Журнал обчисл. прикл. матем.
Вип. 1, 2005, стор. 70–76

J. Comput. Appl. Math.
No. 92, 2005, pp. 70–76

INVESTIGATION OF DIFFERENCE SCHEMES OF FINITE ELEMENT METHOD FOR SECOND-ORDER UNSTEADY-STATE EQUATIONS

UDC 518.61

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Abstract. The work deals with construction and investigation of the difference schemes for solution of Cauchy abstract problem for a second-order equation. We consider a parametric family of schemes. The presence of parameters in the scheme allows to regularise the schemes with the view of optimising the implementation algorithm and scheme accuracy. Such schemes may be applied for solving systems of second-order ordinary differential equations and hyperbolic-type partial differential equations.

Introduction. The scheme which is similar to considered scheme was constructed in [1] on the ground of the finite element method and Bubnov-Galyorkin's procedure. In [2], the finite element schemes for unsteady-state equations comes from the quadratic functional of total system energy. Such schemes require solving the equations for unknown values of solutions either on all levels $t = t_n$, $n = 0, 1, \dots$ or on several levels $n = n_0, n_0 + 1, \dots, n_0 + k$. Such a way for discretising the problem require a big number of arithmetic operations for implementing these schemes.

Using the finite element method on the basis of Bubnov-Galyorkin-Petrov's procedure in [3], new vector two-level difference schemes are constructed which approximate Cauchy abstract problem for second-order equations

$$D\ddot{u} + Au = f(t), \quad u(0) = u_0, \quad \dot{u}(0) = u_1, \quad 0 \le t \le T. \tag{1}$$

Here, A, D are constant operators (matrices), $A^* = A > 0$, $D^* = D > 0$, acting in Hilbert space H; $\ddot{u} = \frac{d^2u}{dt^2}$, $\dot{u} = \frac{du}{dt}$.

On the assumption of interchangeability of operators $A, D : AD = DA$ a convergence property of the solution of the scheme (2) to the sufficiently smooth solution of the problem (1) is shown in [3]. In the same place, an implementation algorithm is proposed, based on factorisation of the operator on upper level.

In the present work, the new parametric family of the schemes of fourth and sixth accuracy orders is constructed and investigated by developing the ideas of [3]. The convergence property is proved and an economic algorithm of implementing such schemes is proposed, without assumption on interchangeability of operators A, D.

1. Construction of Difference Scheme. Let us define the generalised solution of the equation (1) as the function $u(t) \in C^1[0,T]$ satisfying, for any range $(t_n, t_{n+1}) \in [0,T]$, the identity [3]

$$\int\limits_{t_n}^{t_{n+1}} (-D\dot{u}\dot{v} + Auv)dt + D\dot{u}v\Big|_{t_n}^{t_{n+1}} = \int\limits_{t_n}^{t_{n+1}} f(t)v(t)dt, \quad \forall v(t) \in C^1[0,T]. \tag{2}$$

Let $\omega_\tau = \{t_n, \tau = t_{n+1} - t_n, n = 0, 1, 2, \dots\}$ be a net on an interval $t \in [0, T]$ (for simplicity sake, we consider the grid uniform). We seek an approximate solution of the problem (1) in a form of third-power Hermitean spline [1]:

$$y(t) = y^n \varphi_{00}^n(t) + \dot{y}^n \varphi_{10}^n(t) + y^n \varphi_{01}^n(t) + \dot{y}^{n+1} \varphi_{11}^n(t), \tag{3}$$

where:

$$y^n = y(t_n), \quad \dot{y}^n = \frac{dy}{dt}(t_n), \quad \varphi_{00}^n(t) = 2\xi^3 - 3\xi^2 + 1, \quad \varphi_{10}^n(t) = 3\xi^2 + 2\xi^3,$$

$$\varphi_{01}^n(t) = \tau(\xi^3 - 2\xi^2 + \xi), \quad \varphi_{11}^n(t) = \tau(\xi^3 - \xi^2), \quad \xi = \frac{t - t_n}{\tau}.$$

By selecting various weighting functions $v(t)$, namely

$$v(t) = v_1(\xi) = p_1 v_1^{(1)}(\xi) + p_2 v_1^{(2)}(\xi)$$

$$v(t) = v_2(\xi) = s_1 v_2^{(1)}(\xi) + s_2 v_2^{(2)}(\xi), \quad \xi = \frac{t - t_n}{\tau},$$

$$v_1^{(1)}(\xi) = 1, \quad v_1^{(2)}(\xi) = \xi^2 - \xi, \quad v_2^{(1)}(\xi) = \tau\left(\xi - \frac{1}{2}\right),$$

$$v_2^{(2)}(\xi) = \tau\left(\xi^3 - \frac{3}{2}\xi^2 + \frac{1}{2}\xi\right),$$

where parameters p_1, p_2, s_1, s_2 are defined by correlation:

$$p_1 = 6 - 60\gamma, \quad p_2 = 30 - 360\gamma, \quad s_1 = 180\beta - 40\alpha, \quad s_2 = 1680\beta - 280\alpha,$$

we shall get the following vector difference scheme

$$\begin{cases} \left(D - \gamma\tau^2 A\right) \dfrac{\hat{\dot{y}} - \dot{y}}{\tau} + A \dfrac{\hat{y} + y}{2} = \varphi_1, \\ \left(D - \alpha\tau^2 A\right) \dfrac{\hat{y} - y}{\tau} - \left(D - \beta\tau^2 A\right) \dfrac{\hat{\dot{y}} + \dot{y}}{2} = \varphi_2. \end{cases} \tag{4}$$

Here,

$$\varphi_1 = \frac{1}{\tau} \int_{t_n}^{t_{n+1}} f(t) v_1\left(\frac{t - t_n}{\tau}\right) dt = \int_0^1 f(t_n + \tau\xi) v_1(\xi) d\xi,$$

$$\varphi_2 = \frac{1}{\tau} \int_{t_n}^{t_{n+1}} f(t) v_2\left(\frac{t - t_n}{\tau}\right) dt = \int_0^1 f(t_n + \tau\xi) v_2(\xi) d\xi,$$

$$y^0 = u_0, \quad \dot{y}^0 = u_1.$$

In the first equation of system (4), γ is an arbitrary parameter, as distinct from [3], where $\gamma = \frac{1}{12}$.

Let us choose parameters α, β, γ for reasons of accuracy and simplicity of scheme implementation algorithm.

We assume, for simplifying the investigation of the scheme accuracy, that A, D are numbers, and the equation (1) is a second-order homogeneous $f(t) \equiv 0$ ordinary differential equation (test equation) which has an exact solution:

$$u(t) = a_1 \cos \lambda t + a_2 \sin \lambda t, \quad \lambda = \sqrt{\frac{A}{D}}. \tag{5}$$

Constants a_1, a_2 are determined by entry conditions.

We will seek the solution of difference equations (4) ($\varphi_1 = \varphi_2 \equiv 0$) in the form

$$y = y^n = Yq^n, \quad \dot{y} = \dot{y}^n = \dot{Y}q^n, \tag{6}$$

with amplitudes Y and $\dot{Y}$. By substituting (6) to (4), we shall get a homogeneous system with respect to Y and $\dot{Y}$, for which the condition of the occurrence of nontrivial solutions (the determinant is equal to zero) is given by:

$$\left(1-\alpha z^2\right)\left(1-\gamma z^2\right)(q-1)^2+\frac{z^2}{4}\left(1-\beta z^2\right)(q+1)^2=0, \quad z=\tau\lambda.$$

The last equation allows to define the scheme transition module q:

$$q_{1,2}=\frac{\left(1-\beta z^2\right)\left(1-\gamma z^2\right)-\frac{z^2}{4}\left(1-\alpha z^2\right)\pm\sqrt{d}}{\left(1-\beta z^2\right)\left(1-\gamma z^2\right)+\frac{z^2}{4}\left(1-\alpha z^2\right)}, \tag{7}$$

where $d=-z^2\left(1-\alpha z^2\right)\left(1-\beta z^2\right)\left(1-\gamma z^2\right)$.

The condition of scheme stability (4) is given by [3]

$$\left(D-\alpha\tau^2 A\right)\left(D-\beta\tau^2 A\right)\left(D-\gamma\tau^2 A\right)\geq 0.$$

For satisfying this, it is sufficient that

$$\tau^2\leq\frac{D}{mA}, \quad m=\max\{\alpha,\beta,\gamma\}>0. \tag{8}$$

At this, $d\leq 0$ and $|q|=1$, so that we may present q as:

$$q_{1,2}=\cos\varphi\pm i\sin\varphi. \tag{9}$$

By comparing (7) and (9), we shall get an equation to define φ:

$$\cos\varphi=1-\frac{\frac{z^2}{2}\left(1-\alpha z^2\right)}{\left(1-\alpha z^2\right)\left(1-\gamma z^2\right)+\frac{z^2}{4}\left(1-\beta z^2\right)}.$$

From here,

$$\varphi=2\arcsin\left[\frac{z}{2}\sqrt{\frac{\left(1-\alpha z^2\right)}{\left(1-\alpha z^2\right)\left(1-\gamma z^2\right)+\frac{z^2}{4}\left(1-\beta z^2\right)}}\right].$$

From (6), (9), we will find:

$$y=y^n=b_1\cos\frac{\varphi}{\tau}t_n+b_2\sin\frac{\varphi}{\tau}t_n. \tag{10}$$

The distinction of the exact and approximate solutions is characterised by value $\vartheta=\frac{\varphi}{\tau\lambda}$ (see (5) and (10)). The closer is ϑ to 1, the more accurate the approximate solution is.

Let us expand ϑ by powers of z^2:

$$\vartheta=1+r_1 z^2+r_2 z^4+O(z^6), \tag{11}$$

$$r_1=\frac{1}{2}\left(\alpha+\gamma-\beta-\frac{1}{6}\right),$$

$$r_2=\frac{1}{12}\left(\beta-6\alpha\gamma+\frac{1}{40}\right)+\left(\beta+\gamma-\alpha-\frac{1}{6}\right)\bar{r}_2.$$

By minimising $|r_1|$ and $|r_2|$, we may improve the quality of the approximate solution. We shall demand from all schemes considered below that the condition $r_1=0$ is to be satisfied. From here,

$$\alpha+\gamma=\beta+\frac{1}{6} \tag{12}$$

At the same time, $\vartheta=1+r_2\lambda^4\tau^4+O(\tau^6)$, $r_2=\frac{1}{12}\left(\beta-6\alpha\gamma+\frac{1}{40}\right)$.

In this case, we may say that the harmonic propagation speed of the differential equation and difference schemes agree to an accuracy of fourth-order values by step τ.

For obtaining the schemes of sixth order of accuracy we shall require that both (12) and the following condition are satisfied:

$$\beta - 6\alpha\gamma + \frac{1}{40} = 0. \tag{13}$$

As noted above, the investigation of the scheme was made on the assumption of interchangeability of operators A, D. This condition is burdensome. To reject such condition, we shall introduce $w = D^{1/2}y$, $\dot{w} = D^{1/2}\dot{y}$ instead of y, $\dot{y}$, where $D^{1/2}$ is a square root from a positive operator D. Let us also note that $\left(D^{1/2}\right)^* = D^{1/2} > 0$, and there exists an inverse operator $D^{-1/2} = \left(D^{-1/2}\right)^* > 0$.

After making obvious transformations, from the scheme (4), we shall get the scheme

$$\begin{cases} \tilde{D}_\gamma \dfrac{\hat{\dot{w}} - \dot{w}}{\tau} + \tilde{A}\dfrac{\hat{w} + w}{2} = \tilde{\varphi}_1, \\ \tilde{D}_\alpha \dfrac{\hat{w} - w}{\tau} - \tilde{D}_\beta \dfrac{\hat{\dot{w}} + \dot{w}}{2} = \tilde{\varphi}_2, \end{cases} \tag{14}$$

where $\tilde{\varphi}_1 = D^{-1/2}\varphi_1$, $\tilde{\varphi}_2 = D^{-1/2}\varphi_2$, $\tilde{D}_\omega = \tilde{D} - \omega\tau^2\tilde{A}$, $\tilde{D} = E$, $\tilde{A} = D^{-1/2}AD^{-1/2}$. It is obvious that $\tilde{D} = \tilde{D}^* > 0$, $\tilde{A} = \tilde{A}^* > 0$ and $\tilde{D}\tilde{A} = \tilde{A}\tilde{D}$.

2. Algorithm of Scheme Implementation. For implementing the scheme (14), it is necessary to solve a system of two equations for unknowns $\hat{w}$, $\hat{\dot{w}}$

$$\begin{cases} \tilde{D}_\gamma \hat{\dot{w}} + \dfrac{\tau}{2}\tilde{A}\hat{w} = \tilde{\Phi}_1 \equiv \tilde{D}_\gamma \dot{w} - \dfrac{\tau}{2}\tilde{A}w + \tau\tilde{\varphi}_1, \\ -\dfrac{\tau}{2}\tilde{D}\hat{\dot{w}} + \tilde{D}_\alpha \hat{w} = \tilde{\Phi}_2 \equiv -\dfrac{\tau}{2}\tilde{D}_\beta \dot{w} + \tilde{D}_\alpha w + \tau\tilde{\varphi}_2. \end{cases}$$

We shall find by Cramer's formula (considering $\tilde{D}\tilde{A} = \tilde{A}\tilde{D}$):

$$\hat{\dot{w}} = \tilde{\Delta}^{-1}\tilde{\Delta}_1, \quad \hat{w} = \tilde{\Delta}^{-1}\tilde{\Delta}_2, \tag{15}$$

where:

$$\begin{gathered} \tilde{\Delta} = \tilde{D}^2 + \left(\alpha + \gamma - \frac{1}{4}\right)\tau^2\tilde{A}\tilde{D} + \left(\alpha\gamma - \frac{\beta}{4}\right)\tau^4\tilde{A}^2, \\ \tilde{\Delta}_1 = \tilde{D}_\alpha\tilde{\Phi}_1 - \frac{\tau}{2}\tilde{A}\tilde{\Phi}_2, \quad \tilde{\Delta}_2 = \tilde{D}_\gamma\tilde{\Phi}_2 - \frac{\tau}{2}\tilde{D}_\beta\tilde{\Phi}_1. \end{gathered} \tag{16}$$

Let us factorise an operator $\tilde{\Delta}$:

$$\tilde{\Delta} = \tilde{D}_{\omega_1}\tilde{D}_{\omega_2},$$

where ω_1, ω_2 are roots of a quadratic equation

$$\omega^2 + \left(\alpha + \gamma - \frac{1}{4}\right)\omega + \left(\alpha\gamma - \frac{\beta}{4}\right) = 0. \tag{17}$$

Coming back in (15) to old variables $y = D^{-1/2}w$, $\dot{y} = D^{-1/2}\dot{w}$, we shall, after simple transformations, obtain formulas for scheme implementation (4):

$$\Delta\hat{\dot{y}} = \Delta_1, \quad \Delta\hat{y} = \Delta_2,$$

where:

$$\begin{gathered} \Delta = D^{1/2}\tilde{\Delta}D^{1/2} = D^{1/2}\tilde{D}_{\omega_1}\tilde{D}_{\omega_2}D^{1/2} = \\ = D^{1/2}\left(E - \tau^2\omega_1 D^{-1/2}AD^{-1/2}\right) \times \\ \times \left(E - \tau^2\omega_2 D^{-1/2}AD^{-1/2}\right) D^{1/2} = D_{\omega_1}D^{-1}D_{\omega_2}, \\ \Delta_1 = D^{1/2}\tilde{\Delta}_1 = D_\alpha D^{-1}\Phi_1 - \frac{\tau}{2}AD^{-1}\Phi_2, \end{gathered}$$

$$\Delta_2 = D^{1/2}\tilde{\Delta}_1 = D_\gamma D^{-1}\Phi_2 - \frac{\tau}{2}D_\beta D^{-1}\Phi_1,$$
$$\Phi_1 = D_\gamma \dot{y} - \frac{\tau}{2}Ay + \tau\varphi_1, \quad \Phi_2 = -\frac{\tau}{2}D_\beta \dot{y} + D_\alpha y + \tau\varphi_2.$$

On the ground of the last formulas, the calculation algorithm appears as follows:

we solve equations $D\bar{\Phi}_1 = \Phi_1$ and $D\bar{\Phi}_2 = \Phi_2$;

calculate $\Delta_1 = D_\alpha\bar{\Phi}_1 - \frac{\tau}{2}A\bar{\Phi}_2$, $\Delta_2 = D_\gamma\bar{\Phi}_2 - \frac{\tau}{2}D_\beta\bar{\Phi}_1$;

solve equations $D_{\omega_1}x_1 = \Delta_1$ and $D_{\omega_1}x_2 = \Delta_2$;

and, at last, we calculate solutions on a new level by solving the equations

$$D_{\omega_2}\hat{y} = Dx_1, \quad D_{\omega_2}\hat{\dot{y}} = Dx_2.$$

The scheme implementation algorithm would be more economic if $\omega_2 = 0$; this takes place if the following condition is satisfied:

$$\alpha\gamma = \frac{\beta}{4}. \tag{18}$$

Here, $\omega_1 = \alpha + \gamma - \frac{1}{4}$. In this case, it is necessary "to invert" two operators D and D_{ω_1} only.

The necessity to factorise the operator $\tilde{\Delta}$ (and, as consequence, the operator Δ) is connected to the following fact. For the ODE systems (1) resulted from the approximation of hyperbolic equations by the net method or finite element method by spatial variables, the matrix corresponding to A is conditioned poorly and filled badly. The matrix corresponding to the operator Δ will have a greater number of nonzero elements and its conditionality is worsened, while each of factors D, D_{ω_1}, D_{ω_2} in representation Δ to be "inverted" has the same structure and properties as operators D, A do.

Let us consider examples of schemes which satisfy requirements of accuracy (12), (13) and of economy (18).

I The scheme with parameters $\gamma = \frac{1}{12}$, $\alpha = \frac{1}{10}$, $\beta = \frac{1}{60}$ (see [3]) has the accuracy order 4 (the condition (12) is met). The discriminant of the quadratic equation (17) is negative and, therefore, the implementation of such scheme in the field of real numbers by the specified algorithm is impossible (it is reasonable that such scheme can be implemented by inverting the operator Δ on each time step). The condition of scheme stability $\tau^2 \le \frac{10}{\lambda^2}$.

II By choosing $\gamma = \frac{1}{12}$ and satisfying (12) and (18), we shall obtain of order 4 with $\beta = \frac{1}{24}$, $\alpha = \frac{1}{8}$, that coinciding with one of schemes in [3]. Roots of the equation (17): $\omega_1 = -\frac{1}{24}$, $\omega_2 = 0$. The condition of scheme stability $\tau^2 \le \frac{8}{\lambda^2}$.

III Let us introduce a new scheme of fourth accuracy order missing in [3]. The originality of such scheme is that, for its implementation by algorithm, it is necessary to "invert" only one operator D, as $\omega_1 = \omega_2 = 0$. Let us subject the choice of parameters to conditions (12), (18) and

$$\alpha + \gamma = \frac{1}{4}. \tag{19}$$

From here, $\beta = \frac{1}{24}$, and γ, α are roots of a quadratic equation $x^2 - \frac{1}{4}x + \frac{1}{96} = 0$. Therefore, we may choose $\gamma = x_1$, $\alpha = x_2$, or $\alpha = x_1$, $\gamma = x_2$, where $x_1 = \frac{1}{8} + \frac{1}{8}\sqrt{\frac{23}{24}} \approx 0.2473681$, $x_2 = \frac{1}{8} - \frac{1}{8}\sqrt{\frac{23}{24}} \approx 0.0026318$. The condition of scheme stability $\tau^2 \le \frac{1}{x_1\lambda^2} \approx \frac{4}{\lambda^2}$.

IV Now, let us consider scheme of order 6 (this is also a new scheme), which would satisfy the condition (18). The parameters of such scheme are defined from the following system:

$$\alpha + \gamma = \beta + \frac{1}{6}, \quad \beta - 6\alpha\gamma + \frac{1}{40} = 0, \quad \alpha\gamma - \frac{\beta}{4} = 0.$$

The solution is: $\beta = \frac{1}{20}$, $\alpha + \gamma = \frac{13}{60}$, $\alpha\gamma = \frac{1}{80}$. The roots of the equation for α and γ are complex. The scheme may not be implemented even by inverting of the operator Δ.

V We shall point out to one more scheme of 6th accuracy order (new!), which has real roots of quadratic equation:

$$\alpha = \frac{7}{60}, \quad \beta = \frac{1}{30}, \quad \gamma = \frac{1}{12}.$$

The conditions (12), (13) are met for these values of parameters. Where $\omega_{1,2} = -\frac{1}{40} \pm \frac{1}{2}\sqrt{\frac{71}{72}}$. The condition of scheme stability $\tau^2 \le \frac{12}{\lambda^2}$.

3. Stability and Convergence of Schemes. We shall assume that H is a Hilbert space with a scalar product (u, v), $u, v \in H$. Similarly to work [1], we shall present the scheme (14) as

$$\bar{\mathrm{B}} W_t + \tilde{\mathrm{A}} \frac{\hat{W} + W}{2} = \tilde{\Phi}, \quad W(0) = W_0, \tag{20}$$

where:

$$\bar{\mathrm{B}} = \begin{pmatrix} \tilde{D}_\beta \tilde{D}_\gamma & 0 \\ 0 & \tilde{A}\tilde{D}_\alpha \end{pmatrix}, \quad \tilde{\mathrm{A}} = \begin{pmatrix} 0 & \tilde{D}_\beta \tilde{A} \\ \tilde{A}\tilde{D}_\beta & 0 \end{pmatrix}, \quad \tilde{\Phi} = (\tilde{D}_\beta \tilde{\varphi}_1, \tilde{A}\tilde{\varphi}_2),$$

$W_n = (\dot{w}^n, w^n) \in H^2 = H \oplus H$, $\bar{\mathrm{B}}, \tilde{\mathrm{A}}$ are operators from H^2 in H^2.

On the ground of the theorem 1 of [3], as $\tilde{D}^* = \tilde{D} > 0$, $\tilde{A}^* = \tilde{A} > 0$ and $\tilde{D}\tilde{A} = \tilde{A}\tilde{D}$, always supposing

$$\tilde{D} > m\tau^2 \tilde{A}, \quad m = \max\{\alpha, \beta, \gamma\} \tag{21}$$

the scheme (20) is uniformly stable, i.e. the following estimation is true:

$$\left\| W^{n+1} \right\|_{\tilde{\mathrm{A}}} \le \left\| W^n \right\|_{\tilde{\mathrm{A}}} \quad \forall n, \tag{22}$$

where $\|W^n\|^2_{\tilde{\mathrm{A}}} = \|w^n\|^2_{\tilde{A}\tilde{D}_\alpha} + \|\dot{w}^n\|^2_{\tilde{D}_\beta \tilde{D}_\gamma}$.

Let us require, similarly to [3], that a condition stronger than (21) is to be satisfied:

$$\tilde{D} - m\tau^2 \tilde{A} \ge \varepsilon E, \quad 0 < \varepsilon < 1. \tag{23}$$

Then,

$$\|W^n\|^2_{\tilde{\mathrm{A}}} \ge \varepsilon \|w^n\|^2_{\tilde{A}} + \varepsilon^2 \|\dot{w}^n\|^2 = \varepsilon \|y^n\|^2_A + \varepsilon^2 \|\dot{y}^n\|^2_D, \tag{24}$$

as

$$\|w^n\|^2_{\tilde{A}} = (\tilde{A}w^n, w^n) = (D^{-1/2} A D^{-1/2} D^{1/2} y^n, D^{1/2} y^n) = \|y^n\|^2_A,$$

$$\|\dot{w}^n\|^2 = (w^n, w^n) = (D^{1/2} y^n, D^{1/2} y^n) = \|\dot{y}^n\|^2_D.$$

Now, if we apply theorem 2 of [3] to the scheme (14), then, if (23) is satisfied, owing to interchangeability of operators $\tilde{A}\tilde{D} = \tilde{D}\tilde{A}$, we shall obtain the estimation of the solution by the right part

$$\|w^n\|_{\tilde{A}} + \|\dot{w}^n\| \le M(\left\| w^0 \right\|_{\tilde{A}\tilde{D}_\alpha} + \|\dot{w}^n\|_{\tilde{D}_\beta \tilde{D}_\alpha} + \|\tilde{\varphi}^n_1\|_{\tilde{A}^{-1}} + \|\tilde{\varphi}^n_2\|_{\tilde{D}} +$$

$$+ \left\|\tilde{\varphi}^0_1\right\|_{\tilde{A}^{-1}} + \left\|\tilde{\varphi}^0_2\right\|_{\tilde{D}} + \sum_{k=1}^{n} \tau \left(\left\|\tilde{\varphi}^k_{1,\bar{t}}\right\|_{\tilde{A}^{-1}} + \left\|\tilde{\varphi}^k_{2,\bar{t}}\right\| \right),$$

where $M = M(\varepsilon)$. Coming back to old variables y, $\dot{y}$, we see that the following theorem is true:

Theorem 1. *Let $D^* = D > 0$, $A^* = A > 0$. Then, if the following condition is satisfied:*

$$D - m\tau^2 A \geq \varepsilon D, \quad 0 < \varepsilon < 1,$$

the following estimation is true for the solution of scheme (4):

$$\|y^n\|_A + \|\dot{y}^n\|_D \leq M(\|y^0\|_{AD^{-1}D_\alpha} + \|\dot{y}^0\|_{D_\beta D^{-1}D_\alpha} + \|\varphi_1^n\|_{A^{-1}} + \|\varphi_2^n\|_{D^{-1}} +$$

$$+ \|\varphi_1^0\|_{A^{-1}} + \|\varphi_2^0\|_{D^{-1}} + \sum_{k=1}^{n} \tau(\|\varphi_{1,\bar{t}}^k\|_{A^{-1}} + \|\varphi_{2,\bar{t}}^k\|_{D^{-1}}).$$

For $\gamma = \frac{1}{12}$ and if the condition (12) is satisfied, an approximation error of scheme (4) on sufficiently smooth solutions has fourth order. Therefore, the following theorem is true:

Theorem 2. *Let conditions of the theorem 1 be satisfied, and let scheme parameters satisfy the conditions $\gamma = \frac{1}{12}$, $\alpha - \beta = \frac{1}{12}$. Then, the solution of scheme (4) converges to a sufficiently smooth solution of problem (1) with fourth order so that following estimations are true:*

$$\|y^n - u(t_n)\|_A \leq M\tau^4, \quad \|\dot{y}^n - \dot{u}(t_n)\|_D \leq M\tau^4.$$

The issue on convergence and accuracy of fourth order schemes with $\gamma \neq \frac{1}{12}$ and of sixth order schemes remains open (schemes III, IV, V from item 2).

For investigating of such schemes, we shall reduce the system of two-level equations (14) to two three-level schemes

$$\tilde{D}_\gamma \tilde{D}_\alpha w_{\bar{t}t} + \tilde{D}_\beta \tilde{A} \frac{\hat{w} + 2w + \breve{w}}{4} = \tilde{D}_\beta \frac{\tilde{\varphi}_1 + \breve{\tilde{\varphi}}_1}{2} + \tilde{D}_\gamma \tilde{\varphi}_{2,\bar{t}}, \tag{25}$$

$$\tilde{D}_\alpha \tilde{D}_\gamma \dot{w}_{\bar{t}t} + \tilde{A} \tilde{D}_\beta \frac{\hat{\dot{w}} + 2\dot{w} + \breve{\dot{w}}}{4} = \tilde{D}_\alpha \tilde{\varphi}_{1,\bar{t}} - \tilde{A} \frac{\tilde{\varphi}_2 + \breve{\tilde{\varphi}}_2}{2}. \tag{26}$$

The approximation error of such difference equations is the following value:

$$\psi = (\gamma + \alpha - \frac{1}{6} - \beta)O(\tau^2) + (\beta - 6\alpha\gamma + \frac{1}{40})O(\tau^4) + O(\tau^6),$$

that is, a value of order $O(\tau^4)$ if the condition (12) is satisfied, and a value of order $O(\tau^6)$ if (12) and (13) are satisfied simultaneously.

The stability of such difference schemes results from the general stability theory of three-level schemes [4]. The stability condition coincides with the condition (21), or $D > m\tau^2 A$, $m = \max\{\alpha, \beta, \gamma\}$. Here, the interchangeability of initial operators A and D is not assumed as operators $\tilde{A}$ and $\tilde{D}_\omega$ are interchangeable by construction. The accuracy order of schemes (25) and (26) as well as of their analogues for variables $y, \dot{y}$ coincides with the approximation order.

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Received 9/02/2005