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AN ALTERNATING BOUNDARY INTEGRAL BASED METHOD FOR INVERSE POTENTIAL FLOW AROUND IMMERSED BODIES

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Abstract. We propose and investigate an alternating iterative procedure for the inverse problem of calculating the stream function on immersed bodies in an irrotational two-dimensional potential flow, given the stream function and its normal derivative on the surrounding channel walls. In the procedure, mixed boundary-value problems are solved in the channel with either a Dirichlet or a Neumann condition imposed on the immersed bodies, to generate a sequence of approximations to the stream function. Convergence of these approximations to the stream function is shown in an appropriate norm, and it is proven that the procedure is a regularizing method.

1. Introduction

The Laplace equation can be used as a model of potential flow of incompressible fluids in terms of the stream function for two-dimensional regions. This is then a model for flows which are irrotational at every point in the flow field. Different types of boundary conditions can be prescribed on the solid boundaries surrounding the flow. However, the value of the stream function is unknown on any immersed object and has to be calculated along with the solution. We propose and investigate a technique for finding this value on the immersed objects given the knowledge of both the solution and its normal derivative on the channel walls surrounding the flow.

To formulate this situation in mathematical terms, let the region $D_0 \subset \mathbb{R}^2$ be the strip

$$D_0 := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \in \mathbb{R}, 0 < x_2 < L\}$$

modelling a channel with walls $\Gamma_0 = \Gamma_0^1 \bigcup \Gamma_0^2$, where $\Gamma_0^1 := \{x_{1,\infty}(t) = (t, 0), t \in \mathbb{R}\}$ and $\Gamma_0^2 := \{x_{2,\infty}(t) = (t, L), t \in \mathbb{R}\}$. For simplicity, we put $L = \pi$. Moreover, let D_ℓ with $\bar{D}_\ell \subset D_0$, $\ell = 1, \ldots, n$, be the immersed objects, which are bounded disjoint domains with closed boundaries Γ_ℓ given through parametric representations

$$\Gamma_\ell := \{x_\ell(t) = (x_{1\ell}(t), x_{2\ell}(t)) : 0 \le t \le 2\pi\}, \ \ell = 1, \ldots, n, \tag{1.1}$$

where $x_\ell : \mathbb{R} \to \mathbb{R}^2$ are C^2-smooth and 2π–periodic with $|x'_\ell(t)| > 0$ for all $t \in [0, 2\pi]$ and $\ell = 1, \ldots, n$. Put $D := D_0 \setminus \bigcup_{\ell=1}^n \bar{D}_\ell$; see Figure 1 for an example of the configuration.

The stream function for the inviscid incompressible flow in the above two-dimensional channel satisfies the two-dimensional Laplace equation, and to find the numerical value of the stream function on the immersed objects we consider the Cauchy problem of finding a bounded function $u \in C^2(D) \bigcap C^1(\bar{D})$ satisfying the Laplace equation

$$\Delta u = 0 \quad \text{in} \quad D \tag{1.2}$$

† *Key words.* Alternating method, Cauchy problem, Green's functions, Laplace equation, potential flow, trigonometric- and sinc-quadrature rules.

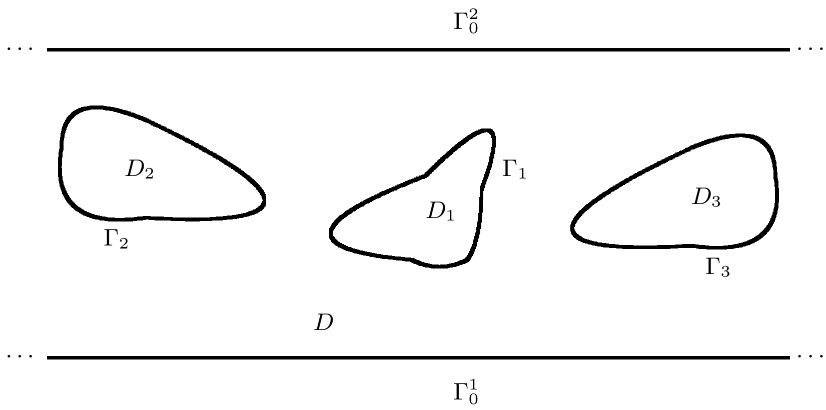


Fig. 1. The channel with walls Γ_0^1 and Γ_0^2, and three immersed objects

and the boundary conditions

$$u = f_1 \quad \text{and} \quad \frac{\partial u}{\partial \nu} = f_2 \quad \text{on} \quad \Gamma_0, \tag{1.3}$$

with u being uniformly bounded at infinity. Due to this condition at infinity uniqueness is clear from results for elliptic Cauchy problems, see, for example, [1] and [2]. We shall assume that the data is such that there exists a solution. Even though there is a unique solution giving the sought values on the objects, this solution does not in general depend continuously on the data, i.e. the problem is ill-posed in the sense of Hadamard making classical methods inappropriate.

There are various techniques for finding the solution to potential flow problems, see, for example, [9] and [12]. We propose an iterative procedure based on the alternating method presented in [6]. The alternating method was originally proposed for bounded domains and we therefore extend it to our unbounded fluid flow channel setting. The situation of the Cauchy problem in a half-plane was recently considered in [4]. The alternating method has successfully been applied in several engineering problems in, for example, fluid flow and heat conduction.

In each iteration of the procedure, mixed direct problems are solved in the solution domain D, where an initial guess of the normal derivatives on the immersed bodies are used to start the process. For the direct mixed fluid flow problems in this study, we propose and investigate a numerical method and even though the solution domain is unbounded, this method does not need any artificial boundary. Instead, the mixed problems are each reduced to a boundary integral equation over the immersed bodies. This approach makes the implementation of the alternating method very efficient.

The outline of the paper is the following. In Section 2, the alternating method and the necessary direct mixed problems are introduced. In Section 3, a numerical method for the direct problems is introduced and well-posedness of the equations solved are proved, see Theorem 3.1 and Theorem 3.2. Convergence of the alternating procedure is discussed in Section 4, and it is shown that the method is regularizing, see further Theorem 4.1. In the final section, i.e. Section 5, numerical examples are given showing that accurate numerical values of the flow on the immersed obstacles can be obtained in an efficient way.

2. An alternating method for the Cauchy problem (1.2)–(1.3)

To formulate the alternating iterative procedure for solving (1.2)–(1.3), we introduce the following mixed boundary value problem:

$$\Delta u = 0 \quad \text{in} \quad D, \tag{2.1}$$

$$\frac{\partial u}{\partial \nu} = h_\ell \quad \text{on} \quad \Gamma_\ell, \quad \ell = 1, \dots, n, \quad \text{and} \quad u = f_1 \quad \text{on} \quad \Gamma_0, \tag{2.2}$$

and also the mixed boundary value problem

$$\Delta u = 0 \quad \text{in} \quad D, \tag{2.3}$$

$$u = g_\ell \quad \text{on} \quad \Gamma_\ell, \quad \ell = 1, \dots, n, \quad \text{and} \quad \frac{\partial u}{\partial \nu} = f_2 \quad \text{on} \quad \Gamma_0. \tag{2.4}$$

By a solution to (2.1)–(2.2) or (2.3)–(2.4), we mean a classical solution which is uniformly bounded at infinity, i.e.

$$u(x) = O(1), \quad |x| \to \infty. \tag{2.5}$$

Let f_1 and f_2 be the given functions in (1.3). The alternating iterative procedure for constructing the solution to (1.2)–(1.3) runs as follows:

- The first approximation u_0 to the solution u of (1.2)–(1.3), is constructed by solving problem (2.1)–(2.2) with $h_\ell = h_\ell^0$, where h_ℓ^0 for $\ell = 1, \dots, n$, is an arbitrary initial guess on the immersed boundary Γ_ℓ.
- Having constructed u_{2k}, we find u_{2k+1} by solving problem (2.3)–(2.4) with $g_\ell = u_{2k}|_{\Gamma_\ell}$, $\ell = 1, \dots, n$.
- Then we find the element u_{2k+2} by solving problem (2.1)–(2.2) with

  $$h_\ell^k = \frac{\partial u_{2k+1}}{\partial \nu}|_{\Gamma_\ell},$$

  where $\ell = 1, \dots, n$.

3. Numerical approximation for the mixed problems

3.1. Mixed Dirichlet-Neumann boundary value problem

We construct the solution to the mixed boundary value problem (2.3)–(2.4), which is assumed to be regular at infinity.

For the planar strip D_0 the corresponding Green's function has the form

$$N(x, y) = \frac{1}{4\pi} \ln \left[\frac{1}{\cosh(x_1 - y_1) - \cos(x_2 - y_2)} \right] + \frac{1}{4\pi} \ln \left[\frac{1}{\cosh(x_1 - y_1) - \cos(x_2 + y_2)} \right].$$

Note that in the planar case the above introduced Green's function is unbounded at infinity. Using the single-layer potential approach with the Green's function N for the strip D_0, we can seek the solution of the mixed problem (2.3)–(2.4) in the form

$$u(x) = \sum_{\ell=1}^{n} \int_{\Gamma_\ell} \varphi_\ell(y) N(x, y)\, ds(y) + \int_{\Gamma_0} f_2(y) N(x, y)\, ds(y) + \alpha\,, \quad x \in D, \tag{3.1}$$

with unknown densities φ_ℓ on Γ_ℓ and a constant α. In order to satisfy the boundedness condition at infinity, the side condition

$$\sum_{\ell=1}^{n} \int_{\Gamma_\ell} \varphi_\ell(y)\, ds(y) = 0$$

is imposed.

As is evident from the approach (3.1), some additional conditions on the boundary function f_2 have to be imposed. Firstly, for the existence of the integral, the function f_2 is required to have an appropriate asymptotic behaviour on the real axis, and, secondly, the integral of f_2 over the boundary Γ_0 has to vanish to satisfy the regularity condition at infinity. We summarize these properties as the following conditions:

$$f_2(x) = O(|x|^{-1-\varepsilon}),\ \varepsilon > 0,\ |x| \to \infty, \quad \int\limits_{\Gamma_0} f_2(y)\, ds(y) = 0. \tag{3.2}$$

Using the continuity of the single-layer potentials and the properties of the Green's function, the problem (2.3)–(2.4) can, by using the representation (3.1), be reduced to the integral equations

$$\begin{cases} \displaystyle\sum_{\ell=1}^{n} \int\limits_{\Gamma_\ell} \varphi_\ell(y) N(x,y)\, ds(y) + \alpha = g_k(x) - \int\limits_{\Gamma_0} f_2(y) N(x,y)\, ds(y),\ x \in \Gamma_k,\ k = 1,\dots,n, \\ \displaystyle\sum_{\ell=1}^{n} \int\limits_{\Gamma_\ell} \varphi_\ell(y)\, ds(y) = 0, \end{cases} \tag{3.3}$$

to be solved for the densities φ_ℓ, $\ell = 1,\dots,n$, and the constant α.

The well-posedness of the integral equation (3.3) in corresponding Hölder or Sobolev spaces follows from classical results [8]. For example, we have the following result in the Sobolev space setting:

Theorem 3.1 *For any given $f_2 \in C(\Gamma_0)$ with the property* (3.2)*, and $g_\ell \in H^{1/2}(\Gamma_\ell)$, $\ell = 1,\dots,n$, the system of equations* (3.3) *possesses a unique solution $\varphi_\ell \in H^{-1/2}(\Gamma_\ell)$, $\ell = 1,\dots,n$, and $\alpha \in \mathbb{R}$.*

Here, the space $H^{1/2}(\Gamma_\ell)$ denotes the standard Sobolev trace space on the immersed body ℓ, where $\ell = 1,\dots,n$, and $H^{-1/2}(\Gamma_\ell)$ is the corresponding dual space.

Using the parameterization (1.1), and similarly for the other different boundary parts of the solution domain D, we can transform the system (3.3) into the parametric form

$$\begin{cases} \displaystyle\frac{1}{2\pi} \sum_{\ell=1}^{n} \int_0^{2\pi} \mu_\ell(\tau) H_{k\ell}(t,\tau) d\tau + \alpha = w_k(t), \quad t \in [0,2\pi], \quad k = 1,\dots,n, \\ \displaystyle\sum_{\ell=1}^{n} \int_0^{2\pi} \mu_\ell(\tau)\, d\tau = 0 \end{cases} \tag{3.4}$$

with the 2π-periodic kernels

$$H_{k\ell}(t,\tau) := 2\pi N(x_k(t), x_\ell(\tau)), \quad t,\tau \in [0,2\pi], \quad k \neq \ell,$$

$$H_{\ell\ell}(t,\tau) := -\frac{1}{2} \ln\left(\frac{4}{e} \sin^2 \frac{t-\tau}{2}\right) + H^1_{\ell\ell}(t,\tau),$$

$$H^1_{\ell\ell}(t,\tau) := \begin{cases} 2\pi N(x_\ell(t), x_\ell(\tau)) + \dfrac{1}{2} \ln\left(\dfrac{4}{e} \sin^2 \dfrac{t-\tau}{2}\right) & \text{for} \quad t \neq \tau, \\ -\ln\left(|\sin x_{2\ell}(t)||x'_\ell(t)|\right) - \dfrac{1}{2} & \text{for} \quad t = \tau, \end{cases}$$

the right-hand sides

$$w_k(t) := g_k(x_k(t)) - \sum_{\ell=1}^{2} \int_{-\infty}^{\infty} f_2^\ell(\tau) N(x_k(t), x_{\ell,\infty}(\tau))\, d\tau, \quad t \in [0, 2\pi] \tag{3.5}$$

and the densities $\mu_\ell(t) := \varphi_\ell(x_\ell(t))|x'_\ell(t)|$, $\ell = 1, \dots, n$. Here, we used the notation $f_2^\ell(t) := f_2(x_{\ell,\infty}(t))$, $\ell = 1, 2$.

For the full discretization of the integral equation of the first kind (3.4), which has a logarithmic singularity, we employ a quadrature method together with the quadrature rule [3,8] based on trigonometric interpolation. For this purpose, we choose an equidistant mesh by setting

$$t_i := i\pi/M, \quad i = 0, \dots, 2M-1,\ M \in \mathbb{N}, \tag{3.6}$$

and use the quadrature rules

$$\frac{1}{2\pi} \int_0^{2\pi} f(\tau)\, d\tau \approx \frac{1}{2M} \sum_{j=0}^{2M-1} f(t_j) \tag{3.7}$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} f(\tau) \ln\left(\frac{4}{e} \sin^2 \frac{t-\tau}{2}\right) d\tau \approx \sum_{j=0}^{2M-1} R_j(t)\, f(t_j) \tag{3.8}$$

with weight functions

$$R_j(t) := -\frac{1}{2M} \left\{ 1 + 2 \sum_{m=1}^{M-1} \frac{1}{m} \cos m(t - t_j) + \frac{\cos(t - t_j)}{M} \right\}.$$

These quadrature formulas are obtained by replacing f with its trigonometric interpolation polynomial and then integrating explicitely, see [8]. We point out that in the case of a periodic and analytic function f, we obtain exponential convergence.

For the numerical calculation of the integrals in (3.5), we apply the so-called sinc-quadrature rule

$$\int_{-\infty}^{\infty} f(\tau)\, d\tau \approx h_\infty \sum_{i=-M_1}^{M_1} f(ih_\infty), \quad M_1 \in \mathbb{N}, \quad h_\infty = \frac{c}{\sqrt{M_1}},\ c > 0. \tag{3.9}$$

This quadrature formula is obtained by replacing f with a sinc approximation (see [10]) and then integrating explicitely. In the case of analytic functions f, which satisfy $f(t) = O(e^{-c|t|})$ for $|t| \to \infty$ and some positive constant c, the quadrature (3.9) has exponential convergence. We note that for integrands with a different asymptotic behavior than in (3.9), if it is needed, some special transformations can be used (see [10]).

Thus, after the application of the quadrature method to the integral equations (3.4) and the quadrature rule (3.9) for the computation of the integral in the right-hand side of (3.5), we obtain the following system of linear equations

$$\begin{cases} \displaystyle\sum_{\ell=1}^{n} \sum_{j=0}^{2M-1} \tilde{\mu}_{\ell j} \tilde{H}_{k\ell}(t_i, t_j) + \alpha = \tilde{w}_k(t_i)\,,\ i = 0, \dots, 2M-1,\ k = 1, \dots, n \\ \displaystyle\sum_{\ell=1}^{n} \sum_{j=0}^{2M-1} \tilde{\mu}_{\ell j} = 0 \end{cases} \tag{3.10}$$

to be solved for $\tilde{\mu}_{\ell j} \approx \mu_\ell(t_j)$ with matrix coefficients

$$\tilde{H}_{k\ell}(t_i,t_j) := \begin{cases} \dfrac{1}{2M} H_{k\ell}(t_i,t_j) & \text{for} \quad k \neq \ell, \\ -\dfrac{1}{2} R_j(t_i) + \dfrac{1}{2M} H^1_{\ell\ell}(t_i,t_j) & \text{for} \quad k = \ell, \end{cases}$$

and the right-hand side

$$\tilde{w}_k(t_j) := g_k(x_k(t_j)) - h_\infty \sum_{i=-M_1}^{M_1} \sum_{\ell=1}^{2} f_2^\ell(ih_\infty) N(x_k(t_j), x_{\ell,\infty}(ih_\infty)), \quad j = 0,\dots,2M-1.$$

A convergence and error analysis for this numerical scheme is described in [3] in a Hölder space setting, and in [8] in a Sobolev space setting. This analysis exhibits the dependence of the convergence rate on the smoothness of the boundary curves Γ_ℓ of the immersed bodies, i.e. the proposed method belongs to the class of algorithms without "saturation effect".

In the alternating procedure we have to obtain the normal derivative on the boundary part Γ_k, and this derivative can be calculated from the above boundary integral formulation. Indeed, from the properties of the single-layer potentials, the normal derivative of the solution on Γ_k can be calculated by the formula

$$\frac{\partial u}{\partial \nu}(x_k(t)) = -\frac{\mu_k(t)}{2|x_k'(t)|} + \frac{1}{2\pi} \sum_{\ell=1}^{n} \int_0^{2\pi} \mu_\ell(\tau) L^1_{\ell k}(t,\tau)\, d\tau + \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{\ell=1}^{2} f^\ell(\tau) L^2_{\ell k}(t,\tau)\, d\tau, \; t \in [0,2\pi], \tag{3.11}$$

where ν is the outward unit normal to Γ_k, $k = 1,\dots,n$, and the kernels are given as

$$L^1_{kk}(t,t) := \frac{\cot x_{2k}(t) x'_{1k}(t) |x'_k(t)|^2 + x'_{2k}(t) x''_{1k}(t) - x'_{1k}(t) x''_{2k}(t)}{2|x'_k(t)|^3},$$

$$L^1_{\ell k}(t,\tau) := \frac{\left[\begin{array}{c} 2x'_{2k}(t) \sinh(x_{1k}(t) - x_{1\ell}(\tau))(\cosh(x_{1k}(t) - x_{1\ell}(\tau)) - \cos x_{2k}(t) \cos x_{2k}(\tau) \\ -x'_{1k}(t)(2\cosh(x_{1k}(t) - x_{1\ell}(\tau)) \sin x_{2k}(t) \cos x_{2\ell}(\tau) - \sin 2x_{2k}(t)) \end{array}\right]}{\left[\begin{array}{c} 2|x'_k(t)|(\cos(x_{2k}(t) - x_{2\ell}(\tau)) - \cosh(x_{1k}(t) - x_{1\ell}(\tau))) \\ \times(\cosh(x_{1k}(t) - x_{1\ell}(\tau)) - \cos(x_{2k}(t) + x_{2\ell}(\tau))) \end{array}\right]}$$

for $t \neq \tau$ and $\ell, k = 1,\dots,n$,

$$L^2_{1k}(t,\tau) := -\frac{\sin x_{2k}(t) x'_{1k}(t) + \sinh(\tau - x_{1k}(t)) x'_2(t)}{|x'_k(t)|(\cos x_{2k}(t) - \cosh(\tau - x_{1k}(t)))}$$

and

$$L^2_{2k}(t,\tau) := -\frac{\sin x_{2k}(t) x'_{1k}(t) - \sinh(\tau - x_{1k}(t)) x'_2(t)}{|x'_k(t)|(\cos x_{2k}(t) + \cosh(\tau - x_{1k}(t)))}.$$

The approximation for (3.11) can be obtained by using the quadratures (3.7) and (3.9).

3.2. Mixed Neumann-Dirichlet boundary value problem

The Neumann-Dirichlet mixed problem, i.e. to find a function u that solves the mixed boundary value problem (2.1)–(2.2), which is regular at infinity, can be numerically solved by the approach of the previous section but with some additional changes. For the sake of completeness we include some of the details.

The Green's function for the Dirichlet problem in the strip D_0 has the form:

$$G(x,y) = \frac{1}{4\pi} \ln \left(\frac{\cosh(x_1 - y_1) - \cos(x_2 + y_2)}{\cosh(x_1 - y_1) - \cos(x_2 - y_2)} \right)$$

and, as one can check, it is bounded at infinity. Thus, for the boundary value problem (2.1)–(2.2), we again employ the single layer potential approach

$$u(x) = \sum_{\ell=1}^{n} \int_{\Gamma_\ell} \varphi_\ell(y) G(x,y)\, ds(y) - \int_{\Gamma_0} f_1(y) \frac{\partial G(x,y)}{\partial \nu(y)}\, ds(y), \quad x \in D. \tag{3.12}$$

Then (2.1)–(2.2) is reduced to the following system of integral equations of the second kind

$$-\frac{1}{2}\varphi_k(x) + \sum_{\ell=1}^{n} \int_{\Gamma_\ell} \varphi_\ell(y) \frac{\partial G(x,y)}{\partial \nu(x)}\, ds(y) = g_k(x) + \int_{\Gamma_0} f(y) \frac{\partial^2 G(x,y)}{\partial \nu(x) \partial \nu(y)}\, ds(y), \quad x \in \Gamma_k \tag{3.13}$$

for $k = 1, \ldots, n$. We note here that it is necessary to have the following asymptotic behaviour for the boundary function

$$f_1(x) = O(|x|^{1-\epsilon}), \quad \epsilon > 0, \quad |x| \to \infty. \tag{3.14}$$

As for the system of integral equations for the Dirichlet-Neumann problem, we have a well-posedness result also in the Neumann-Dirichlet case.

Theorem 3.2 *For any given $f_1 \in C(\Gamma_0)$ with the property (3.14), and $g_\ell \in L^2(\Gamma_\ell)$, $\ell = 1, \ldots, n$, the system of integral equations (3.13) possesses a unique solution $\varphi_k \in L^2(\Gamma_k)$, $\ell = 1, \ldots, n$.*

Taking into account the parametric representation (1.1) together the similar representations introduced in Section 1, we can rewrite equation (3.13) in the parametric form

$$-\frac{1}{2} \frac{\mu_k(t)}{|x_k'(t)|} + \frac{1}{2\pi} \sum_{\ell=1}^{n} \int_0^{2\pi} \mu_\ell(\tau) K_{k\ell}^1(t,\tau) d\tau = g_k(x_k(t)) + \sum_{\ell=1}^{2} \int_{-\infty}^{\infty} f_1^\ell(\tau) K_{k\ell}^2(t,\tau)\, d\tau,\ t \in [0, 2\pi], \tag{3.15}$$

where $k = 1, \ldots, n$, $\mu_k(t) := \varphi_k(x_k(t))|x_k'(t)|$ and

$$K_{kk}^1(t,t) := \frac{-\cot x_{2k}(t) x_{1k}'(t) |x_k'(t)|^2 + x_{2k}'(t) x_{1k}''(t) - x_{1k}'(t) x_{2k}''(t)}{2|x_k'(t)|^3},$$

$$K_{k\ell}^1(t,\tau) := \frac{\left[\begin{array}{c} x_{2k}'(t) \sinh(x_{1k}(t) - x_{1\ell}(\tau)) \sin x_{2k}(t) \sin x_{2k}(\tau) \\ + x_{1k}'(t)(\cosh(x_{1k}(t) - x_{1\ell}(\tau)) \cos x_{2k}(t) - \cos x_{2\ell}(\tau)) \sin x_{2\ell}(\tau) \end{array} \right]}{\left[\begin{array}{c} |x_k'(t)|(\cos(x_{2k}(t) - x_{2\ell}(\tau)) - \cosh(x_{1k}(t) - x_{1\ell}(\tau))) \\ \times(\cosh(x_{1k}(t) - x_{1\ell}(\tau)) - \cos(x_{2k}(t) + x_{2\ell}(\tau))) \end{array} \right]}$$

for $t \neq \tau$ and $\ell, k = 1, \dots, n$, and

$$K^2_{k\ell}(t,\tau) := \frac{\left[\begin{array}{c} x'_{2k}(t)((-4-2\cos 2x_{2k}(t)-2\cosh(2(x_{1k}(t)-\tau)))\sin x_{2k}(t)\sinh(\tau-x_{1k}(t)) \\ +(-1)^\ell 2\sin 2x_{2k}(t)\sinh(2(x_{1k}(t)-\tau))) + (-1)^\ell x'_{1k}(t)(4+4\cos^2 x_{2k}(t) \\ +2\cos 2x_{2k} + 2\cos 2x_{2k}(t) + 2(2+\cos 2x_{2k}(t))\cosh(2(x_{1k}(t)-\tau)) \\ +2\cos x_{2k}(t)\cosh(\tau - x_{1k}(t))(6+\cos 2x_{2k}(t) + \cosh(2(x_{1k}(t)-\tau)))) \end{array}\right]}{8\pi|x'_k(t)|(\cos x_{2k}(t) + (-1)^\ell \cosh(\tau - x_{1k}(t))^4)}$$

for $\ell = 1, 2$.

The numerical solution of the integral equation (3.15) is obtained by the Nyström method using the quadrature rules (3.7) and (3.9). The error and convergence analysis of this method can be found in [8].

From (3.12) we obtain the following parametric representation of the solution on Γ_k for the mixed problem (2.1)–(2.2)

$$u(x_k(t)) = \frac{1}{2\pi}\sum_{\ell=1}^{n}\int_0^{2\pi} \mu_\ell(\tau) Q^1_{k\ell}(t,\tau)d\tau - \sum_{\ell=1}^{2}\int_{-\infty}^{\infty} f_1^\ell(\tau) Q^2_{k\ell}(t,\tau)\,d\tau, \tag{3.16}$$

where $t \in [0, 2\pi]$, $k = 1, \dots, n$, and the kernels have the form

$$Q^1_{k\ell}(t,\tau) := 2\pi G(x_k(t), x_\ell(\tau)), \quad t,\tau \in [0,2\pi], \quad k \neq \ell,$$

$$Q^1_{\ell\ell}(t,\tau) := -\frac{1}{2}\ln\left(\frac{4}{e}\sin^2\frac{t-\tau}{2}\right) + \tilde{Q}^1_{\ell\ell}(t,\tau),$$

$$\tilde{Q}^1_{\ell\ell}(t,\tau) := \begin{cases} 2\pi G(x_\ell(t), x_\ell(\tau)) + \frac{1}{2}\ln\left(\frac{4}{e}\sin^2\frac{t-\tau}{2}\right) & \text{for} \quad t \neq \tau, \\ \ln(2|\sin x_{2\ell}(t)|/|x'_\ell(t)|) - \frac{1}{2} & \text{for} \quad t = \tau, \end{cases}$$

and

$$Q^2_{k\ell}(t,\tau) := \frac{-\sin x_{2k}(t)}{2\pi(\cosh(\tau - x_{1k}(t)) + (-1)^\ell \cos x_{2k}(t))}$$

for $\ell = 1, 2$, respectively. For the approximation of (3.16) the quadratures (3.7)–(3.9) are used.

4. Convergence of the alternating procedure

We have the following convergence result:

Theorem 4.1 *Assume that problem (1.2)–(1.3) has a bounded solution, where f_1 satisfies (3.14), and f_2 satisfies (3.2). Let u_k be the k-th approximate solution constructed in the alternating procedure described in Section 2. Then the approximation tends to the correct function value on each immersed body, more precisely*

$$\lim_{k\to\infty} \|u - u_k\|_{H^{1/2}(\Gamma_\ell)} = 0 \tag{4.1}$$

for $\ell = 1, \dots, n$, and any sufficiently smooth initial data elements h_0^ℓ, which start the procedure.

Here, $H^{1/2}(\Gamma_\ell)$ denotes the standard Sobolev trace space on the immersed body ℓ, where $\ell = 1, \dots, n$.

Note that it is possible also to obtain convergence for the flow in the channel, i.e. to show that $u - u_k$ tends to zero in the domain D in an appropriate norm.

Proof. The above convergence result for the iterative procedure in the case of exact data follows along the lines of the original ideas given for bounded domains in [6] and [7]. Extensions to a semi-infinite domain with one submerged object was given in [4]. For the sake of completeness, we briefly outline the main steps in obtaining convergence for the above channel setting situation and with multiple submerged bodies.

We put the given boundary functions on the immersed obstacles into a vector, for example, with $\boldsymbol{h} = (h_1, \dots, h_n)$, we say that u is a solution to (2.1)–(2.2) with $\boldsymbol{h}$ and f_1, if h_ℓ is the ℓ-th component of $\boldsymbol{h}$, and similar for problem (2.3)–(2.4).

We let U_0 be the solution to (2.1)–(2.2), with a given (sufficiently smooth) function $\boldsymbol{h} = (h_1, \dots, h_n)$ and $f_1 = 0$. Similarly, let U_1 be the solution to (2.3)–(2.4) with $f_2 = 0$ and $\boldsymbol{g} = \Big(U_0|_{\Gamma_1}, \dots, U_0|_{\Gamma_n}\Big)$. We define the operator B by

$$B\boldsymbol{h} = \left(\frac{\partial U_1}{\partial \nu}|_{\Gamma_1}, \dots, \frac{\partial U_1}{\partial \nu}|_{\Gamma_n}\right), \tag{4.2}$$

and it is clear that B is well-defined. In the similar way, let $\tilde{U}_0$ be the solution to (2.1)–(2.2) with $\boldsymbol{h} = 0$, and let $\tilde{U}_1$ be the solution to (2.3)–(2.4) with $\boldsymbol{g} = (\tilde{U}_0|_{\Gamma_1}, \dots, \tilde{U}_0|_{\Gamma_n}\Big)$. With the notation

$$\boldsymbol{G}(f_1, f_2) = \left(\frac{\partial \tilde{U}_1}{\partial \nu}|_{\Gamma_1}, \dots, \frac{\partial \tilde{U}_1}{\partial \nu}|_{\Gamma_n}\right). \tag{4.3}$$

it follows that the Cauchy problem (1.1)-(1.2) is equivalent with finding a solution (a fixed point) to equation

$$B\boldsymbol{h} + \boldsymbol{G}(f_1, f_2) = \boldsymbol{h}. \tag{4.4}$$

Thus, to investigate the convergence of the iterative alternating procedure it is enough to investigate the properties of the operator B. We introduce the inner product

$$(\boldsymbol{h}, \boldsymbol{g}) = \int_D \nabla u \cdot \nabla v \, dx, \tag{4.5}$$

where u solves (2.1)–(2.2) with $\boldsymbol{h} = (h_1, \dots, h_n)$ and $f_1 = 0$, and similarly v solves (2.1)–(2.2) with $h_\ell = g_\ell$, where $\boldsymbol{g} = (g_1, \dots, g_n)$ and $f_1 = 0$. Note that the strip domain D is a Poincaré domain, i.e. the Poincaré inequality holds therein, see further [5]. Thus, it is straightforward to check that $(\cdot, \cdot)$ is a well-defined inner product. We use $\| \cdot \|$ for the corresponding norm.

Therefore, following [6], it can be shown, employing Green's formula, that B is self-adjoint, non-negative, non-expansive, and the number one is not an eigenvalue; thus convergence follows; for further details see [4].

In the case of noisy data f_1^δ and f_2^δ, where $\delta > 0$, and

$$\|\boldsymbol{G}(f_1^\delta, f_2^\delta) - \boldsymbol{G}(f_1, f_2)\| \le \delta, \tag{4.6}$$

using the properties of B, the discrepancy principle can be employed for any given δ as a stopping rule for fixed point iterations for equation (4.4), see [11, Chapt. 3, Sect. 3]. Thus, if $k = k(\delta)$ is the smallest integer with

$$\|\boldsymbol{h}_{k+1}^\delta - \boldsymbol{h}_k^\delta\| \le b\delta \tag{4.7}$$

for given $b > 1$, then $\boldsymbol{h}^{\delta}_{k(\delta)}$ converges to the exact solution of (4.4) when $\delta \to 0$, i.e. the proposed alternating procedure is a regularizing method. In (4.7), the ℓ-th component of $\boldsymbol{h}^{\delta}_{k}$ is

$$(\boldsymbol{h}^{\delta}_{k})_{\ell} = \frac{\partial u_{2k-1}}{\partial \nu}|_{\Gamma_\ell},$$

and these elements are obtained as described in (i)–(iii), where $(f^{\delta}_1, f^{\delta}_2)$ replaces the exact data (f_1, f_2) in (2.2) and (2.4).

5. Numerical examples

Ex. 1. We consider the channel strip D_0 with one circular immersed body (see Fig. 2)

$$\Gamma_1 := \{x_1(t) = (0.5\cos t, 0.5\sin t + 1.5), t \in [0, 2\pi]\}.$$

The boundary function on the inclusion has the form:

$$g_1(t) = \cos(t), \quad t \in [0, 2\pi].$$

The given flux on Γ_0 is $f_2 = 0$. Boundary values f_1 are calculated as the trace of the corresponding potential on Γ_0. To avoid the "inverse crime" these values are calculated on a finer mesh and we shall also add noise to the data.

The results of the Cauchy data reconstructions on the body Γ_1 are presented in Fig. 3

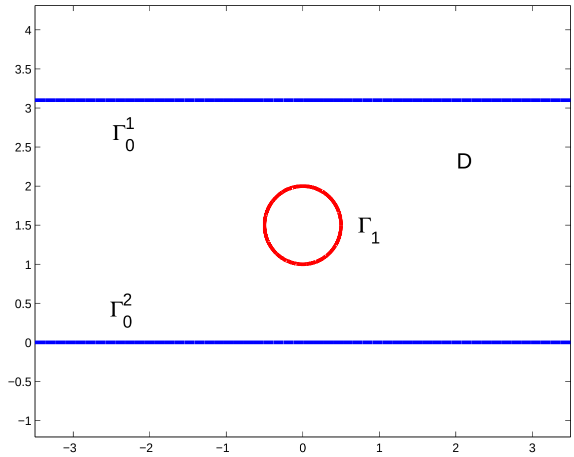


Fig. 2. Strip with a circular inclusion

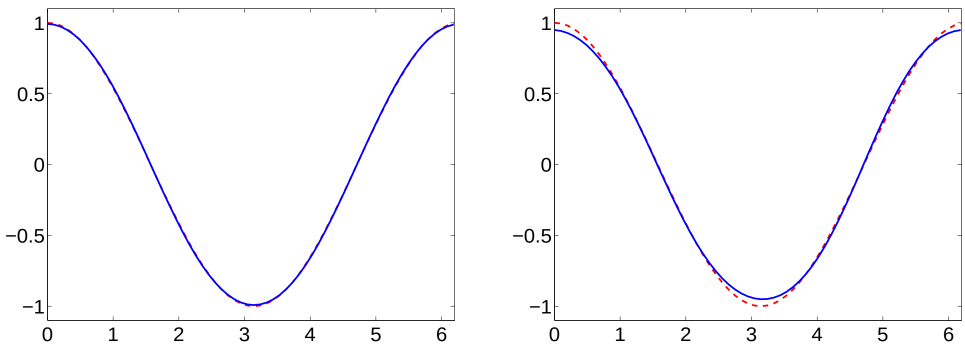


Fig. 3. Reconstruction of the boundary function on Γ_1 in Ex. 1

and Fig. 4, where the function value and the normal derivative are shown both for exact and noisy data. We used the following discretization parameters $M = 32$ and $M_1 = 1000$. In the case of exact data, a very accurate reconstruction is obtained both of the function and the normal derivative but the number of iterations needed is rather large. We then added 5% random pointwise errors to the values of f_1, and, as expected, the reconstructions are less accurate and the normal derivative is affected most by the noise in the Cauchy data. The discrepancy principle, as discussed in the previous Section, was used to terminate the iterations.

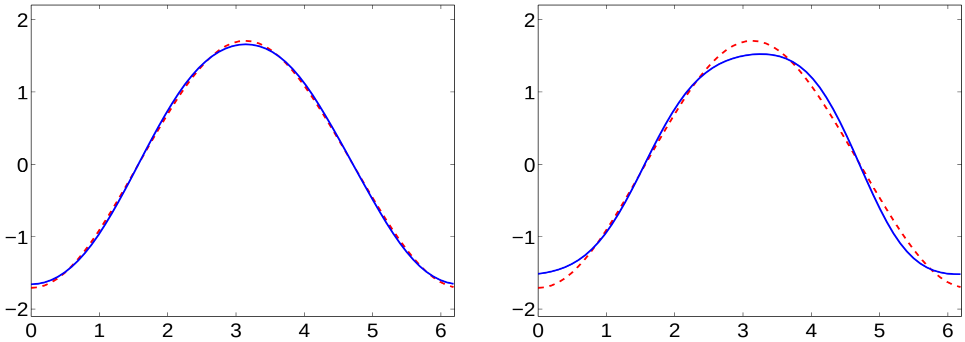
The corresponding L^2 errors

$$e_k := \|u_{2k} - g\|_{L^2(\Gamma_1)}$$

and

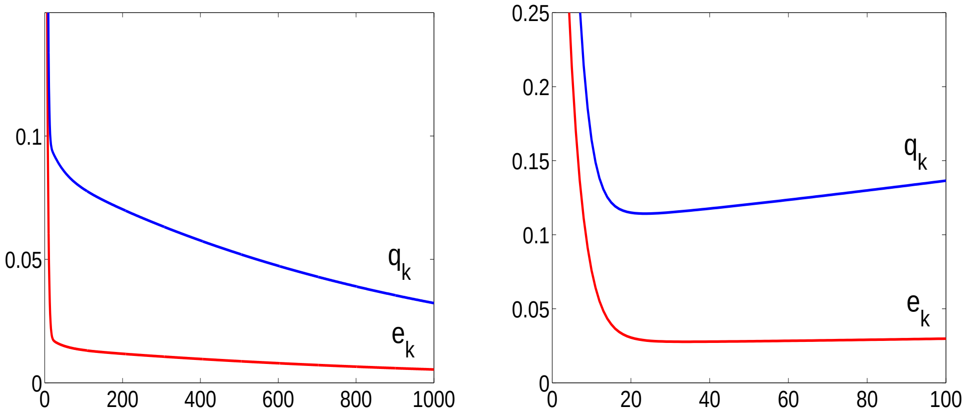
$$q_k := \|\frac{\partial u_{2k}}{\partial \nu} - \frac{\partial u}{\partial \nu}\|_{L^2(\Gamma_1)}$$

are reflected in Fig. 5.



a) Exact data, $k^* = 1000$ b) 5% noise, $k^* = 24$

Fig. 4. Reconstruction of the normal derivative on Γ_1 for Ex. 1



a) Exact data b) 5% noise

Fig. 5. L^2-errors e_k and q_k for the circular inclusion in Ex. 1

Ex. 2. Next, we consider the case of the channel strip D_0 with one immersed kite shaped object (see Fig. 6) described by

$$\Gamma_1 := \{x_1(t) = (0.6 \cos t + 0.39 \cos 2t - 0.39 + 2.11, 0.8 \sin t + 1.5), t \in [0, 2\pi]\}.$$

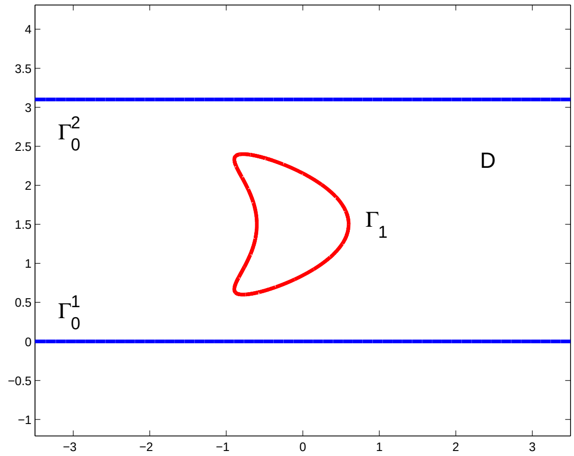
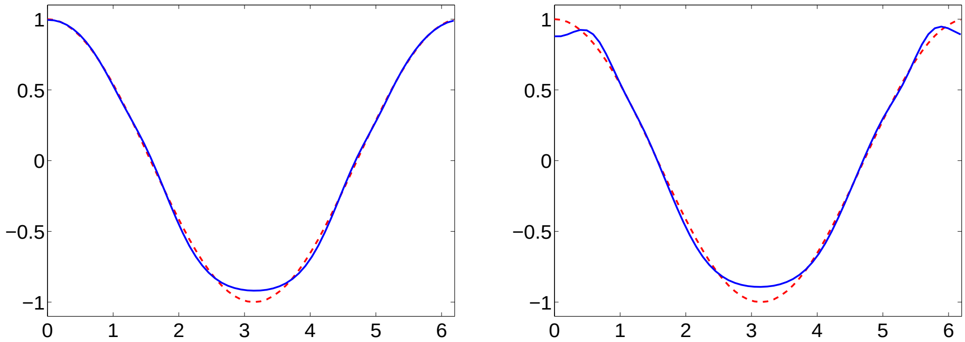


Fig. 6. Strip with the kite inclusion

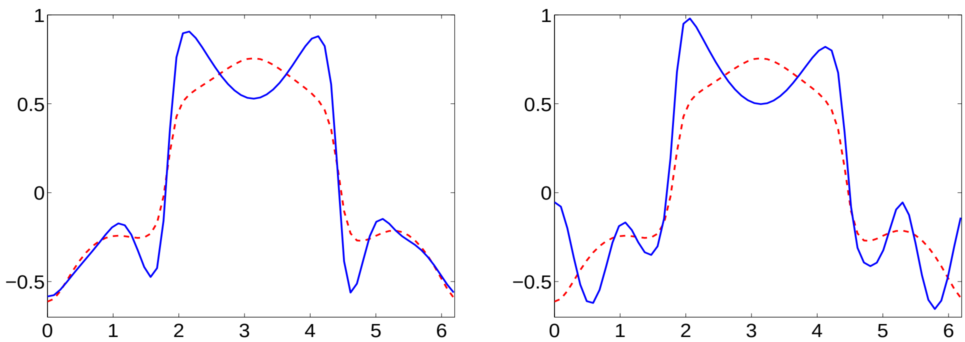
All other input data and method parameters are as in the previous example.

Again, in the case of exact data, even with this more complicated shape of the immersed body, an accurate reconstruction of the function is obtained, see Fig. 7a). However, the reconstruction of the normal derivative is not that accurate, see Fig. 8a). Noise in the data influence as in the previous example, see Fig. 7b) and Fig 8b). The corresponding L^2 errors e_k and q_k are given in Fig. 9.



a) Exact data, $k^* = 1000$ b) 5% noise, $k^* = 348$

Fig. 7. Reconstruction of the boundary function on Γ_1 in Ex. 2



a) Exact data, $k^* = 1000$ b) 5% noise, $k^* = 335$

Fig. 8. Reconstruction of the normal derivative on Γ_1 in Ex. 2

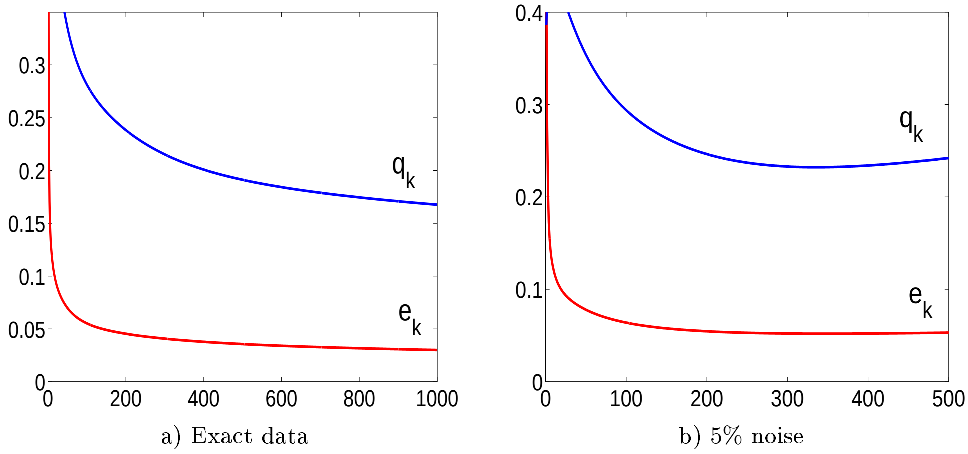


Fig. 9. L^2-errors e_k and q_k for the kite shaped inclusion in Ex. 2

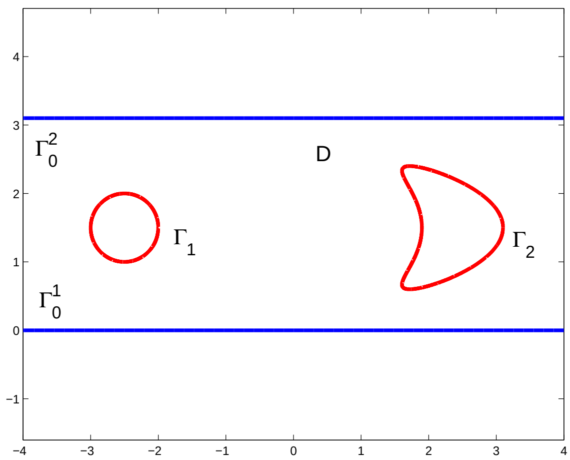


Fig. 10. Strip with two inclusions

Ex. 3. We consider the channel strip with two immersed bodies: one the circular (Γ_1) and one kite shaped Γ_2 (see Fig. 10)

$$\Gamma_1 := \{x_1(t) = (0.5\cos t - 2.5, 0.5\sin t + 1.5), t \in [0, 2\pi]\}$$

and

$$\Gamma_2 := \{x_2(t) = (0.5\cos t + 2.5, 0.5\sin t + 1.5), t \in [0, 2\pi]\}.$$

The boundary functions on the inclusions have the form:

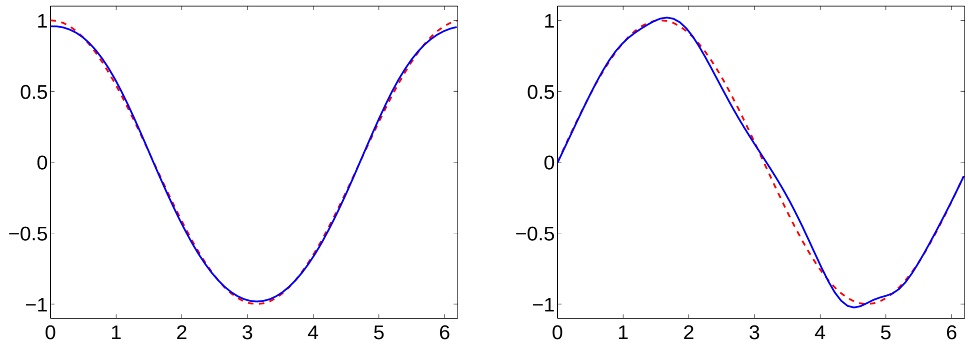
$$g_1(t) = \cos(t), \quad g_2(t) = \sin(t), \quad t \in [0, 2\pi].$$

The given flux on Γ_0 is

$$f_2 = 0.$$

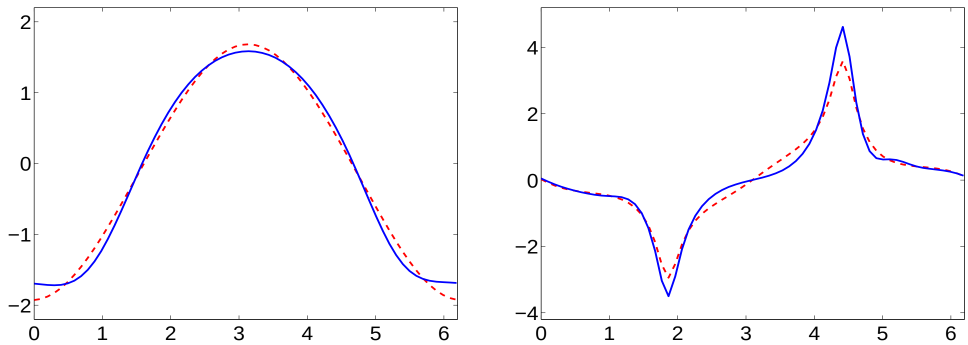
The boundary value f_1 are calculated as the trace of the corresponding potential on Γ_0. Again, to avoid the “inverse crime”, these values are calculated on a finer mesh and we also add noise to the data.

The results of the reconstruction of the Cauchy data on Γ_1 and Γ_2 in the case of exact data are presented in Fig. 11–13, and in the case of noisy data in Fig. 14–16.



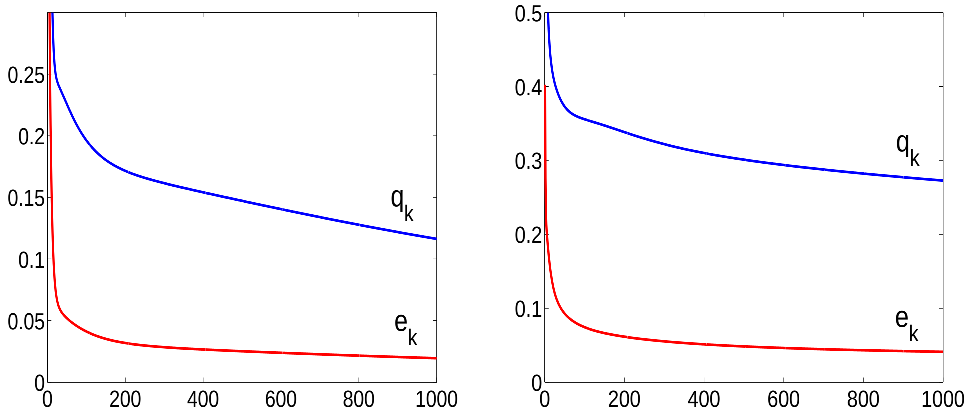
a) circle shaped boundary Γ_1 b) kite shaped boundary Γ_2

Fig. 11. Reconstruction of boundary functions for exact data, $k^* = 1000$, in Ex. 3



a) circle shaped boundary Γ_1 b) kite shaped boundary Γ_2

Fig. 12. Reconstruction of normal derivatives for exact data, $k^* = 1000$, in Ex. 3



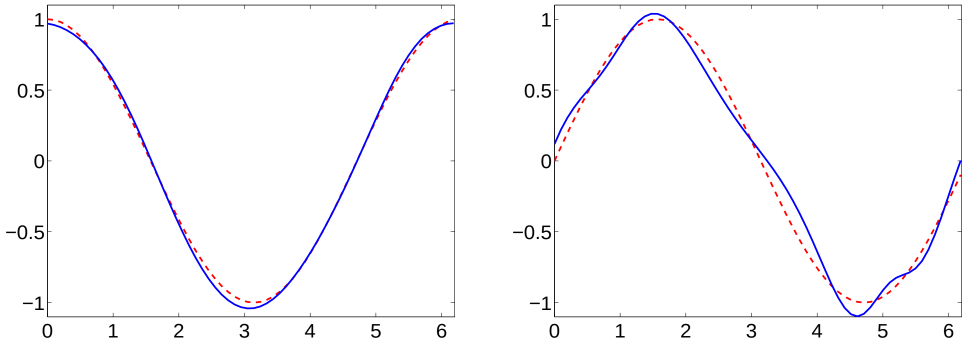
a) circle shaped boundary Γ_1 b) kite shaped boundary Γ_2

Fig. 13. L^2-errors e_k and q_k for exact data in Ex. 3

For exact data, both the function value and its normal derivative are accurately reconstructed, see Fig. 11–12. Note that the reconstructions are slightly less accurate on the kite shaped obstacle. The L^2 errors e_k and q_k for exact data are given in Fig. 13.

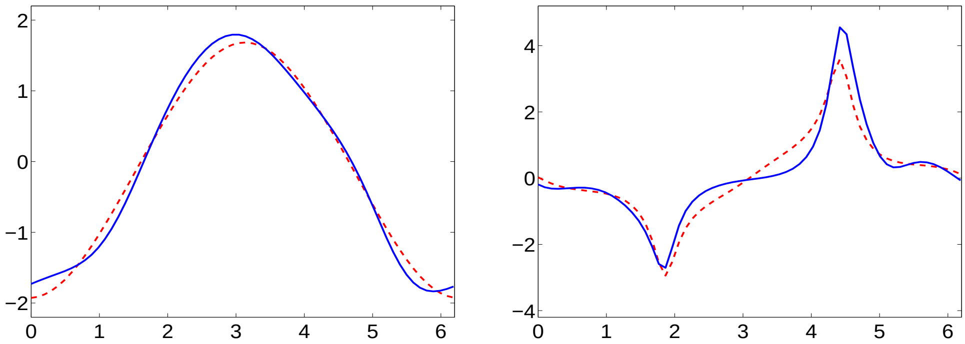
For the case of 5% random pointwise errors added to the values of f_1, the reconstructions of the function and its normal derivative on the immersed bodies are given in

Fig. 14–15. The discrepancy principle was used to stop the iterations. Again, less accurate reconstructions are obtained on the kite shaped obstacle. The L^2 errors e_k and q_k for noisy data are given in Fig. 16.



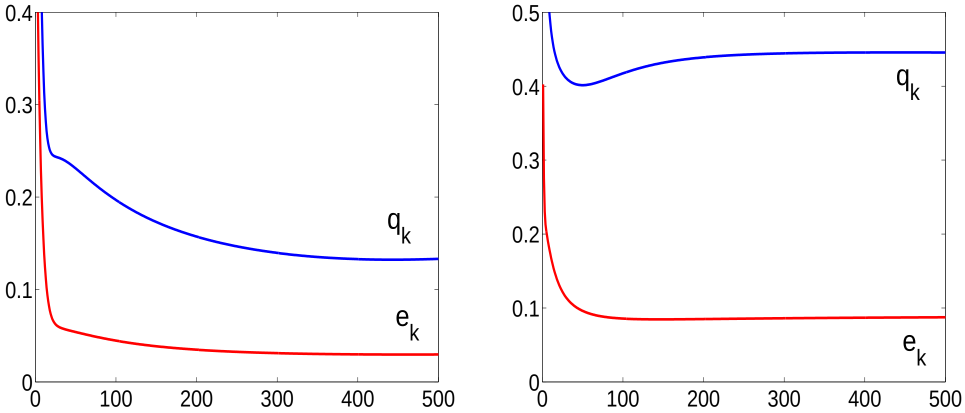
a) circle shaped boundary Γ_1, $k^* = 461$ b) kite shaped boundary Γ_2, $k^* = 146$

Fig. 14. Reconstruction of boundary functions for 5% noise in Ex. 3



a) circle shaped boundary Γ_1 , $k^* = 443$ b) kite shaped boundary Γ_2, $k^* = 50$

Fig. 15. Reconstruction of normal derivatives for 5% noise in Ex. 3



a) circle shaped boundary Γ_1 b) kite shaped boundary Γ_2

Fig. 16. L^2-errors e_k and q_k for 5% noise in Ex. 3

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