f\left(t_k\right)\right\|_X\leq M\left\|A\left(t_k\right)f\left(t_k\right)\right\|_X\tau. \tag{3.45}$$

Taking into account (3.42)–(3.45) the equality (3.41) yields

$$\|\varphi_k - \tilde{\varphi}_k\|_{C_\tau(X)} \leq$$
$$\leq \tau M \left\{ \|A(0) f(x)\|_{C(X)} + \|f'(x)\|_{C(X)} + \|A(t_k) f(t_k)\|_{C_\tau(X)} \right\} \tag{3.46}$$

where

$$\|V(t)\|_{C(X)} = \max_{t\in[0,1]} \|V(t)\|_X .$$

Let us go over to the estimating of the operator

$$\left[\tilde{B}_k - B_k\right] \tilde{B}_k^{-1} = \left[U(t_{k+1}, t_k) - (I + \tau A(t_k))^{-1}\right] A^{-1}(t_k) [I$$
$$+\tau A(t_k)] = I_1 + I_2, \tag{3.47}$$

where

$$I_1 = \left[\exp\{-\tau A(t_k)\} - (I + \tau A(t_k))^{-1}\right] A^{-1}(t_k) [I + \tau A(t_k)],$$

$$I_2 = \int_{t_k}^{t_{k+1}} U(t_{k+1}, \eta) [A(t_k) - A(\eta)] \exp\{-(\eta - t_k) A(t_k)\} d\eta \times$$
$$\times A^{-1}(t_k) [I + \tau A(t_k)].$$

The operator I_1 can be transformed to

$$I_1 = \left[I - \int_{t_k}^{t_{k+1}} \exp\{-(t_k - \eta) A(t_k)\} A(t_k) d\eta - (I + \tau A(t_k))^{-1} \right] \times$$

$$\times A^{-1}(t_k) [I + \tau A(t_k)] = \tau - \int_{t_k}^{t_{k+1}} \exp\{-(t_k - \eta) A(t_k)\} d\eta [I + \tau A(t_k)]$$

$$= - \int_{t_k}^{t_{k+1}} \exp\{-(t_k - \eta) A(t_k)\} d\eta + \tau \exp\{-\tau A(t_k)\},$$

which implies the estimate

$$\|I_1\|_{X\to X} \leq M\tau.$$

For the operator I_2 we have

$$\|I_2\|_{X\to X} \leq \int_{t_k}^{t_{k+1}} \|U(t_{k+1}, t_\eta)\|_{X\to X} \left\|[A(t_k) - A(\eta)] A^{-1}(t_k)\right\|_{X\to X} \times$$
$$\times \|\exp\{-(\eta - t_k) A(t_k)\} [I + \tau A(t_k)]\|_{X\to X} d\eta \leq$$
$$\leq M \int_{t_k}^{t_{k+1}} (\eta - t_k) \left(1 + \frac{\tau}{\eta - t_k}\right) d\eta \leq \frac{3}{2} M\tau^2.$$

These last two inequalities imply the estimate

$$\max_{0\leq k\leq N-1} \left\| \left[\tilde{B}_k - B_k\right] \tilde{B}_k^{-1} \right\|_{X\to X} \leq M\tau. \tag{3.48}$$

Estimates (3.46), (3.48) together with Theorem 3.2 lead to the following second main result of this section.

Theorem 3.3. *Let the assumptions of Theorem 3.2 be fulfilled and the vector-valued function $f(t) \in D(A(0))$ be continuously differentiable on $[0,1]$, then for difference scheme (3.38) the following accuracy estimate holds*

$$\|A(t_k) z_k\|_{C_\tau(X)} \leq \tau \ln \frac{1}{\tau} M, \tag{3.49}$$

where the constant M is independent of τ.

Remark 3.4. The investigation technique of this section is closed to that of [1] but the main results, namely Theorems 3.2, 3.3 are new.

Remark 3.5. Due to the factor $\ln \frac{1}{\tau}$ on the right-hand side of estimate (3.36)(i.e. the multiplicative constant on the right-hand side depends weakly on the step-size) it would be more correct to call this estimate as an estimate of the "almost" strong stability. We have obtained this estimate as well as the accuracy estimate (3.49) in a strong norm. One can try to get similar estimates without the logarithmic factor in the weaker norm of the Banach space X. But when using the estimate

$$\max_{0 \leq k \leq N-1} \left\| \left[\tilde{B}_k - B_k \right] \tilde{B}_k^{-1} \right\|_{X \to X} \leq M\tau$$

one needs estimates for $\left\| \tilde{B}_k y_k \right\|_{C_\tau(X)}$, i.e. the assertion of Theorem 3.1, which leads again to presence of the factor $\ln \frac{1}{\tau}$ in the resulting estimates.

4. ρ-STABILITY OF THREE-LEVEL DIFFERENCE SCHEMES AND STRONGLY P-POSITIVE OPERATORS

Second order differential equations with operator coefficients are a powerful mathematical tool in the description and study of evolutionary partial differential equations arising in various fields of applications. In the numerical solution of evolution problems, the problem of stability of numerical methods with respect to initial data is of great importance. Considering these methods as difference schemes with operator coefficients provides a suitable model for stability analysis.

In this section we discuss some results from [15] concerning difference schemes for the following initial value problem:

$$\frac{d^2u}{dt^2} + Au = 0, \quad t \in (0,T], \quad u(0) = u_0, \ u^{'}(0) = u_1, \tag{4.1}$$

where $u : R_+ \to X$ is a vector-valued function, A is a linear, densely defined, closed operator with domain $D(A)$ in a Banach space X with norm $\|\cdot\| \equiv \|\cdot\|_X$. In particular, the equation (4.1) with the Laplace operator $A = -\Delta$ is the well-known wave equation.

Due to the presence of the second-order time derivative in (4.1), difference schemes for the numerical solution of this problem have at least three time levels, i.e. they involve approximate values y_n for $u(t_n)$ at three neighboring points of the time grid $\omega_\tau = \{t_i : i = 0, 1, 2, \ldots, t_0 = 0, t_i - t_{i-1} = \tau\}$.

Unfortunately, known stability results do not include certain important classes of difference schemes with unbounded operator coefficients in a Banach space. Such results are also important for finite difference and finite element approximations of partial differential operators since the norms of these approximations tend to infinity if the discretization parameter tends to zero.

The aim of this section is to discuss some stability results for regularized three-level difference schemes with unbounded operator coefficients in a Banach space. Note that the initial difference scheme (without regularization) can be unstable.

We consider the following family of three-level difference schemes:

$$(I+\alpha A)y_{\bar{t}t,n}+\beta A y_{\overset{\circ}{t},n}+Ay_n=0, \qquad n=1,2,\dots, \tag{4.2}$$

with given y_0, y_1, where

$$y_{\overset{\circ}{t},n}=\frac{y_{n+1}-y_n}{2\tau}, \qquad y_{\bar{t}t,n}=\frac{y_{n+1}-2y_n+y_{n-1}}{\tau^2},$$

α,β are parameters and A is a linear, densely defined, closed operator in a Banach space X. If $\alpha=\beta=0$ then we have the explicit difference scheme for the equation (4.1) which is unstable in the case of an unbounded operator A.

The difference scheme (4.2) can be written down as

$$[I+(\alpha+\frac{\beta\tau}{2})A]y_{n+1}-2[I+(\alpha-\frac{\tau^2}{2})A]y_n+[I+(\alpha-\frac{\beta\tau}{2})A]y_{n-1}=0, \qquad n=1,2,\dots \tag{4.3}$$

In order to get an explicit formula for the solution of (4.3) let us consider the scalar recurrence equation

$$au_{n+1}-2bu_n+cu_{n-1}=0$$

with constant coefficients a,b,c. Setting $u_n=r^n\tilde{u}_n$ we get

$$\tilde{u}_{n+1}-2\frac{b}{ar}\tilde{u}_n+\frac{c}{ar^2}\tilde{u}_{n-1}=0$$

and with $r=\sqrt{ca^{-1}}, x=b\sqrt{ca^{-1}}$ we have

$$\tilde{u}_{n+1}-2x\tilde{u}_n+\tilde{u}_{n-1}=0.$$

This is the recurrence equation which satisfies both Chebyshev polynomials of first kind $T_n(x)$ and of second kind $U_n(x)$ (see [20]). Since U_{n-1} and U_{n-2} are linear independent and by definition $U_{-2}(x)=-1,\ U_{-1}(x)=0$, we can write down $u_n,\ n=0,1,\dots$ with initial values $u_0,\ u_1$ as follows

$$u_n=(\sqrt{ca^{-1}})^n[-U_{n-2}(x)u_0+(\sqrt{ca^{-1}})^{-1}U_{n-1}(x)u_1].$$

Thus, denoting

$$\chi(\alpha,\beta,A)\equiv\chi(A)=[I+(\alpha-\frac{\tau^2}{2})A]\left\{[I+(\alpha+\frac{\beta\tau}{2})A][I+(\alpha-\frac{\beta\tau}{2})A]\right\}^{-1/2},$$

$$Q(\alpha,\beta,A)\equiv Q(A)=\left\{[I+(\alpha+\frac{\beta\tau}{2})A]^{-1}[I+(\alpha-\frac{\beta\tau}{2})A]\right\}^{1/2}$$

the solution of (3) can be obviously represented by

$$y_n=Q^n(A)[-U_{n-2}(\chi)y_0+Q^{-1}(A)U_{n-1}(\chi)y_1]. \tag{4.4}$$

Next, we introduce two definitions which we will use in our analysis.

Definition 4.1. Given a function $\rho=\rho(\tau)$ and a real $\sigma\geq 0$ we say that the scheme (2) is ρ-stable with respect to initial data in the domain $D(A^\sigma)$ of the operator A^σ if there exists a constant M independent of n such that the estimate

$$\|y_n\|\leq M\rho^n\left(\|\frac{y_0+y_1}{2}\|_\sigma+\|\frac{y_1-y_0}{\tau}\|_\sigma\right) \tag{4.5}$$

holds for any $y_0,y_1\in D(A^\sigma)$ with $\|u\|_\sigma=\|A^\sigma u\|$.

Here and below we denote by $M,M_1,\dots,C,C_1,\dots,c,c_1,\dots$ various positive constants.

A principal role in our stability analysis will play the strongly P-positive operators which have been introduced in [4]. Let Γ_0 be a counter-clock wise oriented parabola $y^2 = \frac{c_0^2}{2}x + c_1$, $c_0,\ c_1 > 0$ (see Fig. 1). We denote by Ω_{Γ_0} the domain lying inside of Γ_0. Now, we are in the position to give the definition of the strong P-positivity.

Definition 4.2. We say that an operator $A : D(A) \subset X \to X$ is strongly P-positive if its spectrum $\Sigma(A)$ lies in the domain Ω_{Γ_0} and on Γ_0 and outside of Γ_0 the estimate

$$\|(z-A)^{-1}\|_{X\to X} \leq \frac{M}{1+\sqrt{|z|}} \tag{4.6}$$

holds with a positive constant M.

Remark 4.3. The form of the path $\Gamma_0 \equiv \Gamma_0(z)$ for a bounded z is not essential for our analysis. What important is the behavior of the resolvent and of $\Sigma(A)$ at infinity, i.e. that Γ is a parabola and the estimate (4.6) holds for $|z| \to \infty$.

Example 4.4. This example shows that there exist important classes of the partial differential operators which are strongly P-positive [6].

Let $V \subset X \equiv H \subset V^*$ be a triple of Hilbert spaces and let $a(\cdot,\cdot)$ be a sesquilinear form on V. We denote by c_e the constant from the imbedding inequality $\|u\|_X \leq c_e\|u\|_V, :$ $\forall u \in V$. Assume that $a(\cdot,\cdot)$ is bounded, i.e.,

$$|a(u,v)| \leq c\|u\|_V\|v\|_V \quad \text{for all } u,v \in V$$

The boundedness of $a(\cdot,\cdot)$ implies the well-posedness of the continuous operator $A : V \to V^*$ defined by

$$a(u,v) ={}_{V^*}< Au, v >_V \quad \text{for all } \in V.$$

As usual, one can restrict A to a domain $D(A) \subset V$ and consider A as an (unbounded) operator in H. The assumptions

$$\begin{aligned} \Re e\ a(u,u) &\geq \delta_0\|u\|_V^2 - \delta_1\|u\|_X^2 \text{ for all } u \in V, \\ |\Im m\ a(u,u)| &\leq \kappa\|u\|_V\|u\|_X \text{for all } u \in V \end{aligned} \tag{4.7}$$

guarantee that the numerical range $\{a(u,u) : u \in X \text{ with } \|u\|_X = 1\}$ of A (and $sp(A)$) lies in Ω_{Γ_0}, where the parabola Γ_0 depends on the constants $\delta_0, \delta_1, \kappa, c_e$. Actually, if $a(u,u) = \xi_u + i\eta_u$ then we get

$$\begin{aligned} \xi_u = \Re e\ a(u,u) &\geq \delta_0\|u\|_V^2 - \delta_1 \geq \delta_0 c_e^{-2} - \delta_1, \\ |\eta_u| = |\Im m\ a(u,u)| &\leq \kappa\|u\|_V. \end{aligned}$$

It implies

$$\xi_u > \delta_0 c_e^{-2} - \delta_1, \quad \|u\|_V^2 \leq \frac{1}{\delta_0}(\xi_u + \delta_1), \quad |\eta_u| \leq \kappa\sqrt{\frac{\xi+\delta_1}{\delta_0}}. \tag{4.8}$$

The first and the last inequalities in (4.8) mean that the parabola $\Gamma_\delta = \{z = \xi + i\eta : \xi = \frac{\delta_0}{\kappa}\eta^2 - \delta_1,\ \eta \in (-\infty,\infty)\}$ contains the numerical range of A. Supposing that $\Re e\ sp(A) > \gamma_1 > \gamma_0 = b_0$ one can easily see that there exists another parabola (called the spectral parabola) $\Gamma_S = \{z = \xi + i\eta : \xi = a_0\eta^2 + b_0,\ \eta \in (-\infty,\infty)\}$ with $a_0 = \frac{(\gamma_1-\gamma_0)\delta_0}{(\gamma_1+\delta_1)\kappa}$ in the right-half plane containing $sp(A)$ (see Fig. 1). Note that $\delta_0 c_e^{-2} - \delta_1 > 0$ is the sufficient condition for $\Re e\, sp(A) > 0$ and in this case one can choose $\gamma_1 = \delta_0 c_e^{-2} - \delta_1$.

Analogously to [4] it can be shown that inequality (4.6) holds true in $\mathbb{C}\backslash\Omega_{\Gamma_S}$ (see the discussion in [4][pp. 330-331]).

Example 4.5. This example shows how one can derive the parameters of the parabola from the coefficients of the elliptic second order partial differential operator [6].

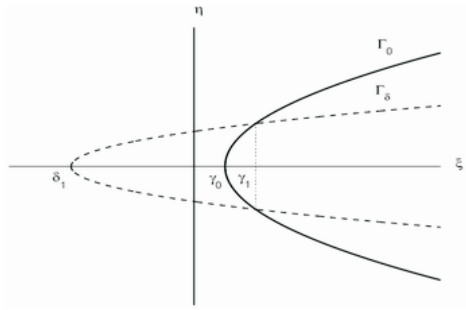

FIGURE 1. Parabolae Γ_δ and Γ_0

Let us consider the strongly elliptic differential operator

$$\mathcal{L} := -\sum_{j,k=1}^{d} \partial_j a_{jk} \partial_k + \sum_{j=1}^{d} b_j \partial_j + c_0 \qquad (\partial_j := \frac{\partial}{\partial x_j}) \tag{4.9}$$

with smooth (in general complex) coefficients a_{jk}, b_j and c_0 in a domain Ω with a smooth boundary. For the ease of presentation, we consider the case of Dirichlet boundary conditions. We suppose that $a_{pq} = a_{qp}$ and the following ellipticity condition holds

$$\sum_{i,j=1}^{d} a_{ij}\, y_i\, y_j \geq C_1 \sum_{i=1}^{d} y_i^2.$$

This operator is associated with the sesquilinear form

$$a(u,v) = \int_\Omega \left(\sum_{i,j=1}^{d} a_{ij}\, \partial_i u\, \overline{\partial_j v} + \sum_{j=1}^{d} b_j\, \partial_j u\, \overline{v} + c_0 u \overline{v} \right) d\Omega.$$

Lemma 4.6. *The spectrum of the operator $\mathcal{L}$ lies inside of the parabola $\Gamma_\delta = \{z = \xi + i\eta : \xi = \frac{\delta_0}{\kappa}\eta^2 - \delta_1,\ \eta \in (-\infty, \infty)\}$ with parameters defined by $\delta_0 = C_1, \delta_1 = C_2, \kappa = C_3$ where*

$$C_2 := \inf_{x\in\Omega} \left| \frac{1}{2} \sum\nolimits_j \frac{\partial b_j}{\partial x_j} - c_0 \right|, \qquad C_3 := \sqrt{d} \max_{x,j} |b_j(x)|, \tag{4.10}$$

C_1 is the constant from the inequality of the strong ellipticity and C_F is the constant from the Friedrich's inequality.

Proof. Let $\lambda = \xi + i\eta$ be an eigenvalue corresponding to the eigenfunction $u = u_r + iu_i$. Then we have

$$a(u,u) = \int_\Omega \left(\sum_{i,j=1}^{d} a_{ij}\, \partial_i u\, \overline{\partial_j u} + \sum_{j=1}^{d} b_j\, \partial_j u\, \overline{u} + c_0 u \overline{u} \right) d\Omega$$

$$= \int_\Omega \left(\sum_{i,j=1}^{d} a_{ij}\, \partial_i u\, \overline{\partial_j u} + \sum_{j=1}^{d} b_j (\partial_j u_r + i u_i)(u_r - i u_i) + c_0 |u|^2 \right) d\Omega$$

$$= \int_\Omega \left(\sum_{i,j=1}^{d} a_{ij}\, \partial_i u\, \overline{\partial_j u} + \sum_{j=1}^{d} b_j [\partial_j u_r u_r - i \partial_j u_r u_i + i \partial_j u_i u_r + \partial_j u_i u_i] + c_0 |u|^2 \right) d\Omega$$

$$= \int_\Omega \left(\sum_{i,j=1}^d a_{ij}\, \partial_i u\, \overline{\partial_j u} + \sum_{j=1}^d b_j [\frac{1}{2}\partial_j (u_r^2 + u_i^2) - i\partial_j u_r u_i + i\partial_j u_i u_r] + c_0 |u|^2 \right) d\Omega$$

$$= \int_\Omega \left(\sum_{i,j=1}^d a_{ij}\, \partial_i u\, \overline{\partial_j u} - \sum_{j=1}^d \frac{1}{2}\partial_j b_j |u|^2 - b_j [i\partial_j u_r u_i - i\partial_j u_i u_r] + c_0 |u|^2 \right) d\Omega$$

$$= (\xi + i\eta)\|u\|_0^2 \tag{4.11}$$

By separation of the real and imaginary parts we further get

$$\Re\, e\; a(u,u) = \int_\Omega \left(\sum_{i,j=1}^d a_{ij}\, \partial_i u\, \partial_j \overline{u} + (c_0 - \frac{1}{2}\sum_{j=1}^d \partial b_j)|u|^2 \right) d\Omega,$$

$$\Im\, m\; a(u,u) = -\int_\Omega b_j [\partial_j u_r u_i - \partial_j u_i u_r] d\Omega \tag{4.12}$$

$$= \frac{1}{2i}\sum_{j=1}^d \int_\Omega b_j\, (u \cdot \partial_j \overline{u} - \overline{u}\partial_j u)\, d\Omega,$$

where

$$\|u\|_0^2 = \|u\|_{L_2(\Omega)}^2 = \int_\Omega |u|^2 d\Omega. \tag{4.13}$$

Let $|u|_1^2 = \sum_j |\partial_j u|^2$ be the semi-norm of the Sobolev space $H^1(\Omega)$, $\|\cdot\|_k$ be the norm of the Sobolev space $H^k(\Omega)$ $(k = 0, 1, \dots)$ with $H^0(\Omega) = L_2(\Omega)$, and choose $\|u\|_0 = 1$. Using the elipticity condition

$$\sum_{i,j=1}^d a_{ij}\, \partial_i u\, \partial_j \overline{u} \geq C_1 \sum_{j=1}^d |\partial_j u|^2, \tag{4.14}$$

the Cauchy-Bunjakovski inequality

$$\int_\Omega u \sum_{j=1}^d b_j \partial_j u d\Omega \leq \|u\|_0 \sqrt{\int_\Omega \left(\sum_{j=1}^d b_j \partial_j u \right)^2} \tag{4.15}$$

and the inequality $\left(\sum_{j=1}^d s_j\right)^2 \leq \sqrt{d}\sum_{j=1}^d s_j^2$ we get the estimates (compare with (4.7))

$$\begin{aligned} \xi = \Re\, e\; a(u,u) &\geq C_1 |u|_1^2 - C_2 \|u\|_0^2, \\ |\eta| = |\Im\, m\; a(u,u)| &\leq C_3 \|u\|_0 |u|_1 \end{aligned} \tag{4.16}$$

with

$$\begin{aligned} \delta_0 = C_1, C_2 = \delta_1 = \max_{x\in\Omega} \left| c_0 - \frac{1}{2}\sum_{j=1}^d \partial_j b_j \right|, \\ C_3 = \kappa = \sqrt{d}\max_{j,x} |b_j(x)|. \end{aligned} \tag{4.17}$$

Let C_F be the constant from the Friedrichs inequality

$$\|u\|_0 \geq C_F |u|_1 \qquad \text{for all } u \in H_0^1(\Omega).$$

This constant can be estimated by $C_F \leq c_0 (mes\ \Omega)^{1/d}$, where $c_0 = 2\frac{d-1}{d-2}$ for $d > 2$ and $c_0 = 3/2$ for $d \leq 2$ (see [10], p.71). Now, the first inequality in (4.16) implies

$$\xi = \Re\, e\; a(u,u) \geq \delta_0 |u|_1^2 - \delta_1 \|u\|_0^2 \geq \delta_0 C_F^{-2} - \delta_1. \tag{4.18}$$

It is easy to show that in this case with $V = H_0^1(\Omega), H = L_2(\Omega)$ it holds $\xi_u \geq C_1|u|_1^2 - C_2\|u\|_0 \geq C_1C_F - C_2,\ |\eta_u| \leq C_3|u|_1 \leq C_3\sqrt{(\xi_u + C_2)/C_1}$, so that the parabola Γ_δ is defined by the parameters $\delta_0 = C_1, \delta_1 = C_2, \kappa = C_3$ and the lower bound of $sp(A)$ can be estimated by $\gamma_1 = C_1C_F^{-2} - C_2 > \gamma_0$. Now, the spectral parabola Γ_S in the right half-plane can be constructed as above by putting $a_0 = \frac{(\gamma_1-\gamma_0)\delta_0}{(\gamma_1+\delta_1)\kappa}$.

The strongly elliptic partial differential operators with $\Re\ \Sigma(A) > 0$ are important examples of both strongly P-positive and strongly positive operators (also sectorial operators or infinitesimal generators of holomorphic semigroups). The framework of the strongly P-positivity is important for studying cosine families of operators related to the equation (4.1) (see e.g. [7], [2]). It was shown in [4] that the strong positiveness of the operator A provides some algorithmic representations of a cosine family generated by A as well as the existence, stability and approximation results for (4.1) in the case when the initial data belong to the domain of some fractional power of the operator A. Contrary to the known necessary and sufficient conditions under which an operator A generates a cosine family [7] our condition is easier to prove.

Example 4.7. This is an example of a strongly P-positive operator in a "genuine" Banach space, i.e. in the case which can not be reduced to a Hilbert space [15].

Let us consider the one-dimensional operator $A : L_1(0,1) \to L_1(0,1)$ with the domain $D(A) = \{u | u \in H_0^2(0,1)\}$ in the Sobolev space $H_0^2(0,1)$ defined by

$$Au = -u'' \quad \forall\ u \in D(A).$$

The eigenvalues $\lambda_k = k^2\pi^2, k = 1, 2, \dots$ of A lie on the real axis inside of the path

$$\Gamma = \begin{cases} z = \eta^2 \pm i\eta, & \eta \geq 1\ , \\ z = 1 \pm i\eta^2, & |\eta| \leq 1. \end{cases}$$

The Green function for the problem

$$(zI - Au) \equiv u''(x) + zu(x) = -f(x),\ x \in (0,1); u(0) = u(1) = 0$$

is

$$G(x,\xi) = \frac{1}{\sqrt{z}\sin\sqrt{z}} \begin{cases} \sin\sqrt{z}x \sin\sqrt{z}(1-\xi) & x \leq \xi, \\ \sin\sqrt{z}\xi \sin\sqrt{z}(1-x) & x \geq \xi, \end{cases}$$

i.e. we have

$$u(x) = (zI - A)^{-1}f = \int_0^1 G(x,\xi)f(\xi)d\xi.$$

In order to show that the estimate (4.6) holds true it is sufficient to estimate the Green function on the parabola $z = \eta^2 \pm i\eta = \sqrt{\eta^4 + \eta^2}(\cos\phi \pm i\sin\phi)$, where

$$\cos\phi = \frac{\eta}{\sqrt{\eta^2 + 1}},\ \sin\phi = \frac{1}{\sqrt{\eta^2 + 1}}.$$

Actually, we have $\sqrt{z} = \sqrt{[4]}\eta^4 + \eta^2(\cos\frac{\phi}{2} \pm i\sin\frac{\phi}{2}) = a \pm ib$ with

$$\cos\frac{\phi}{2} = \frac{\sqrt{\eta^2 + \sqrt{\eta^4 + \eta^2}}}{\sqrt{2}\sqrt{[4]}\eta^4 + \eta^2}, \sin\frac{\phi}{2} = \frac{\sqrt{\sqrt{\eta^4 + \eta^2} - \eta^2}}{\sqrt{2}\sqrt{[4]}\eta^4 + \eta^2},$$

$$a = \frac{\sqrt{\eta^2 + \sqrt{\eta^4 + \eta^2}}}{\sqrt{2}}, b = \frac{\sqrt{\sqrt{\eta^4 + \eta^2} - \eta^2}}{\sqrt{2}}.$$

The following estimates hold for $x \leq \xi$ and for η large enough

$$\left|\frac{\sin\sqrt{z}x \sin\sqrt{z}(1-\xi)}{\sqrt{z}\sin\sqrt{z}}\right| =$$

$$= \frac{[\sin^2 ax + \sinh^2 bx]^{\frac{1}{2}}[\sin^2 a(1-\xi) + \sinh^2 b(1-\xi)]^{\frac{1}{2}}}{\sqrt[4]{\eta^4 + \eta^2[\sin^2 a + \sinh^2 b]^{\frac{1}{2}}}} \le \frac{c}{\eta}$$

with an absolute constant c. The case $\xi \le x$ can be considered analogously. The last estimate implies that $\|(zI-A)^{-1}f\|_{L_1} \le \frac{M}{1+\sqrt{|z|}}\|f\|_{L_1}$, i.e. the operator A is strongly P-positive in $X \equiv L_1(0,1)$. The same estimates for the Green function imply the strong P-positivity of A also in $L_\infty(0,1)$.

In the next section we will show that the strong P-positivity of the unbounded operator A is one of the sufficient conditions for the ρ-stability of the regularized scheme (2) whereas the explicit scheme (2) with $\alpha = \beta = 0$ is unstable.

5. A sufficient condition for ρ-stability of three-level difference schemes with strongly P-positive operator coefficients

For the sake of simplicity we set in (4.2)

$$\beta = \tau, \qquad \alpha = \frac{\tau^2}{2}.$$

In this case the scheme (4.2) takes the form

$$(I + \frac{\tau^2}{2}A)y_{\bar{t}t} + \tau A y_{\overset{\circ}{t}} + Ay = 0 \tag{5.1}$$

or

$$[I + \tau^2 A]y_{n+1} - 2y_n + y_{n-1} = 0, \qquad n = 1, 2, \ldots \tag{5.2}$$

and the operators χ, Q are given by $\chi(A) = Q(A) = [I + \tau^2 A]^{-1/2}$.

The next theorem represents the first main result of this section (see also [15]).

Theorem 5.1. *Let A be a strongly P-positive operator with the domain $D(A)$ having a spectrum $\Sigma(A)$ placed inside of a parabola $y^2 = \frac{c_0^2}{2}x$, $c_0 \equiv const > 0$, $\Re\Sigma(A) > \gamma$, $\tau < \sqrt{2}c_0^{-1}$. Then the difference scheme (4.2) with $\beta = \tau$, $\alpha = \frac{\tau^2}{2}$ is ρ-stable with respect to initial data in $D(A^\sigma)$ with $\rho = (1 - \frac{c_0\tau}{\sqrt{2}})^{-1/2}$.*

Proof. We choose the integration path enveloping the spectrum of A as shown in Fig. 2.

Using the Dunford-Cauchy integral we represent the solution of (5.2) as follows

$$y_n = \frac{1}{2\pi i}\int_\Gamma -Q^n(z)U_{n-2}(\chi(z))(z-A)^{-1}dz y_0 + \tag{5.3}$$

$$+\frac{1}{2\pi i}\int_\Gamma Q^{n-1}(z)U_{n-1}(\chi(z))(z-A)^{-1}dz y_1$$

or in view of the elementary relations

$$y_0 = \frac{y_0+y_1}{2} + \frac{y_0-y_1}{2}, \qquad y_1 = \frac{y_0+y_1}{2} - \frac{y_0-y_1}{2}$$

we get

$$y_n = \frac{1}{4\pi i}\int_\Gamma f_n^{(+)}(z)(z-A)^{-1}dz(y_0+y_1) - \tag{5.4}$$

$$-\frac{1}{4\pi i}\int_\Gamma f_n^{(-)}(z)(z-A)^{-1}dz(y_0-y_1),$$

where

$$f_n^{(+)}(z) = Q^{n-1}(z)[U_{n-1}(\chi) - Q(z)U_{n-2}(\chi)], \tag{5.5}$$

$$f_n^{(-)}(z) = Q^{n-1}(z)[U_{n-1}(\chi) + Q(z)U_{n-2}(\chi)].$$

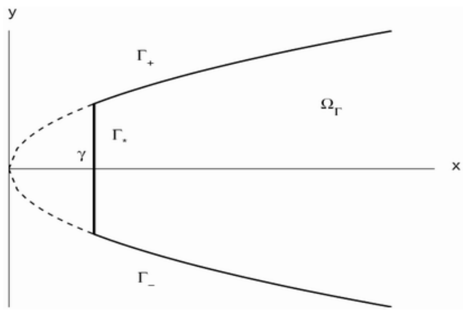

FIGURE 2. Integration path Γ

Taking into account the form of the path Γ we can transform the integrals in (5.4) as follows (we use the notations $z = x + iy = x + ic_0\sqrt{\frac{x}{2}}$, $\bar{z} = x - ic_0\sqrt{\frac{x}{2}}$, $dz = (1 + ic_0/(2\sqrt{2x}))dx$, $d\bar{z} = (1 - ic_0/(2\sqrt{2x}))dx$)

$$I_n^{(\pm)} \equiv \frac{1}{4\pi i}\int_\Gamma f_n^{(\pm)}(z)(z-A)^{-1}dz =$$

$$= \frac{1}{4\pi i}\int_\gamma^\infty [f_n^{(\pm)}(\bar{z})(\bar{z}-A)^{-1} - f_n^{(\pm)}(t)(z-A)^{-1}]dx -$$

$$-\frac{c_0}{8\sqrt{2}\pi}\int_\gamma^\infty [f_n^{(\pm)}(\bar{z})(\bar{z}-A)^{-1} - f_n^{(\pm)}(z-A)^{-1}]\frac{dx}{\sqrt{x}} -$$

$$-\frac{1}{4\pi}\int_0^{c_0\sqrt{\frac{\gamma}{2}}} f_n^{(\pm)}(\gamma+iy)(\gamma+iy-A)^{-1}dy - \frac{1}{4\pi}\int_0^{c_0\sqrt{\frac{\gamma}{2}}} f_n^{(\pm)}(\gamma-iy)(\gamma-iy-A)^{-1}dy =$$

$$-\frac{1}{2\pi}\int_\gamma^\infty \mathrm{Im} f_n^{(\pm)}(z)(\bar{z}-A)^{-1}dx + \frac{c_0}{2\pi\sqrt{2}}\int_\gamma^\infty f_n^{(\pm)}(z)\sqrt{x}(\bar{z}-A)^{-1}(z-A)^{-1}dx +$$

$$+\frac{c_0 i}{4\sqrt{2}\pi}\int_\gamma^\infty \mathrm{Im} f_n^{(\pm)}(z)(\bar{z}-A)^{-1}\frac{dx}{\sqrt{x}} - \frac{c_0^2 i}{8\pi}\int_\gamma^\infty f_n^{(\pm)}(z-A)^{-1}(\bar{z}-A)^{-1}dx +$$

$$+\frac{i}{2\pi}\int_0^{c_0\sqrt{\frac{\gamma}{2}}} f_n^{(\pm)}(\gamma+iy)y(\gamma+iy-A)^{-1}(\gamma-iy-A)^{-1}dy -$$

$$-\frac{1}{2\pi}\int_0^{c_0\sqrt{\frac{\gamma}{2}}} \mathrm{Re} f_n^{(\pm)}(\gamma+iy)(\gamma-iy-A)^{-1}dy. \tag{5.6}$$

In what follows we will use the relations

$$U_{n-1}(\chi) - Q(\chi)U_{n-2}(\chi) =$$

$$= U_{n-1}(\chi) - \chi U_{n-2}(\chi) = T_{n-1}(\chi) =$$

$$= \frac{1}{2}[(\chi + \sqrt{\chi^2-1})^{n-1} + (\chi - \sqrt{\chi^2-1})^{n-1}],$$

$$U_{n-1}(\chi) + Q(\chi)U_{n-2}(\chi) =$$

$$= U_{n-1}(\chi) + \chi U_{n-2}(\chi) = \frac{1}{\sqrt{\chi^2-1}} \times$$

$$\times [(2\chi + \sqrt{\chi^2-1})(\chi + \sqrt{\chi^2-1})^{n-1} - (2\chi - \sqrt{\chi^2-1})(\chi - \sqrt{\chi^2-1})^{n-1}],$$

$$|U_{n-1}(\chi) - \chi U_{n-2}(\chi)| \le [\Phi(\chi)]^{n-1} \tag{5.7}$$

$$|U_{n-1}(\chi) + \chi U_{n-2}(\chi)| \le [\Phi(\chi)]^{n-1}\frac{|2\chi + \sqrt{\chi^2-1}| + |2\chi - \sqrt{\chi^2-1}|}{|\sqrt{\chi^2-1}|}$$

with

$$\Phi(\chi) = \max\{|\chi + \sqrt{\chi^2 - 1}|, |\chi - \sqrt{\chi^2 - 1}|\}, \qquad (5.8)$$

where $T_n(x)$ are Chebyshev polynomials of first kind [20].

It follows from (5.4)-(5.8) that

$$\|y_n\| \leq C \left\{ \left[\int_\gamma^\infty \frac{\Phi_n(z_\Gamma)}{|z_\Gamma|^\sigma} dx + \int_0^{c_0\sqrt{\frac{\gamma}{2}}} \Phi_n(z_\gamma) dy \right] \left\| A^\sigma \frac{y_0 + y_1}{2} \right\| + \right.$$

$$\left. \left[\int_\gamma^\infty \frac{\Phi_n(z_\Gamma)\Phi_{(1)}(z_\Gamma)}{|z_\Gamma|^\sigma} dx + \int_0^{c_0\sqrt{\frac{\gamma}{2}}} \Phi_n(z_\gamma)\Phi_{(1)}(z_\Gamma) dy \right] \left\| A^\sigma \frac{y_1 - y_0}{2} \right\| \right\}, \qquad (5.9)$$

where

$$z_\Gamma = x + ic_0\sqrt{\frac{x}{2}}, \qquad z_\gamma = \gamma + iy,$$

$$\Phi_n(z) = [|\chi(z)|\Phi(\chi(z))]^{n-1}, \qquad (5.10)$$

$$\Phi_{(1)}(z) = \frac{\max\{|2\chi(z) + \sqrt{\chi^2(z) - 1}|, |2\chi(z) - \sqrt{\chi^2(z) - 1}|\}}{|\sqrt{\chi^2(z) - 1}|}.$$

First of all we have to estimate

$$|\chi(z)|\Phi(\chi(z)) = \max\left\{|1 + i\tau\sqrt{z}|^{-1}, |1 - i\tau\sqrt{z}|^{-1}\right\}. \qquad (5.11)$$

Let $z = x + iy = \rho e^{i\theta}$, $x > 0, y \in (-\infty, \infty)$, $\rho = \sqrt{x^2 + y^2}$, $\cos\theta = \frac{x}{\rho}$, $\sin\theta = \frac{y}{\rho}$, then

$$|q_\pm(z)|^{-2} \equiv |1 \pm i\tau\sqrt{z}|^2 = |1 \pm i\tau(x^2 + y^2)^{1/4} e^{i\theta/2}|^2 =$$

$$= [1 \mp \tau(x^2 + y^2)^{1/4} \sin\frac{\theta}{2}]^2 + \tau^2(x^2 + y^2)^{1/2}\cos^2\frac{\theta}{2} =$$

$$= [1 \mp 2\tau(x^2 + y^2)^{1/4} \sin\frac{\theta}{2}] + \tau^2(x^2 + y^2)^{1/2}.$$

Since the angle θ lies in the first or in the fourth quadrant, we have

$$\sin\frac{\theta}{2} = \frac{y}{\sqrt{\sqrt{x^2 + y^2} + x}} \frac{1}{\sqrt{2}(x^2 + y^2)^{1/2}}$$

and

$$|q_\pm(z)|^{-2} = 1 \pm \sqrt{2}\tau \frac{y}{\sqrt{\sqrt{x^2 + y^2} + x}} + \tau^2(x^2 + y^2)^{1/2}.$$

We see that for $z = z_\Gamma$

$$|q_\pm(z)|^{-2} = 1 \pm \sqrt{2}\tau \frac{c_0\sqrt{x/2}}{\sqrt{x + \sqrt{x^2 + c_0^2 x/2}}} + \tau^2(x^2 + c_0^2 x/2)^{1/2} \geq 1 - \tau\frac{c_0}{\sqrt{2}} \qquad (5.12)$$

and for $z = z_\Gamma$

$$|q_\pm(z)|^{-2} = 1 \pm \sqrt{2}\tau \frac{y}{\sqrt{\gamma + \sqrt{\gamma^2 + y^2}}} + \tau^2(\gamma^2 + y^2)^{1/2} \geq$$

$$\geq 1 - \sqrt{2}\tau \frac{c_0\sqrt{\gamma/2}}{\sqrt{\gamma + \sqrt{\gamma^2 + c_0^2\gamma/2}}} \geq 1 - \tau\frac{c_0}{\sqrt{2}}, \qquad (5.13)$$

$$0 \leq y \leq c_0\sqrt{\gamma/2}, \qquad \tau \leq \sqrt{2}/c_0.$$

Next, we consider $\Phi_{(1)}(z)$ for $z = z_\Gamma$ and $z = z_\gamma$. It is easy to see that

$$\Phi_{(1)}(z) = \frac{1}{\tau} \max\{|\frac{2}{\sqrt{z}} + i\tau|, |\frac{2}{\sqrt{z}} - i\tau|\}, \tag{5.14}$$

$$\Phi_{(1)}(z_\Gamma) \le \frac{1}{\tau}\left[\frac{2}{\sqrt{x^2 + c_0^2 x/2}} + \tau\right] \le \frac{1}{\tau}\left[\frac{2}{\gamma} + \tau\right], \tag{5.15}$$

$$\Phi_{(1)}(z_\gamma) \le \frac{1}{\tau}\left[\frac{2}{\sqrt{\gamma^2 + y^2}} + \tau\right] \le \frac{1}{\tau}\left[\frac{2}{\gamma} + \tau\right]. \tag{5.16}$$

Now, we are in the position to estimate $\|y_n\|$. Taking into account (5.9)-(5.15) we get

$$\|y_n\| \le c\left(1 - \frac{c_0\tau}{\sqrt{2}}\right)^{-n/2}\left(\left\|\frac{y_0 + y_1}{2}\right\|_\sigma + \left\|\frac{y_1 - y_0}{\tau}\right\|_\sigma\right), \tag{5.17}$$

where $\sigma > 0$, $\|u\|_\sigma = \|A^\sigma u\|$. The proof is complete.

Remark 5.2. From the stability estimate one gets

$$\|y_n\| \le Ce^{\frac{c_0\tau n}{2\sqrt{2}}}\left(\|\frac{y_0 + y_1}{2}\|_\sigma + \|\frac{y_1 - y_0}{\tau}\|_\sigma\right) \equiv Ce^{\frac{c_0 T}{2\sqrt{2}}}\left(\|\frac{y_0 + y_1}{2}\|_\sigma + \|\frac{y_1 - y_0}{\tau}\|_\sigma\right),$$

i.e. the stability constant increases exponentially with respect to the length of the time interval $T = n\tau$.

Remark 5.3. The explicit scheme (4.2) with $\alpha = \beta = 0$, i.e.

$$y_{\bar{t}t} + Ay = 0$$

or in the index form

$$y_{n+1} - 2[I - \frac{\tau^2}{2}A]y_n + y_{n-1} = 0$$

is unstable if A is an unbounded operator in a Banach space E.

Actually, choosing $y_0 = 0$ we get

$$y_2 = 2[I - \frac{\tau^2}{2}A]y_1.$$

Since A is unbounded there exists $y_1^{(k)}$ with $\|y_1^{(k)}\| = 1$ such that $\|(I - \frac{\tau^2}{2}A)y_1^{(k)}\| \ge k$ for any arbitrarily large k. Thus, the estimate (4.5) can not be valid for all y_0, y_1, i.e. the scheme is not stable with respect to the initial data.

Note, that the scheme (5.2) related to the differential equation (4.1) is of the first order of approximation with respect to τ whereas the unstable explicit scheme is of the second order. The following regularized difference scheme

$$\left(I + \frac{\tau^2}{2}A\right)y_{\bar{t}t,n} + Ay_n = 0, \qquad n = 1, 2, \ldots \tag{5.18}$$

with given y_0, y_1 is a special case of (4.2) for $\beta = 0$, $\alpha = \frac{\tau^2}{2}$. This scheme has the second order of approximation with respect to τ. It can be also interpreted as the regularized explicit scheme. The next main result of this section deals with the stability of this scheme and can be proved analogously to Theorem 5.1 (see [15]).

Theorem 5.4. *Let A be strongly P-positive with a spectrum $\Sigma(A)$ inside of the parabola $y^2 = c_0^2 x/2$ and $\beta = 0$, $\alpha = \frac{\tau^2}{2}$,$\tau \leq 2\sqrt{2}c_0^{-1}$, $\Re\Sigma(A) > \gamma$. Then the difference scheme (4.2) is ρ-stable with respect to initial data in $D(A^\sigma)$, $\sigma > 0$ with $\rho = (1 + \frac{c_0\tau}{\sqrt{2}} + \frac{c_0^2\tau^2}{4})^{\frac{1}{2}}$.*

Remark 5.5. One can see from the conditions $\tau < \sqrt{2}c_0^{-1}$ for the scheme (5.2) and $\tau < 2\sqrt{2}c_0^{-1}$ for the scheme (5.18) that the opening of the parabola determines the upper limit of the time-step τ for which these schemes are ρ-stable.

Let us consider the following inhomogeneous difference scheme

$$(I + \alpha A)y_{\bar{t}t,n} + \beta A y_{\overset{\circ}{t},n} + Ay_n = f_n, \qquad n = 1, 2, \ldots \tag{5.19}$$

$$y_0 = y_1 = 0.$$

Below we define a type of stability that plays an important role for inhomogeneous problems.

Definition 5.6. Given a function $\rho(\tau)$ we say that the scheme (5.19) is ρ-stable with respect to the right hand side in $D(A^\sigma)$ with some real $\sigma > 0$ if there exists a constant $M > 0$ independent of n, such that the estimate

$$\|y_n\| \leq M\rho^n \sum_{p=0}^{n-1} \tau \|f_p\|_\sigma$$

holds for any discrete function $f_p \in D(A^\sigma)$.

The solution of (5.19) can be represented as

$$y_n = \left\{[I + (\alpha + \frac{\beta\tau}{2})A][I + (\alpha - \frac{\beta\tau}{2}A]\right\}^{-1/2} \tau^2 \sum_{p=0}^{n-1} Q^{n-p}(A)U_{n-p-1}(\chi(A))f_p =$$

$$\frac{\tau^2}{2\pi i}\sum_{p=0}^{n-1}\left\{\int_\Gamma [(1 + (\alpha + \frac{\beta\tau}{2})z)(1 + (\alpha - \frac{\beta\tau}{2})z)]^{-1/2} \times\right.$$

$$\left.\times Q^{n-p}(z)U_{n-p-1}(\chi(z))(z - A)^{-1}dz\right\} f_p,$$

$$n = 2, 3, \ldots; \qquad y_0 = 0, y_1 = 0.$$

Using the inequality

$$|Q^k(z)U_{k-1}(\chi(z))| \leq \frac{|Q(z)\Phi(\chi(z))|^k}{\sqrt{|1 - \chi^2(z)|}},$$

the estimates (5.12),(5.14),(5.15) and following the idea of the proof of Theorems 5.1, 5.1 we get the following statement.

Theorem 5.7. *Under assumptions of Theorem 5.1 ($\alpha = \dfrac{\tau^2}{2}, \beta = \tau$) or Theorem 5.4 ($\beta = 0, \alpha = \dfrac{\tau^2}{2}$) the corresponding difference schemes from the family (5.19) are ρ-stable with respect to the right hand side in $D(A^\sigma)$ with $\rho(\tau) = (1 - \dfrac{c_0\tau}{\sqrt{2}})^{-1/2}$ and $\rho(\tau) = (1 + \dfrac{c_0\tau}{\sqrt{2}} + \dfrac{c_0^2\tau^2}{4})^{\frac{1}{2}}$ respectively.*

Example 5.8. In some sense, this example shows the sharpness of our results. Indeed, let us consider the scalar problem

$$\frac{d^2u}{dt^2} + Au = 0, \quad t \in (0,T],$$
$$u(0) = u_0, \quad u'(0) = i\sqrt{A}u_0,$$

where $A = x - ic_0\sqrt{\frac{x}{2}}$ can be viewed for $x \to \infty$ as an "unbounded" strongly P-positive operator with the spectrum inside of the parabola $y^2 = (c_0 + \varepsilon)^2 x/2$ with an arbitrarily small positive ε. The solution of the problem is the function

$$u(x,t) = \exp\left\{ i\sqrt{x - ic_0\sqrt{\frac{x}{2}}}\; t \right\} u_0$$
$$= \exp\left\{ i\sqrt[4]{x^2 + c_0^2\frac{x}{2}} \left(i\cos\frac{\varphi}{2} - \sin\frac{\varphi}{2} \right) t \right\} u_0,$$

where

$$\cos\varphi = \frac{x}{\sqrt{x^2 + c_0^2\frac{x}{2}}} > 0, \quad \sin\varphi = -\frac{c_0\sqrt{\frac{x}{2}}}{\sqrt{x^2 + c_0^2\frac{x}{2}}} < 0.$$

It is easy to see that

$$|u(x,t)| = \rho_d(x,t)|u_0|$$

with

$$\rho_d(x,t) = \exp\left\{ \sqrt[4]{x^2 + c_0^2\frac{x}{2}}\, \frac{1}{\sqrt{2}} \sqrt{1 - \frac{x}{\sqrt{x^2 + c_0^2\frac{x}{2}}}}\; t \right\}$$
$$= \exp\left\{ \frac{t}{\sqrt{2}} \sqrt{\sqrt{x^2 + c_0^2\frac{x}{2}} - x} \right\}.$$

For the solution of the corresponding difference scheme (5.18) Theorem 5.4 provides the estimate

$$|y_n(x)| = |y(x, n\tau)| \le c\rho^n(\tau)|u_0|$$

with $\rho(\tau) = \left(1 + \frac{(c_0+\varepsilon)\tau}{\sqrt{2}} + \frac{(c_0+\varepsilon)^2\tau^2}{4}\right)^{\frac{1}{2}}$. In particular we have for a fixed $t = n\tau$ and $x \to \infty$

$$|u(\infty,t)| = \lim_{x\to\infty} |u(x,t)| = \rho_d(\infty,t)|u_0|,$$
$$|y(t)| \le c\rho(\tau)^{\frac{t}{\tau}}|u_0|,$$

where $\rho_d(\infty,t) = \lim_{x\to\infty}\rho_d(x,t) = \exp\frac{c_0 t}{2\sqrt{2}}$. It is easy to check that

$$\lim_{\tau\to 0}\rho(t)^{\frac{t}{\tau}} = \lim_{\tau\to 0}\left[1 + \frac{(c_0+\varepsilon)\tau}{\sqrt{2}} + \frac{(c_0+\varepsilon)^2\tau^2}{4}\right]^{\frac{t}{2\tau}} = \exp\left\{\frac{(c_0+\varepsilon)t}{2\sqrt{2}}\right\},$$

i.e. the parabola containing the spectrum of A defines the behavior of the difference solution asymptotically in t in exactly the same way as the exact solution.

Example 5.9. Let us consider the difference scheme (5.18) with A as in Example 3 for $x = 10^4, c_0 = 10^2, y_0 = 1, y_1 = 1 + \tau i\sqrt{x - ic_0\sqrt{\frac{x}{2}}}$ and $n = 1, 2, \ldots, 100$. One can see that the absolute value of the solution as function of t computed by (5.18) is stable when the sufficient stability condition $\tau < 2\sqrt{2}/c_0$ holds (Fig. 3). The next figure shows that the instability can occur if the condition $\tau < 2\sqrt{2}/c_0$ is violated (see Fig. 4).

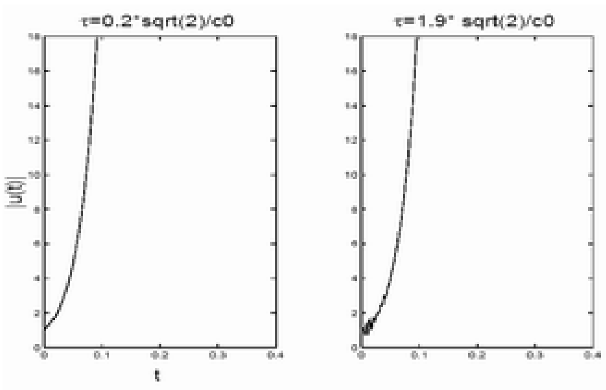

FIGURE 3. Solution of scheme (5.18): the sufficient stability condition is fulfilled

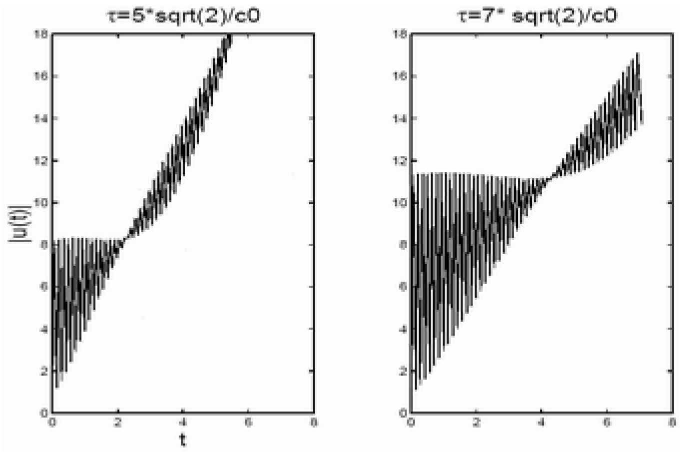

FIGURE 4. Solution of scheme (5.18): the sufficient stability condition is violated

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