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CONSTRUCTING OF H-ADAPTIVE FINITE ELEMENT METHOD FOR PIEZOELECTRICITY PROBLEM

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Abstract. Based on a classic Galerkin scheme a piecewise-quadratic a posteriori error estimator (AEE) has been built for finite element method piecewise-linear approximation of a stationary piezoelectricity problem for elastic deformation and electric potential. In addition to energy error characteristics on a triangular element, the proposed AEE allows to calculate approximate point-values of errors in center of its sides. Using this AEE and Delaunay triangulation constructing algorithm an h-adaptive FEM scheme has been built, which convergence computing characteristics are illustrated by results of the solved model problem.

1. Introduction

Among various applications of piezoelectrics, see for example [5–8, 13], their application as sensors of engineering construction deformation state plays a significant role. In such cases piezoelectric plays the role of closed patch, which gets load of the under-control device and passes its characteristic as electric potentials to the data analyzing center, which responds to the received information content. Reliability of such a control system largely depends on the piezoelectric physical-mechanical structure knowledge completeness and special energy transducing features between mechanical and electrical fields. Due to the complexity of analyzing physical-mechanical fields interaction in the piezoelectric, it is important to create computing experiment means, which guarantee results with predefined accuracy.

Therefore, the goal of that paper is constructing of an adaptive FEM scheme, which creates the mentioned set of computer modeling instruments in the piezoeffect problem. In section 2 we formulate boundary and appropriative variational piezoelectricity problem, which was studied in [9]. In section 3, based upon a classic Galerkin procedure, a problem of finding FEM approximation in terms of displacement and electric potential and a problem of the a posteriori error estimator of their errors are formulated. Although the piezoelectricity problem relates to the mixed variational problems, in that case it was possible to use an AEE constructing technique, borrowed by us from [3, 4, 10, 12] for the elastostatic problem. In sections 4 and 5 it is proposed a concrete realization of the abstract scheme using displacement and potential piece-linear approximation on a triangle mesh, and also a piecewise-quadratic error approximation, which enable its calculation on the finite element border. In the last section the numeric experiment results confirm reliability of the proposed methodology.

2. Formulation of piezoelectricity stationary problem

The following physical fields' interaction in piezoelectricity boundary problem will be observed below. Let observed piezoelectric occupy a limited bounded domain Ω of points $x = (x_1, \ldots, x_d)$ of Euclidean space $\mathbb{R}^d$ with continuous Lipschitz boundary Γ and normal unit vector $n = \{n_i\}_{i=1}^{d}$ to Γ, where $n_i = cos(n, x_i)$. We need to find an elastic

† *Key words.* Piezoelectricity, finite element method (FEM), a posteriori error estimator (AEE), h-adaptation of triangulation.

displacement vector $u = \{u_i(x)\}_{i=1}^d$ and electric potential $p = p(x)$ which satisfy equation of elastostatics and electric field (hereinafter we will assume summation for repeated indexes)

$$\begin{cases} -\sigma_{ij,j} = \rho f_i, \\ \sigma_{ij} = c_{ijkm}\varepsilon_{km}(u) - e_{kij}E_k(p), \\ \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}) \text{ in } \Omega, \end{cases} \tag{2.1}$$

$$\begin{cases} D_{m,m} = \rho_*, \\ D_k = g_{km}E_m(p) + e_{kij}\varepsilon_{ij}(u), \\ E_k(p) = -p_{,k} \text{ in } \Omega, \end{cases} \tag{2.2}$$

and boundary conditions

$$\begin{cases} u_i = 0 \text{ on } \Gamma_u,\ \Gamma_u \subset \Gamma,\ mes\,(\Gamma_u) > 0, \\ \sigma_{ij}n_j = \hat{\sigma}_i \text{ on } \Gamma_\sigma,\ \Gamma_\sigma = \Gamma \setminus \Gamma_u, \\ p = 0 \text{ on } \Gamma_p,\ \Gamma_p \subset \Gamma,\ mes\,(\Gamma_p) > 0, \\ D_k n_k = \hat{D} \text{ on } \Gamma_d,\ \Gamma_d = \Gamma \setminus \Gamma_p. \end{cases} \tag{2.3}$$

Here $\rho = \rho\,(x)$ - piezoelectric mass density, $f_i = \{f_i(x)\}$ - vector of volume forces, $\rho_* = \rho_*(x)$- intrinsic charge density, $\sigma_{ij}(x)$ and ϵ_{ij} - symmetric stress and strain tensors components, $D_k(x)$ and $E_k(x)$ - induction and electric field tension vector components respectively. Components $c_{ijkm}(x)$ describe piezoelectric elasticity tensor with common properties of symmetry and elipticity, $e_{kij}(x)$ and $g_{ij}(x)$ define piezoelectricity and dielectric susceptibility modules, $\hat{\sigma} = \{\hat{\sigma}_i(x)\}$ and $\hat{D} = \hat{D}(x)$ define surface force and electrical induction vectors respectively. Detailed information about piezoelectric physical characteristics and piezoeffect model can be found in monographs [5–8, 13] and others.

Let us define the spaces of the acceptable elastic displacements and electric potentials

$$\begin{aligned} V &= \{v \in \left[H^1(\Omega)\right]^d : v = 0 \text{ on } \Gamma_u\}, \\ Q &= \{q \in H^1(\Omega) : q = 0 \text{ on } \Gamma_p\}, \end{aligned} \tag{2.4}$$

respectively and let us denote the space $\Phi := V \times Q$ and it conjugated space $\Phi' := V' \times Q'$. Taking into account piezoelectric boundary problem(2.1)-(2.3) let us introduce its corresponding variational form [9]

$$\begin{cases} \text{given } (l, r) \in \Phi';\ \text{find } \phi = (u, p) \in \Phi \text{ such that} \\ c(u, v) - e(p, v) = < l, v >,\ \forall v \in V, \\ g(p, q) + e(q, u) = < r, q >,\ \forall q \in Q, \end{cases} \tag{2.5}$$

where bilinear and linear forms are defined by the following expressions

$$c(u, v) = \int_\Omega c_{ijkm}\varepsilon_{ij}(u)\epsilon_{km}(v)dx,$$

$$e(q, v) = \int_\Omega e_{kij}E_k(u)\varepsilon_{ij}(v)dx\ \forall u, v \in V,$$

$$< l, v >= \int_\Omega \rho f_i v_i dx + \int_{\Gamma_\sigma} \hat{\sigma}_i v_i d\gamma\ \forall u, v \in V,$$

$$g(p, q) = \int_\Omega g_{km}E_k(p)E_,(q)dx,$$

$$< r, q >= \int_\Omega \rho_* q dx + \int_{\Gamma_d} q\hat{D}_i n_i d\gamma\ \forall p, q \in Q.$$

In order to simplify our notations, let us define on space Φ a bilinear form

$$\begin{aligned} \Pi(\phi,\varphi) &:= c(u,v) + g(p,q) \\ &\quad + [e(q,u) - e(p,v)] \ \ \forall\ \phi = (u,p),\ \ \varphi = (v,q) \in \Phi \end{aligned} \tag{2.6}$$

and a linear functional

$$< \chi, \varphi > = < l, v > + < r, q > \ \ \forall\ \varphi = (v,q) \in \Phi. \tag{2.7}$$

Then, variational problem of piezoelectricity(2.5) can be written in a short form

$$\begin{cases} \text{given } \chi \ \in \Phi'; \\ \text{find vector } \phi = (u,p) \in \Phi \text{ such that} \\ \Pi(\phi,\varphi) = < \chi, \varphi > \ \ \forall\ \varphi = (v,q) \in \Phi. \end{cases} \tag{2.8}$$

Taking into account the properties of bilinear forms of the problem (2.5) [9], it is useful to define energy norms at acceptable functions spaces

$$\begin{cases} ||u||_V = c^{1/2}(u,u) \ \ \forall\ u \in V, \\ ||p||_Q = g^{1/2}(p,p) \ \ \forall\ p \in Q \end{cases} \tag{2.9}$$

and

$$||\varphi||_\Phi = \sqrt{||v||_V^2 + ||q||_Q^2} \ \ \forall\ \varphi = (v,q) \in \Phi. \tag{2.10}$$

3. Problem of FEM approximation error

By imitating technique [4], let us describe the main steps of AEE constructing, related to piezoelectricity problems and announced in [2]. It is useful to employ AEE as an indicator of adaptation necessity. Application of Galerkin scheme implies transference of variational problem (2.8) solution finding from space Φ to its finite subspace $\Phi_h \subset \Phi$, $\dim \Phi_h = N(h) < +\infty$, selected in a certain way. Thus, a problem discretized by Galerkin (2.8) is the following

$$\begin{cases} \text{given } \chi \ \in \Phi' \text{ to find } \phi_h = (u_h, p_)) \in \Phi_h \text{ such that} \\ \Pi(\phi_h,\varphi) = < \chi, \varphi > \ \ \forall\ \varphi = (v,q) \in \Phi_h. \end{cases} \tag{3.1}$$

Now, due to (3.1), we can formulate variational problem about finding FEM discretization error:

$$\begin{cases} \text{given approximation } \phi_h \ \in \Phi_h; \\ \text{find error } e := \phi - \phi_h = (\gamma,\xi) \in E := \Phi \setminus \Phi_h \text{ such that} \\ \Pi(e,\varphi) = < \rho(\phi_h), \varphi > \ \ \forall\ \varphi \in E, \\ \text{where } \ < \rho(\phi_h), \varphi > := < \chi, \varphi > - \Pi(\phi_h,\varphi)\ \forall\ \varphi \in E. \end{cases} \tag{3.2}$$

Now, let us discretize a problem (3.2) by means of already used Galerkin scheme in a certain finite subspace of error space. Actually, in such a way we come up to problem of an a posteriori error estimator of FEM approximation:

$$\begin{cases} \text{given approximation } \phi_h \ \in \Phi_h \text{ and } E_h \subset E,\ \dim E_h < +\infty; \\ \text{find estimator } e_h := (\gamma_h, \xi_h) \in E_h \text{ such that} \\ \Pi(e_h,\varphi) = < \rho(\phi_h), \varphi > \ \ \forall\ \varphi \in E_h. \end{cases} \tag{3.3}$$

In case the problem of an a posteriori estimator (3.3) has been already solved, then the value of energy norm

$$||e_h||^2_\Phi = \Pi(e_h, e_h) =< \rho(\phi_h), e_h > \tag{3.4}$$

tells us about approximate value of the found FEM approximation error. In consideration of the intention to realize described scheme on the basis of FEM technology, it should be denoted, that it gives information about local error distribution in domain Ω in a natural way. More precisely, if we use $\Im_h$ division of the domain Ω to finite elements for the calculation, $\Im_h = \{K\}$, then a posteriori error estimator e_h in the piecewise way is defined on each element K. Thus, its norm (3.4) can be represented in a natural way as a sum of

$$||e_h||^2_\Phi = \sum_{K\in\Im_h} \eta^2_K(e_h) \tag{3.5}$$

where $\eta_K(e_h)$ is called an error indicator on element K, which are determined by the norm of estimator on it

$$\begin{aligned} \eta^2_K(e_h) = |e_h|^2_K := \\ \int_K [c_{ijkm}\varepsilon_{ij}(\gamma_h)\varepsilon_{km}(\gamma_h) + g_{km}E_k(\xi_h)E_m(\xi_h)]dx \quad \forall\ K \in \Im_h. \end{aligned} \tag{3.6}$$

This important feature of the FEM scheme allows creating FEM approximations error control system with the help of goal oriented refinement/coarsening of finite elements on the basis of different criteria, for example, to get equal pre-defined error level for all elements of a triangulation [1].

4. Piecewise-linear FEM approximation

Below, we will use division $\Im_h$ of domain Ω into triangle finite elements K, $h_K :=$ diam$K$, $h := \max_{K\in\Im_h} h_K$. Considering triangle K with coordinates $A_i = (x_i, y_i),\ i = 1, 2, 3$, see 1, let us define barycentre coordinates on it [11]

$$\begin{aligned} &L_i(x, y) := \frac{(x_j y_m - x_m y_j) + (y_j - y_m)x - (x_j - x_m)y}{2|K|}, \\ &i = 1, 2, 3, \quad i \to j \to m \to i, \end{aligned} \tag{4.1}$$

with the help of which we will define piecewise linear approximations of the displacement and electric potential, for example,

$$p(x, y) \cong p^K(x, y) := \sum_{i=1}^{3} p^K_i L_i(x, y) \quad \forall\ (x, y) \in K \quad \forall\ K \in \Im_h. \tag{4.2}$$

In this way we implicitly define the simplest of all available finite spaces of approximation $\Phi_h \subset \Phi$. Its elements have piecewise constant gradients on division $\Im_h$, so it can significantly simplify calculation of linear algebraic equations system coefficients of the problem (3.1). Due to repeated forming and solving of that equations in process of $h-$adaptation, the argument of 'simple calculations' can play the main role in practical application.

5. A posteriori error estimator

Having selected such a structure of displacement and potential approximations, constructing of error approximation space E_h for solving problem of estimator (3.3) requires some sharpness. Let us define desired a posteriori error estimator on triangle K by the expression

$$e_K(x,y) := 4L_1(x,y)L_2(x,y)\begin{pmatrix}\gamma_K^{12}\\ \xi_K^{12}\end{pmatrix} + 4L_2(x,y)L_3(x,y)\begin{pmatrix}\gamma_K^{23}\\ \xi_K^{23}\end{pmatrix}$$
$$+ 4L_3(x,y)L_1(x,y)\begin{pmatrix}\gamma_K^{31}\\ \xi_K^{31}\end{pmatrix} \quad \forall\,(x,y)\in K \;\; \forall\, K\in\Im_h, \tag{5.1}$$

its coefficients $e_K^{ij} = (\gamma_K^{ij}, \xi_K^{ij})^T$ will be consecutively calculated in the following way. For example, to find value $e_K^{12} = (\gamma_K^{12}, \xi_K^{12})^T$ we create quadrangle $Q := K \cup K'$, where finite element K' has a joint side A_1A_2 with triangle K, look pic. 1. Having marked its barycenter coordinates by $L_i' = l_i'(x,y)$, we consider piecewise defined quadratic bubble-function

$$b_Q(x,y) := \begin{cases} 4L_1(x,y)L_2(x,y) & \forall\,(x,y)\in K,\\ 4L_1'(x,y)L_2'(x,y) & \forall\,(x,y)\in K', \end{cases} \tag{5.2}$$

with $\sup b_Q = \overline{Q}$.

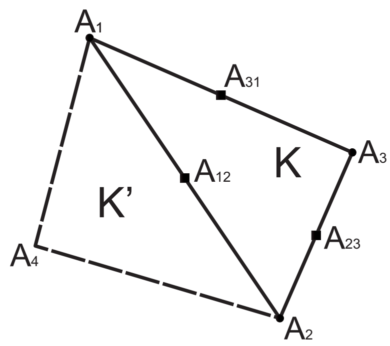


Fig. 1. Creating quadrangle $Q = K \cup K'$ for calculating e_K^{12}

Function (5.2) is equal to zero on all sides of quadrangle Q and equal to 1 in the center $A_{12} = \left(\frac{1}{2}(x_1+x_2), \frac{1}{2}(y_1+y_2)\right)$ of the segment A_1A_2. Now, assuming that searched error estimator on quadrangle Q has a form of

$$e_Q(x,y) := b_Q(x,y)e_K^{12} \quad \forall\,(x,y)\in Q$$

we can find vector e_K^{12} as a solution of the problem (3.3) with E_h space basis in the following form

$$\varphi^1 = (b_Q, 0, 0),\; \varphi^2 = (0, b_Q, 0),\; \varphi^3 = (0, 0, b_Q).$$

Let us denote, that taking into account the carried out construction, the calculated vector value e_K^{12} gives us point-value approximation to the FEM approximation error in the center A_{12} of the segment A_1A_2,

$$e(A_{12}) \cong e_h(A_{12}) = e_K^{12}. \tag{5.3}$$

In the same manner we calculate the rest of coefficients e_K^{23}, e_K^{31} of a posteriori error estimator of the FEM approximation (5.1) on the finite element K.

6. Adaptation criterion and strategy

Taking into account proposed in p.4 postprocess FEM approximation error calculation way by its nature, we can formulate problem of finding approximation with pre-guarantied accuracy of approach to the exact solution $\phi \in \Phi$ of the variational problem (2.8). To be more precise, we will orientate ourselves on solving of the following problem [3]:

$$\left\{\begin{array}{l}\text{given admissible level } \theta = const > 0 \text{ of the relative}\\ \text{approximation error in energy norm } ||\cdot||_\Phi\\ \text{of the admissible function space } \Phi;\\ \text{to find triangulation } \Im_h = \{K\} \text{ for constructing of}\\ \text{approximation space } \Phi_h \subset \Phi \text{ such, that}\\ \text{Galerkin's approximation } \phi_h \in \Phi_h \text{ founded in problem (3.1),}\\ \text{satisfies accuracy condition}\\ \dfrac{\eta_K(e_h)}{\sqrt{M_h^{-1}\left[||\phi||_\Phi^2 + \eta^2(e_h)\right]}} \times 100\% \leq \theta \quad \forall\, K \in \Im_h,\\ \text{where } M_h \text{ - number of finite element of the triangulation } \Im_h.\end{array}\right. \tag{6.1}$$

The above formulated problem (6.1) was solved by us in iterations of the consecutive local adaptation of current triangulations as follows. If a declared accuracy criterion from (6.1) is not executed on element K, then we add its mass center to existing node set. After declaration of all new nodes, next division $\Im_h$ is generated accordingly Delaunay triangulation algorithm. This recurrent process ends at the division, which has no element that break the condition (6.1).

7. Analysis of numerical experiment results

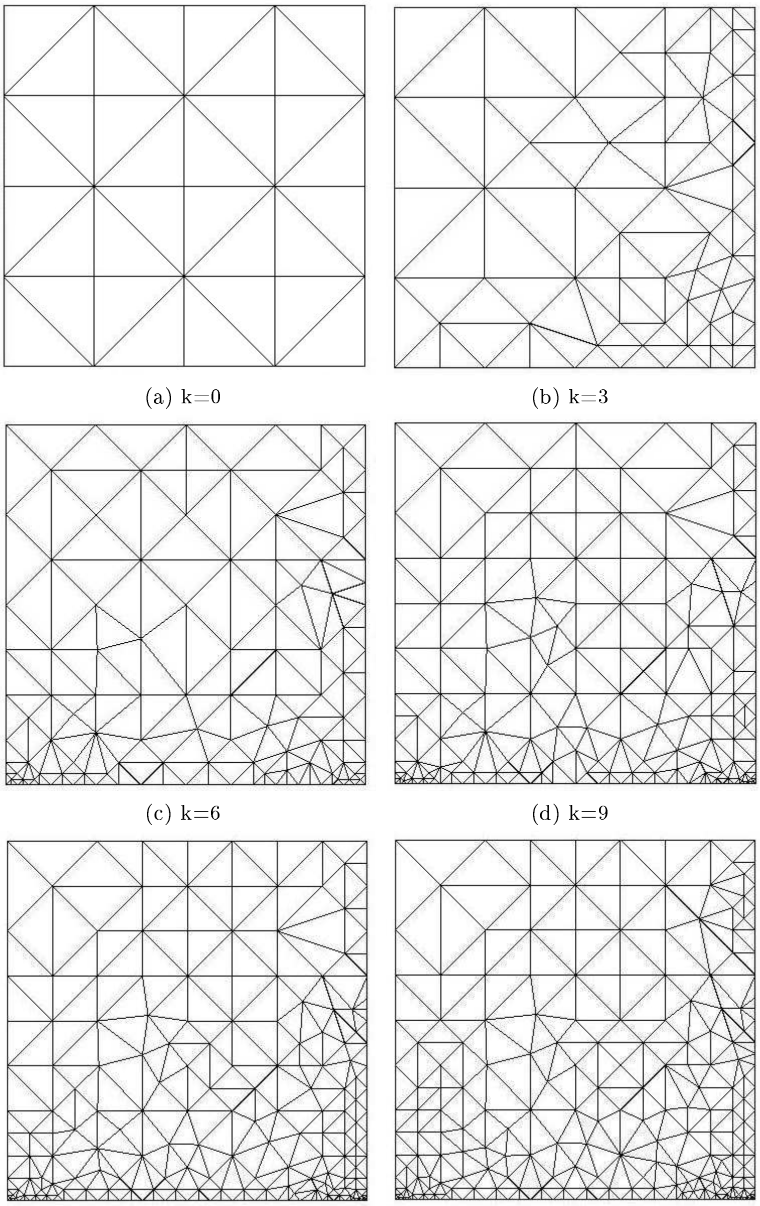
Developed $h-$ adaptive scheme from sections 2-5 was applied to the series of the piezoelectricity model problems. Let us denote, that the convergence pace p_h can be calculated according to the formula

$$p_h := 2\frac{\ln||e_h^m|| - \ln||e_h^k||}{\ln N_k - \ln N_m}, \; 0 \leq m, \; k = m+1, m+2, \ldots .$$

Let us examine lithium-niobite plate with dimensions 1×1 cm, with one side clamped and grounded, that is $\Omega = (0, 0.01) \times (0, 0.01)$, $u(x, 0) = 0$, $p(x, 0) = 0$. The force $g = 5 \cdot 10^6 \; H/m^2$ is applied on the vertical side $x = 0.01$

The initial triangulation consists of 32 finite elements. Using of linear approximations generates 25 nodes. Let us set an accuracy level equal to 5%.

Figure 2 shows triangulations received as a result of mesh adaptation in places of the largest errors. As it was expected, the largest error values are recognized by AEE on the clamped side elements. Actually, the local triangle mesh refinement occurs consecutively on their plane, what finally identifies, that the structure of the searched solutions contains features near the ends of the clamped side. Moreover, the difference among built meshes structures near that edge strictly shows the difference of local solution behavior in these regions: node distribution control system refines mesh near loaded plate edge.



(a) k=0 (b) k=3

(c) k=6 (d) k=9

(e) k=12 (f) k=15

Fig. 2. Built triangulations

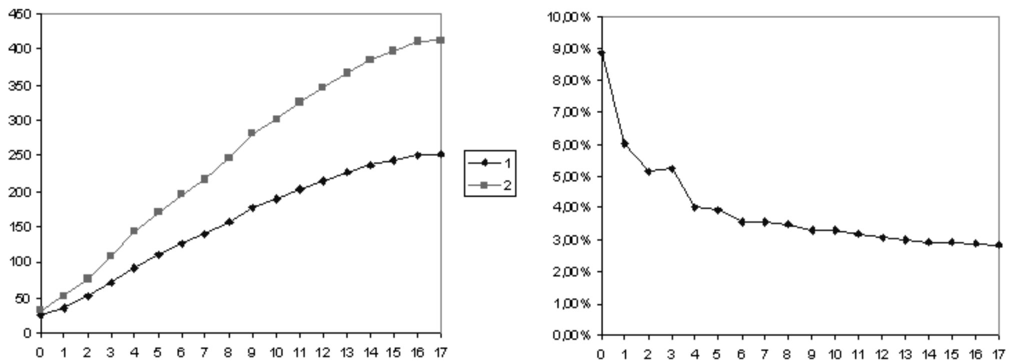
Regarding convergence features of the proposed FEM $h-$ adaptive scheme, we can come to a conclusion based on the data from table 1.

Tabl. 1. Step-by-step results of adaptation . Here k - adaptation step number, M_h - quantity of finite elements, N_h - quantity of triangulation $\Im_h$ nodes

k	N_h	M_h	$\min h_k$	$\max h_k$	$\lvert\phi_h\rvert_\Phi \times 10^{-6}$	$\lvert e_h\rvert_\Phi \times 10^{-6}$	$\delta_h\%$	p_h
0	25	32	0,35355	0,35355	294878	26243	8,9%	
1	36	52	0,17678	0,35355	307219	18547	6,0%	1,90
2	52	77	0,12500	0,35355	312277	16146	5,2%	1,33
3	71	109	0,08839	0,35355	313242	16349	5,2%	0,91
4	92	144	0,04419	0,35355	315418	12628	4,0%	1,12
5	111	171	0,03125	0,35355	315680	12497	4,0%	0,99
6	126	196	0,02210	0,25000	316217	11248	3,6%	1,05
7	139	216	0,02210	0,25000	316551	11283	3,6%	0,98
8	157	247	0,01563	0,25000	316964	11042	3,5%	0,94
9	178	281	0,01105	0,25000	317252	10519	3,3%	0,93
10	189	301	0,01105	0,25000	317407	10447	3,3%	0,91
11	203	326	0,01105	0,25000	317548	10132	3,2%	0,91
12	215	346	0,00781	0,25000	317744	9739	3,1%	0,92
13	227	367	0,00552	0,25000	317823	9487	3,0%	0,92
14	237	385	0,00552	0,25000	317964	9262	2,9%	0,92
15	243	397	0,00552	0,25000	318041	9282	2,9%	0,91
16	251	411	0,00552	0,25000	318089	9105	2,9%	0,92
17	252	412	0,00552	0,25000	318089	9091	2,9%	0,93

Graphics on fig. 3a show that adaptation close to linear nodes and finite elements quantity growth on the initial steps slow down their growing on the final steps.

According to this we have monotonic growth of calculated energy norms of the FEM approximation, which on the final steps are convergent with 6-figure precision. Similarly, the sequence of relative errors $\delta_h := \dfrac{\|e_h\| \cdot 100\%}{\sqrt{\|\phi_h\|_\Phi^2 + \|e_h\|_\Phi^2}}$ creates non-increasing numerical sequence, which is reducing with the triangulation refinement, fig. 3b.



(a) 1 - M_h, 2 - N_h (b) δ_h

Fig. 3. Triangulation process convergence

8. Conclusions

In the paper, an $h-$ adaptive scheme of FEM is proposed for solving of two-dimensional stationary piezoelectricity problem with guaranteed pre-defined accuracy approach calculation. For these problems variational form is formulated, Galerkin discretization is performed and appropriative numeric schemes for solution finding are built. A problem of error is formulated. On the basis of the bubble-function properties and the problem of error a posteriori error estimators are built. Received schemes are realized as application software. Illustrative efficiency and reliability confirming of proposed schemes have been demonstrated by a number of numerical experiments using developed application software.

Bibliography

1. Abramov J., Lipina O., Shynkarenko G., Yamelynets A. Pieace-linear $h-$adaptive finite element method approximation for onedimentional boundary value problems // Visnyk Lviv. un-tu Ser. prykl. matem ta inform. 2006. Vol. 11. P. 3-17. (in Ukrainian)
2. Chaban F., Shynkarenko H. Finite element method approximations for the boundary value problems of piezoelectricity // Modern Analysis and Application: Book of abstracts. Odessa, 2007. P. 32-33.
3. Kvasnycya G., Shynkarenko G. Finite element method adaptive approximation for elastostatic problems // Visnyk Lviv. un-tu Ser. prykl. matem ta inform. 2003. Vol. 5. P. 12-23. (in Ukrainian)
4. Kvasnycya G., Shynkarenko G. Comparision of simple a posteriori error estimators of finite element method for elastostatic problems // Visnyk Lviv. un-tu Ser. prykl. matem ta inform. 2003. Vol. 7. P. 162-174. (in Ukrainian)
5. Korotkina M. *Electromagnetoelastostatics.* Moskwa: Moskow. Hosudarst. Un-tet., 1988. (in Russian)
6. Maugin G. *Continuum mechanichs of electromagnetic solids.* North Holland, Amsterdam: Elsevier, 1988.
7. Nowackiy V. *Electromagnetic effects in solid bodies.* Moskwa: Mir, 1986. (in Russian)
8. Parton V., Kudryavcev B. *Electromagnitoelastostatic of piezoelectric and electrocoductivity bodies.* Moskwa: Nauka, 1988. (in Russian)
9. Shynkarenko G. Mesh-projection approximation for the variational problems of piroelectricity. I. Problems definition and staty-state forced vibration analyzis. Diferencial. uravneniya. 1993. Vol. 29. N 7. P. 1252-1260. (in Russian)
10. Verfurth R. *A Review of A Posteriori Estimation and Adaptive Mesh-Refinement Techniques, Advances in Numerical Mathematics.* New York, Stuttgart: Wiley/Teubner, 1996.
11. Zienkiewicz O.C., Taylor R.L. *The Finite Element Method. Vol. 1. The Basis.* 5th Ed. Oxford: Butterworth-Heinemann, 2000.
12. Zienkiewicz O.C., Zhu J.Z. A simple error estimator and adaptive procedure for practical engineering analysis // Int. J. Numer. Methods Engrg. 1987. Vol. 24. P. 337-357.
13. Zhariy O., Ulitko A. *Introducing into nonstationary vibration and waves mechanic.* Kiev: Hight school, 1989. (in Russian)

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