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ON THE STABILITY OF FINITE-DIFFERENCE SCHEMES FOR ONE-DIMENSIONAL PARABOLIC EQUATIONS SUBJECT TO INTEGRAL CONDITIONS

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M. SAPAGOVAS

Abstract. The author prove the stability of a finite-difference scheme for the one-dimensional parabolic equation with the constant coefficients and nonlocal integral conditions. Stability analysis technique is based on calculation or estimation of the eigenvalues of non-symmetric difference matrix. In case of constant coefficients in integral condition it implies more general stability conditions, compared to that described in literature. The efficiency of the stability analysis presented in the paper is illiustrated by results of numerical experiment.

1. Introduction

The article deals with the stability of difference schemes for one-dimensional parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad x \in (0,1), \quad t \in (0,T) \tag{1}$$

subject to initial condition

$$u(0,x) = \varphi(x), \quad x \in (0,1) \tag{2}$$

as well as integral conditions (instead of common boundary-value conditions)

$$u(0,t) = \int_0^1 \alpha(x) u(x,t) \mathrm{d}x + \mu_1(t), \quad t \in (0,T), \tag{3}$$

$$u(1,t) = \int_0^1 \beta(x) u(x,t) \mathrm{d}x + \mu_2(t), \quad t \in (0,T). \tag{4}$$

The pioneering studies that had a great impact on the subsequent development of theory of differential equations subject to non-local conditions were the articles [1]–[3]. The great part of articles [4]–[10] dealing with non-local problem (1)–(4) employ either finite-difference or finite element technique. Substantiating theoretically application of these techniques and, in particular, studying the stability, one face an essential peculiarity of the problem under consideration, i.e. non-symmetric matrix of system of difference equations. The articles cited above [4]–[9] when dealing with the stability of difference schemes make certain assumptions related to the "smallness" of the coefficients $\alpha(x)$ and $\beta(x)$. These assumptions in most cases are either congruous or equivalent to the following inequalities:

$$|\alpha(x)| \le 1, \quad |\beta(x)| \le 1. \tag{5}$$

Key words and phrases. One-dimensional parabolic equation, nonlocal condition, finite-difference scheme, stability, eigenvalue problem.

Not providing any proof the article [10] states inaccurately, that difference schemes considered there are unconditionally stable, and no constraints on functions $\alpha(x)$ or $\beta(x)$ are imposed.

Article [11] proves that essentially weaker constraints compared to (5) suffice the existence and uniqueness of the solution of implicit difference scheme. Namely, boundedness of absolute value of both $\alpha(x)$ and $\beta(x)$ and fineness of numerical grid in both spatial and time directions imply existence and uniqueness of the solution.

This article explores the stability of implicit second order difference schemes. We prove the absolute stability of the difference scheme in special vector norm whenever $\alpha(x)$ and $\beta(x)$ satisfy completely different conditions than (5). These conditions evolve when exploring the eigenvalues of the matrix of the system of difference equations. For example, in the case $\alpha(x)$ and $\beta(x)$ are constant these new stability condition is $-\infty < \alpha + \beta < 2$, instead of $|\alpha| \leq 1$, $|\beta| \leq 1$, widening significantly the class of constraints (3) and (4) ensuring stability of the difference scheme. Similar stability analysis technique for different non-local conditions than (3), (4) is used in [12], but somehow this article remains unnoticed by specialists or undervalued.

Parabolic equations subject to other non-local conditions than (3), (4) were investigated by a series of authors. It is worth mentioning separately the class of problems for one- or two-dimensional parabolic equations subject to non-local conditions, linking only contour points [13]-[21]. More comprehensive review and bibliography of earlier works on the topic one can find in [6], [12], whereas recent works are overlooked in [10], [11], [22].

Investigation of that differential or difference eigenvalue problems with non-local conditions, which lie in the base of the technique presented in this article, make a self-contained problem area analysed in various articles, such as [12], [20], [21], [23]–[29].

2. Difference schemes

Differential equation (1) we approximate by the following system of difference equations:

$$\frac{u_i^{j+1} - u_i^j}{\tau} = \sigma \Lambda u_i^{j+1} + (1-\sigma)\Lambda u_i^j + \varphi_i^j, \tag{6}$$

where

$$i = \overline{1, N-1};\ j = \overline{0, M-1};\ h = 1/N;\ \tau = T/M,$$

$$\Lambda u_i^j = \frac{u_{i-1}^j - 2u_i^j + u_{i+1}^j}{h^2}$$

and φ_i^j is a value of function $f(x,t)$ at a certain point, close to the point (i,j). We approximate non-local and initial condition (2)–(4) by the following difference equations:

$$\begin{aligned} \sigma u_0^{j+1} + (1-\sigma)u_0^j = \sigma(\alpha, u^{j+1}) + (1-\sigma)(\alpha, u^j) + \\ + \sigma\mu_1^{j+1} + (1-\sigma)\mu_1^j, \end{aligned} \tag{7}$$

$$\begin{aligned} \sigma u_N^{j+1} + (1-\sigma)u_N^j = \sigma(\beta, u^{j+1}) + (1-\sigma)(\beta, u^j) + \\ + \sigma\mu_2^{j+1} + (1-\sigma)\mu_2^j,\ j = \overline{0, M-1}, \end{aligned} \tag{8}$$

$$u_i^0 = \varphi_i, \quad i = \overline{0, N}. \tag{9}$$

where we denote

$$(\alpha, u^j) = h\Big(\frac{\alpha_0 u_0^j + \alpha_N u_N^j}{2} + \sum_{i=1}^{N-1} \alpha_i u_i^j\Big),$$

$$(\beta, u^j) = h\Big(\frac{\beta_0 u_0^j + \beta_N u_N^j}{2} + \sum_{i=1}^{N-1} \beta_i u_i^j\Big).$$

For $\sigma = 1$ and $\varphi_i^j = f_i^{j+1}$, difference scheme (6)–(8) turns to be implicit with approximation error of $O(h^2 + \tau)$. For $\sigma = 0$ and $\varphi_i^j = f_i^j$, the difference scheme becomes explicit with the same approximation error, i.e. $O(h^2 + \tau)$. For $\sigma = 1/2$ and $\varphi_i^j = f_i^{j+1/2}$, we obtain implicit Crank-Nicolson scheme with approximation error of $O(h^2 + \tau^2)$.

For every fixed value of j, we put difference scheme (6)–(9) into some special matrix form. Let us begin with the purely implicit scheme (the case $\sigma = 1$):

$$\frac{u_i^{j+1} - u_i^j}{\tau} = \frac{u_{i-1}^{j+1} - 2u_i^{j+1} + u_{i+1}^{j+1}}{h^2} + f_i^{j+1}, \quad i = \overline{1, N-1}, \tag{10}$$

$$u_0^{j+1} = (\alpha, u^{j+1}) + \mu_1^{j+1}, \tag{11}$$

$$u_N^{j+1} = (\beta, u^{j+1}) + \mu_2^{j+1}. \tag{12}$$

To put this system into a special matrix form, rewrite (11) and (12) as a system of two equations in two unknowns u_0^{j+1} and u_N^{j+1}:

$$\begin{cases} (1 - \dfrac{h\alpha_0}{2})u_0^{j+1} - \dfrac{h\alpha_N}{2}u_N^{j+1} = h\displaystyle\sum_{i=1}^{N-1} \alpha_i u_i^{j+1} + \mu_1^{j+1}, \\ -\dfrac{h\beta_0}{2}u_0^{j+1} + (1 - \dfrac{h\beta_N}{2})u_N^{j+1} = h\displaystyle\sum_{i=1}^{N-1} \alpha_i u_i^{j+1} + \mu_2^{j+1}. \end{cases}$$

This system has a single solution, that is, unknowns u_0^{j+1} and u_N^{j+1} can be expressed linearly in terms of the rest variables u_i^{j+1}, $i = \overline{1, N-1}$ provided the determinant of the system is not a zero:

$$D = \begin{vmatrix} 1 - \dfrac{h\alpha_0}{2} & -\dfrac{h\alpha_N}{2} \\ -\dfrac{h\beta_0}{2} & 1 - \dfrac{h\beta_N}{2} \end{vmatrix} = 1 - \frac{h}{2}(\alpha_0 + \beta_N) + \frac{h^2}{4}(\alpha_0\beta_N - \alpha_N\beta_0) \neq 0. \tag{13}$$

When solving the system, we obtain:

$$u_0^{j+1} = \sum_{i=1}^{N-1} a_i u_i^{j+1} + \bar{\mu}_1^{j+1}, \tag{14}$$

$$u_N^{j+1} = \sum_{i=1}^{N-1} b_i u_i^{j+1} + \bar{\mu}_2^{j+1}, \tag{15}$$

where

$$a_i = \frac{1}{D}\Big(h\alpha_i - \frac{h^2\beta_N}{2}\alpha_i + \frac{h^2\alpha_N}{2}\beta_i\Big),$$

$$b_i = \frac{1}{D}\Big(h\beta_i - \frac{h^2\alpha_0}{2}\beta_i + \frac{h^2\beta_0}{2}\alpha_i\Big),$$

$$\bar{\mu}_1^{j+1} = \frac{\mu_1^{j+1}}{D}(1 - \frac{h\beta_N}{2}) + \frac{\mu_2^{j+1}}{D}\frac{h\alpha_N}{2},$$

$$\bar{\mu}_2^{j+1} = \frac{\mu_2^{j+1}}{D}(1 - \frac{h\alpha_0}{2}) + \frac{\mu_1^{j+1}}{D}\frac{h\beta_0}{2}.$$

Putting expressions for u_0^{j+1} and u_N^{j+1} into system (10) for $i = 1$ and $i = N-1$, we obtain the following form of the system:

$$\begin{cases} \dfrac{u_1^{j+1} - u_1^j}{\tau} = \dfrac{\sum\limits_{i=1}^{N-1} a_i u_i^{j+1} - 2u_1^{j+1} + u_2^{j+1}}{h^2} + f_1^{j+1} + \dfrac{\bar{\mu}_1^{j+1}}{h^2}, \\ \dfrac{u_1^{j+1} - u_1^j}{\tau} = \dfrac{u_{i-1}^{j+1} - 2u_i^{j+1} + u_{i+1}^{j+1}}{h^2} + f_i^j, \quad i = \overline{2, N-2}, \\ \dfrac{u_{N-1}^{j+1} - u_{N-1}^{j+1}}{\tau} = \dfrac{u_{N-2}^{j+1} - 2u_{N-1}^{j+1} + \sum\limits_{i=1}^{N-1} b_i u_i^{j+1}}{h^2} + f_{N-1}^{j+1} + \dfrac{\bar{\mu}_2^{j+1}}{h^2}. \end{cases} \tag{16}$$

Define square matrix A of order $N-1$:

$$A = \frac{1}{h^2} \begin{pmatrix} 2 - a_1 & -1 - a_2 & -a_3 & \dots & \dots & -a_{N-1} \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \vdots & \vdots & & \\ & & & \vdots & \vdots & \\ & & & -1 & 2 & -1 \\ -b_1 & -b_2 & \dots & -b_{N-3} & -1 - b_{N-2} & 2 - b_{N-1} \end{pmatrix}. \tag{17}$$

Finally, system (16) for any fixed $j = \overline{0, M-1}$ obtains the form

$$(E + \tau A)u^{j+1} = u^j + \tau f^{j+1}; \tag{18}$$

where E is identity matrix, $u^{j+1} = (u_1^{j+1}, u_2^{j+1}, \dots, u_{N-1}^{j+1})'$, f^{j+1} is a vector of order $N-1$, whose components are corresponding right-hand terms of system (16).

We note, that system (10)–(12) can be written in a matrix form as a system of order $N+1$ not using any preliminary rearrangements. Here the system is written as a system of order $N-1$. Lemma 1 will clear the purpose of this form. Moreover, system (10)–(12) can be always put into the form (18) provided inequality (13) holds. In case α and β are constants not depending on variable x, inequality (13) takes rather simple form:

$$h \neq \frac{2}{\alpha + \beta}. \tag{19}$$

Lemma 1. *Eigenvalue problem for matrix A*

$$Au = \lambda u \tag{20}$$

can be written as an eigenvalue problem of non-local difference problem:

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + \lambda u_i = 0, \quad i = \overline{1, N-1}, \tag{21}$$

$$u_0 = h\Big(\frac{\alpha_0 u_0 + \alpha_N u_N}{2} + \sum_{i=1}^{N-1} \alpha_i u_i\Big), \tag{22}$$

$$u_N = h\Big(\frac{\beta_0 u_0 + \beta_N u_N}{2} + \sum_{i=1}^{N-1} \beta_i u_i\Big). \tag{23}$$

Proof. Matrix equation (20) is obtained directly from (21)–(23) applying the same procedure as it was applied for system (10)–(12) when putting it into the equivalent form (18).

Next, instead of implicit scheme (6)–(9), consider explicit scheme ($\sigma = 0$):

$$\frac{u_i^{j+1} - u_i^j}{\tau} = \frac{u_{i-1}^j - 2u_i^j + u_{i+1}^j}{h^2} + f_i^j, \quad i = \overline{1, N-1}, \tag{24}$$

$$u_0^j = (\alpha, u^j) + \mu_1^j, \tag{25}$$

$$u_N^j = (\beta, u^j) + \mu_2^j, \tag{26}$$

subject to initial condition $u_i^0 = \varphi_i, i = \overline{0, N}$. Analogously to the implicit scheme, we put this system of difference equations into the matrix form

$$u^{j+1} = (E - \tau A)u^j + \tau f^j, \tag{27}$$

where A is the same matrix defined by (17). As before, here $j = \overline{0, M-1}$, $u^0 = \varphi$.

Let us explore the stability conditions of difference schemes (18) and (27). We stress that matrix A is non-symmetric.

3. The norm of non-symmetric matrix

In order to explore the stability of the difference schemes, define a special vector norm and matrix norm in a following way. Let A be a simple structured matrix, i.e. it has $N-1$ linearly independent eigenvectors. Denote the eigenvectors as $v_1, v_2, \ldots v_{N-1}$. Matrix

$$H = (v_1 v_2 \cdots v_{N-1}),$$

with the columns being vectors v_i, is non-singular. For any square matrix B of order $N-1$, we define the norm [30]:

$$\|B\|_* = \|H^{-1}BH\|_1 = \max_{1 \le i \le N-1} \sum_{j=1}^{N-1} |\tilde{b}_{ij}|, \tag{28}$$

where $\tilde{b}_{ij}$ are the elements of matrix $H^{-1}BH$. The definition of matrix H implies that $H^{-1}BH$ is diagonal matrix with the eigenvalues of matrix A in the main diagonal. Thus,

$$\|A\|_* = \|H^{-1}BH\|_1 = \max_{1 \le i \le N-1} |\lambda_i(A)| = \rho(A). \tag{29}$$

where $\rho(A)$ is a spectral radius of matrix A vector norm consistent with the matrix norm $||A||_*$ is given by [30]:

$$\|u\|_* = \|H^{-1}u\|_1 = \max_{1 \le i \le N-1} |\tilde{u}_i|, \tag{30}$$

where $\tilde{u}_i$ is a component of vector $H^{-1}u$.

In [24], norm $|| \cdot ||_*$ is used to solve the systems of difference equations with non-symmetric matrix. Here we use this norm to investigate stability of the systems of difference equations (18) and (27). First of all we stress, that this way of defining the matrix norm eliminates the difficulties in stability analysis arising from the non-symmetricity of the matrix. Thus, stability analysis of non-symmetric matrices, as in case of symmetric matrices, is reduced to calculation or estimating of eigenvalues of matrix A. However, the vector norm becomes more intricate compared to symmetric case, and interpretation of the norm is not so evident.

4. Stability of difference schemes

We will refer to the following stability definition of the difference schemes [31]: The difference scheme (18) or (27) is stable, if the following inequality holds:

$$\|u^{j+1}\| \leq M_1\|u^0\| + M_2 \max_{0\leq j\leq M-1} \|f^j\| \tag{31}$$

where M_1, M_2, are the constants not depending on h, τ, j.

Theorem 1. *Let A be a simple structured matrix with positive eigenvalues. Then system of difference equations (18) is unconditionally stable.*

Proof. Equality (18) implies:

$$\|u^{j+1}\|_* \leq \|(E+\tau A)^{-1}\|_* \cdot \|u\|_* + \tau\|(E+\tau A)^{-1}\|_* \cdot \|f^{j+1}\|_*. \tag{32}$$

Because all eigenvalues of matrix A are positive and $\tau > 0$, then for any eigenvalue of matrix $E + \tau A$ inequality $\lambda(E + \tau A) > 1$ holds. Therefore,

$$\lambda\Big((E+\tau A)^{-1}\Big) \leq q < 1.$$

Substituting this estimate into (32) we obtain

$$\|u^{j+1}\|_* \leq q\|u^j\|_* + \tau q\|f^{j+1}\|_* \tag{33}$$

or

$$\|u^{j+1}\|_* \leq q^{j+1}\|u^0\|_* + \tau\sum_{s=1}^{j+1} q^s\|f^{j+2-s}\|_*.$$

Hence

$$\|u^{j+1}\|_* \leq \|u^0\|_* + T \max_{1\leq j\leq N-1} \|f^j\|_*. \tag{34}$$

This completes the proof.

Theorem 2. *Let A be a simple structured matrix with positive eigenvalues. If*

$$\rho(A) \leq \frac{c}{h^2},$$

where c is constant not depending on τ or h, then explicit difference scheme (27) is conditionally stable for $\tau \leq \dfrac{2h^2}{c}$.

Proof. Since $0 < \lambda_i(A) \leq c/h^2$, then

$$|\lambda_i((E-\tau A)^{-1})| \leq 1, \quad \text{if } \tau \leq \frac{2h^2}{c}.$$

Denote

$$\max_{1\leq i\leq N-1} |\lambda_i((E-\tau A)^{-1})| = q_1 \leq 1.$$

In the same way as in the proof of theorem 1, we obtain inequality (33), with constant q_1 instead of q. Therefore, in this case inequality (34) holds, too.

This completes the proof.

To have a clear understanding what novelty theorems 1 and 2 give for stability analysis compared to constraints $|\alpha| \leq 1, |\beta| \leq 1$, let us consider the case where α and β both are constants.

5. Analysis of spectrum of matrix A

Let us investigate the eigenvalue problem of matrix A in case both $\alpha = const$, and $\beta = const$. In accordance to lemma 1, the eigenvalues of A can be found solving the eigenvalue problem (21) with non-local conditions (22)–(23). Notice, that problem (21)–(23) for constant $\alpha = const$, and $\beta = const$ is a finite-difference approximation of the differential problem

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \lambda u = 0, \tag{35}$$

$$u(0) = \alpha \int_0^1 u(x)\mathrm{d}x, \tag{36}$$

$$u(1) = \beta \int_0^1 u(x)\mathrm{d}x, \tag{37}$$

Theorem 3. *For constant $\alpha = const$, and $\beta = const$, difference problem (21) – (23) has a trivial eigenvalue if and only if $\alpha + \beta = 2$.*

Proof. To begin with, we will establish conditions for the matrix of the system (21) subject to constraints (22), (23) yielding a trivial eigenvalue $\lambda = 0$. To this end substitute $\lambda = 0$ into equation (21) and put down the general solution of this difference equation:

$$u_i = c_1 ih + c_2 \tag{38}$$

Let us find the values of constants c_1 and c_2 such, that solution (38) satisfies both constraints (22) and (23). To do this, we put (38) into (22) and (23):

$$\begin{cases} -\dfrac{\alpha}{2}c_1 + (1-\alpha)c_2 = 0, \\ (1-\dfrac{\beta}{2})c_1 + (1-\beta)c_2 = 0. \end{cases}$$

For this system to have a non-trivial solution (c_1, c_2), it is necessary and sufficient that the main determinant is zero:

$$D = \begin{vmatrix} -\dfrac{\alpha}{2} & 1-\alpha \\ 1-\dfrac{\beta}{2} & 1-\beta \end{vmatrix} = 0$$

or $\alpha + \beta = 2$. This completes the proof.

Theorem 4. *If $\alpha = const, \beta = const$, $\alpha + \beta > 2$ and $h < \dfrac{2}{\alpha+\beta}$, then problem (21)–(23) has single negative eigenvalue.*

Proof. Let $\lambda < 0$ in equation (21) and determine, when this equation has a non-trivial solution satisfying conditions (22) and (23). Put equation (21) into the form

$$u_{i-1} - 2(1 - \frac{\lambda h^2}{2})u_i + u_{i+1} = 0. \tag{39}$$

For $\lambda < 0$, inequality $1 - \frac{\lambda h^2}{2} > 1$ holds. Denote

$$1 - \frac{\lambda h^2}{2} = \cosh \varphi h. \tag{40}$$

The general solution of equation (39) has a form

$$u_i = c_1 \cdot \cosh i\varphi h + c_2 \sinh i\varphi h. \tag{41}$$

Substituting solution (41) into conditions (22), (23) and performing elementary rearrangements, we obtain

$$\begin{cases} \Big(1 - \dfrac{h\alpha}{2}\dfrac{\sinh\varphi}{\tanh\frac{\varphi h}{2}}\Big)c_1 - \dfrac{h\alpha}{2}\dfrac{\cosh\varphi - 1}{\tanh\frac{\varphi h}{2}}c_2 = 0, \\ \Big(\cosh\varphi - \dfrac{h\beta}{2}\dfrac{\sinh\varphi}{\tanh\frac{\varphi h}{2}}\Big)c_1 - \Big(\sinh\varphi - \dfrac{h\beta}{2}\dfrac{\cosh\varphi - 1}{\tanh\frac{\varphi h}{2}}\Big)c_2 = 0, \end{cases} \tag{42}$$

Equating determinant of the system to zero and performing a series of elementary reductions, we get

$$\sinh\varphi = \frac{h(\alpha+\beta)}{2}\frac{\cosh\varphi - 1}{\tanh\frac{\varphi h}{2}}$$

or

$$\sinh\frac{\varphi}{2}\Big(\tanh\frac{\varphi}{2} - \frac{2}{h(\alpha+\beta)}\tanh\frac{\varphi h}{2}\Big) = 0. \tag{43}$$

Equality

$$\sinh\frac{\varphi}{2} = 0$$

implies $\varphi = 0$. In accordance to (40), this contradicts to the assumption of the theorem $\lambda < 0$. Therefore,

$$\tanh\frac{\varphi}{2} - \frac{2}{h(\alpha+\beta)}\tanh\frac{\varphi h}{2} = 0. \tag{44}$$

Denote

$$f_1(\varphi) = \tanh\frac{\varphi}{2}, \quad f_2(\varphi) = \frac{2}{h(\alpha+\beta)}\tanh\frac{\varphi h}{2}. \tag{45}$$

Let us explore the behaviour of those functions over the interval $(0,\infty)$. Firstly, both functions are monotonous in the interval $(0,\infty)$ since $f_1(\varphi)$ increases from 0 to 1 and $f_2(\varphi)$ increases from 0 to $2/h(\alpha+\beta)$. Moreover,

$$f_1'(0) = \frac{1}{2}, \quad f_2'(0) = \frac{1}{\alpha+\beta}.$$

Therefore, if $\alpha+\beta \le 2$, then $f_1(\varphi) < f_2(\varphi)$ on the whole interval $\varphi \in (0,\infty)$. If $\alpha+\beta \ge 2/h$, then $f_1(\varphi) > f_2(\varphi)$ on the whole interval $\varphi \in (0,\infty)$. Next, if $2 < \alpha+\beta < 2/h$, then functions $f_1(\varphi)$ and $f_2(\varphi)$, being monotonous intersect exactly once over the whole interval $(0,\infty)$. Consequently, there exist a root φ^* of equation (44) if and only if

$$2 < \alpha+\beta < \frac{2}{h}. \tag{46}$$

Eigenvalue $\lambda*$ corresponding to the root φ^* can be calculated using formula (40):

$$1 - \frac{\lambda^* h^2}{2} = \cosh\varphi^* h.$$

This completes the proof.

Theorem 5. *If $\alpha = const$, $\beta = const$ and $\alpha+\beta < 2$, then all eigenvalues of difference problem (21)–(23) are real and different. Moreover, some of the eigenvalues depend on α and β, some of the eigenvalues remain constant for any value of α and β.*

Proof. We prove this theorem using the same technique as in theorem 4. For $\lambda > 0$, coefficient at u_i in equation (39) is less than 1. Let us consider separately the case

$$|1 - \frac{\lambda h^2}{2}| < 1.$$

Denote

$$1-\frac{\lambda h^2}{2}=\cos\varphi h. \tag{47}$$

Then difference equation (39) has a general solution of a form

$$u_i=c_1\cos i\varphi h+c_2\sin i\varphi h. \tag{48}$$

Put this solution into non-local conditions (22) and (23). Performing a series of elementary rearrangements we obtain

$$\begin{cases} \Big(1-\dfrac{h\alpha}{2}\dfrac{\sin\varphi}{\tan\dfrac{\varphi h}{2}}\Big)c_1-h\alpha\dfrac{\sin^2\dfrac{\varphi}{2}}{\tan\dfrac{\varphi h}{2}}c_2=0,\\ \Big(\cos\alpha-\dfrac{h\beta}{2}\dfrac{\sin\varphi}{\tan\dfrac{\varphi h}{2}}\Big)c_1+\Big(\sin\varphi-h\beta\dfrac{\sin^2\dfrac{\varphi}{2}}{\tanh\dfrac{\varphi h}{2}}\Big)c_2=0. \end{cases} \tag{49}$$

Equating the determinant of the system to zero, we get

$$\sin\varphi-h(\alpha+\beta)\frac{\sin^2\frac{\varphi}{2}}{\tan\frac{\varphi h}{2}}=0.$$

or

$$\sin\frac{\varphi}{2}\Big(\cos\frac{\varphi}{2}-\frac{h(\alpha+\beta)}{2}\frac{\sin\frac{\varphi}{2}}{\tan\frac{\varphi h}{2}}\Big)=0. \tag{50}$$

This equation yields two equations:

$$\sin\frac{\varphi}{2}=0 \tag{51}$$

and

$$\cos\frac{\varphi}{2}-\frac{h(\alpha+\beta)}{2}\frac{\sin\frac{\varphi}{2}}{\tan\frac{\varphi h}{2}}=0. \tag{52}$$

The roots of equation (51) does not depend on α or β:

$$\varphi_k=2k\pi,\quad k=1,2,\dots,\Big[\frac{N-1}{2}\Big], \tag{53}$$

where

$$\Big[\frac{N-1}{2}\Big]=\begin{cases}\dfrac{N-1}{2} & \text{if } N-\text{odd},\\ \dfrac{N-1}{2}-1 & \text{if } N-\text{even}.\end{cases}$$

As $k>[\frac{N-1}{2}]$, the eigenvalues obtained of equation (47) begin repeating. Next, put equation (52) into the form

$$\tan\frac{\varphi}{2}=\frac{2}{h(\alpha+\beta)}\tan\frac{\varphi h}{2} \tag{54}$$

Denote

$$f_1(\varphi)=\tan\frac{\varphi}{2},\quad f_2(\varphi)=\frac{2}{h(\alpha+\beta)}\tan\frac{\varphi h}{2}$$

We analyse functions $f_1(\varphi)$ and $f_2(\varphi)$ only over the interval $[0,N\pi]$, since both are periodic functions with common least period $N\pi$.

For $0<\alpha+\beta<2$, function $f_2(\varphi)>0$ over the interval $(0,N\pi)$. Thus, graphs of functions $f_1(\varphi), f_2(\varphi)$ intersect only once over each interval

$$(2k\pi,(2k+1)\pi),\quad k=0,1,\dots,N_1,$$

where

$$N_1 = \begin{cases} \dfrac{N-2}{2} & \text{if } N - \text{odd}, \\ \dfrac{N-3}{2} & \text{if } N - \text{even}. \end{cases}$$

For $\alpha + \beta < 0$ function $f_2(\varphi) < 0$ over the interval $(0, N\pi)$. Thus again, graphs of the functions $f_1(\varphi), f_2(\varphi)$ intersect only once over each interval

$$((2k-1)\pi, 2k\pi), \quad k = 1, 2, \dots, N_2,$$

where

$$N_2 = \begin{cases} \dfrac{N-1}{2} & \text{if} N - -\text{odd}, \\ \dfrac{N}{2} & \text{if } N - \text{even}. \end{cases}$$

For $\alpha + \beta = 0$ the roots of equation (54) have a form

$$\varphi_k = (2k-1)\pi, \quad k = 1, 2, \dots, N_2,$$

Consequently, in all the three cases $(0 < \alpha + \beta < 2, \alpha + \beta < 0, \alpha + \beta = 0)$ the number of roots of equations (54) and (51) is equal to $N - 1$. For every root φ_k, the corresponding eigenvalue λ_k is calculated by equation (47).

This completes the proof.

Remark. At the beginning of the proof of theorem 5 we took a separate case $|1 - \lambda h^2/2| < 1$ of a more general constraint $1 - \lambda h^2/2 < 1$. This is enough to get all of $N - 1$ positive eigenvalues. It can be checked directly, that in case $1 - \lambda h^2/2 \leq -1$ there are no positive eigenvalues at all or they appear only when constraint $\alpha + \beta \geq 2$ is imposed. For example, if $1 - \lambda h^2/2 = -1$ and number N is even, then $\lambda = h^2/2$ is an eigenvalue provided $\alpha + \beta = 2$.

6. Numerical results

We illustrate efficiency of findings we obtained in this article by presenting the numerical results of problem formulated in [5]. Problem (1)–(4) was solved in case of constant coefficients α and β and following expressions for

$$f(x,t) = -30x^4 + 6t^5,$$

$$\mu_1(t) = t^6 - \alpha(t^6 + \frac{1}{7}),$$

$$\mu_2(t) = 1 + t^6 - \beta(t^6 + \frac{1}{7}).$$

These functions are selected in the way that $u(x,t) = x^6 + t^6$ is a solution of problem (1)–(4). According to constraints $|\alpha| \leq 1, |\beta| \leq 1$, the following values of α and β satisfying the constraints are selected in [5]:

$$\alpha = 0.2; \quad \beta = 0,4;$$

$$\alpha = 0.4; \quad \beta = 0,6;$$

One more pair of values do not satisfying the constraints is chosen as well: $\alpha = 1,4$, $\beta = 0,4$.

Table 1

h	$\alpha = 0.5$ $\beta = 1$	$\alpha = -5$ $\beta = 5$	$\alpha = -13$ $\beta = 5$	$\alpha = -32$ $\beta = -32$
0.1	$-0.23 \cdot 10^{-1}$ $-0.53 \cdot 10^{-1}$	$0.81 \cdot 10^{-1}$ $-0.24 \cdot 10^{-1}$	$0.48 \cdot 10^{-1}$ $-0.12 \cdot 10^{-1}$	$0.18 \cdot 10^{-1}$ $-0.94 \cdot 10^{-2}$
0.05	$-0.59 \cdot 10^{-2}$ $-0.13 \cdot 10^{-1}$	$0.20 \cdot 10^{-1}$ $-0.59 \cdot 10^{-2}$	$0.12 \cdot 10^{-1}$ $-0.31 \cdot 10^{-2}$	$0.46 \cdot 10^{-2}$ $-0.23 \cdot 10^{-2}$
0.025	$-0.15 \cdot 10^{-2}$ $-0.33 \cdot 10^{-2}$	$0.51 \cdot 10^{-2}$ $-0.15 \cdot 10^{-2}$	$0.30 \cdot 10^{-2}$ $-0.78 \cdot 10^{-2}$	$0.12 \cdot 10^{-2}$ $-0.60 \cdot 10^{-3}$
0.0125	$-0.40 \cdot 10^{-3}$ $-0.89 \cdot 10^{-3}$	$0.14 \cdot 10^{-2}$ $-0.41 \cdot 10^{-3}$	$0.85 \cdot 10^{-3}$ $-0.22 \cdot 10^{-3}$	$0.33 \cdot 10^{-3}$ $-0.17 \cdot 10^{-3}$

Continuation

h	$\alpha = -30$ $\beta = 5$	$\alpha = -30$ $\beta = 20$	$\alpha = -120$ $\beta = 70$	$\alpha = -1000$ $\beta = 500$
0.1	$0.42 \cdot 10^{-1}$ $-0.10 \cdot 10^{-1}$	$0.92 \cdot 10^{-1}$ $-0.12 \cdot 10^{-1}$	$0.87 \cdot 10^{-1}$ $-0.95 \cdot 10^{-1}$	$0.76 \cdot 10^{-1}$ $-0.89 \cdot 10^{-2}$
0.05	$0.10 \cdot 10^{-1}$ $-0.26 \cdot 10^{-2}$	$0.23 \cdot 10^{-1}$ $-0.29 \cdot 10^{-2}$	$0.22 \cdot 10^{-1}$ $-0.24 \cdot 10^{-2}$	$0.19 \cdot 10^{-1}$ $-0.22 \cdot 10^{-2}$
0.025	$0.26 \cdot 10^{-2}$ $-0.65 \cdot 10^{-3}$	$0.59 \cdot 10^{-2}$ $-0.74 \cdot 10^{-3}$	$0.55 \cdot 10^{-2}$ $-0.61 \cdot 10^{-3}$	$0.47 \cdot 10^{-2}$ $-0.57 \cdot 10^{-3}$
0.0125	$0.74 \cdot 10^{-3}$ $-0.18 \cdot 10^{-3}$	$0.16 \cdot 10^{-2}$ $-0.21 \cdot 10^{-3}$	$0.15 \cdot 10^{-2}$ $-0.17 \cdot 10^{-3}$	$0.13 \cdot 10^{-2}$ $-0.16 \cdot 10^{-3}$

Table 2

h	$\alpha = -30$ $\beta = 31$	$\alpha = -30$ $\beta = 31.9$	$\alpha = 32$ $\beta = -30$	$\alpha = -30$ $\beta = 32.5$	$\alpha = -1000$ $\beta = 1000$
0.1	$0.86 \cdot 10^{0}$ $-0.35 \cdot 10^{-1}$	$0.28 \cdot 10^{1}$ $-0.96 \cdot 10^{-1}$	$-0.39 \cdot 10^{1}$ $-0.12 \cdot 10^{0}$	$0.24 \cdot 10^{2}$ $-0.69 \cdot 10^{0}$	$0.16 \cdot 10^{2}$ $-0.24 \cdot 10^{-1}$
0.05	$0.21 \cdot 10^{0}$ $-0.88 \cdot 10^{-2}$	$0.71 \cdot 10^{0}$ $-0.24 \cdot 10^{-1}$	$-0.96 \cdot 10^{0}$ $-0.30 \cdot 10^{-1}$	$0.56 \cdot 10^{1}$ $-0.16 \cdot 10^{0}$	$0.40 \cdot 10^{1}$ $-0.60 \cdot 10^{-2}$
0.025	$0.54 \cdot 10^{-1}$ $-0.22 \cdot 10^{-2}$	$0.18 \cdot 10^{0}$ $-0.60 \cdot 10^{-2}$	$-0.24 \cdot 10^{0}$ $-0.75 \cdot 10^{-2}$	$0.14 \cdot 10^{1}$ $-0.41 \cdot 10^{-1}$	$0.10 \cdot 10^{1}$ $-0.15 \cdot 10^{-2}$
0.0125	$0.15 \cdot 10^{-1}$ $-0.61 \cdot 10^{-3}$	$0.47 \cdot 10^{-1}$ $-0.16 \cdot 10^{-2}$	$-0.64 \cdot 10^{-1}$ $-0.20 \cdot 10^{-2}$	$0.36 \cdot 10^{0}$ $-0.10 \cdot 10^{-1}$	$0.28 \cdot 10^{0}$ $-0.42 \cdot 10^{-3}$

According to the stability condition $-\infty < \alpha + \beta < 2$ attained in our article, the difference scheme under consideration is stable over much wider set of values of parameters α and β compared to that in [5]. This is clearly seen in table 1, where error data are given. Error there is defined by $\varepsilon_i^j = u(x_i, t_j) - u_i^j$; where $u(x_i, t_j)$ are the values of exact solution of differential problem, and u_i^j are the values at the point (x_i, t_j) of the solution of corresponding difference problem.

For every value of h and for each pair of values of α and β table 1 shows the solution error at two characteristic points: the upper value ε_i^j is calculated at $x = 0,0; t = 1,0$

and the lower value is at $x = 0,5; t = 1,0$. In all of the cases the step τ was calculated according to the formula $\tau = h^2$.

The results in table 1 show, that the difference scheme is stable for rather large absolute values of α and β provided inequality $\alpha + \beta < 2$ holds.

Table 2 is build in the same way. There we can see the results for the absolute values of α or β being much greater than 1 with the sum of these numbers close to upper bound of the stability constraint, i.e. $\alpha + \beta$ is close to 2. The results show stability of the scheme, yet computational error increases significantly as $\alpha + \beta$ increases.

Table 3

h	$\alpha = 10$ $\beta = 10$	$\alpha = 20$ $\beta = 20$	$\alpha = 50$ $\beta = 50$	$\alpha = 1000$ $\beta = 1000$
0.1	$0.22 \cdot 10^{-1}$ $-0.69 \cdot 10^{-2}$	$0.21 \cdot 10^{-1}$ $-0.80 \cdot 10^{-2}$	$0.20 \cdot 10^{-1}$ $-0.85 \cdot 10^{-2}$	$0.19 \cdot 10^{-1}$ $-0.88 \cdot 10^{-2}$
0.05	$0.88 \cdot 10^{-1}$ $-0.28 \cdot 10^{-2}$	$0.82 \cdot 10^{-2}$ $-0.33 \cdot 10^{-2}$	$0.79 \cdot 10^{-2}$ $-0.35 \cdot 10^{-2}$	$0.48 \cdot 10^{-2}$ $-0.22 \cdot 10^{-2}$
0.025	∞ ∞	$0.36 \cdot 10^{-2}$ $-0.14 \cdot 10^{-2}$	$0.35 \cdot 10^{-2}$ $-0.15 \cdot 10^{-2}$	$0.12 \cdot 10^{-2}$ $-0.56 \cdot 10^{-3}$
0.0125	∞ ∞	$0.16 \cdot 10^{-2}$ $-0.66 \cdot 10^{-2}$	$0.16 \cdot 10^{-2}$ $-0.77 \cdot 10^{-2}$	$0.32 \cdot 10^{-2}$ $-0.16 \cdot 10^{-3}$
0.0625	∞ ∞	∞ ∞	∞ ∞	$0.22 \cdot 10^{-3}$ $-0.12 \cdot 10^{-3}$

Table 3 is build in the same way as table 1, too. It shows the numerical results for the values of α and β not satisfying the stability constraint, i.e. $\alpha + \beta > 2$, for any value of h, though instability occurs only when $h < \frac{2}{\alpha+\beta}$ (see Theorem 4).

According to theorem 4, if $\alpha+\beta > 2$ and $h < \frac{2}{\alpha+\beta}$ there exists one negative eigenvalue, therefore inequality $\rho\Big((E + \tau A)^{-1}\Big) < 1$ subject to the value of τ can change to the inequality $\rho\Big((E + \tau A)^{-1}\Big) > 1$. In case h is not a sufficiently small, matrix A has no negative eigenvalue, and inequality $\rho\Big((E + \tau A)^{-1}\Big) < 1$ remains true. Thus, for $h > 2/(\alpha + \beta)$ and $\alpha + \beta > 2$ numerical computations using implicit scheme (10) - (12) can preserve stability. Such a scheme is called "quasistable". Some results in table 1 illustrate this fact.

It is worth mentioning that difference schemes of higher order accuracy, say $O(h^4 + \tau^2)$, for differential equations with constant coefficients subject to non-local conditions can be highly efficient [5].

7. Conclusion

Finite difference method applied to one-dimensional parabolic equations subject to non-local integral conditions yields a system of difference equations with non-symmetric matrix. These systems of difference equations, unlike symmetric difference schemes, lack efficient methods for stability analysis. Stability analysis technique presented in this article is based on calculation or estimation of the eigenvalues of non-symmetric difference matrix. We show that this method is efficient for a certain class of problems with non-local conditions. In case of constant coefficients α and β present in non-local condition it implies more realistic stability constraints, compared to that described in literature.

Undoubtedly, this technique can be generalised both for different types of non-local conditions and two- or three-dimensional parabolic equations. In order to evaluate more comprehensively the efficiency of the technique presented, additional investigation on meaningfulness of usage of the norm $|| \cdot ||_*$ should be accomplished. It is not clear yet, whether the usage of this norm in certain situations yield quantitatively poorer results compared to the results obtained for symmetric difference schemes. The efficiency of the stability analysis presented for differential equations with variable coefficients subject to non-local conditions is still not clear, too.

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Institute of Mathematics And Informatics, Akademijos 4, Lt-08663, Vilnius, Lithuania
E-mail address: mifosap@ktl.mii.lt

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