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ON THE NUMERICAL SOLUTION OF A MIXED INITIAL BOUNDARY VALUE PROBLEM FOR THE HEAT EQUATION IN A DOUBLE-CONNECTED PLANAR DOMAIN

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Abstract. We consider an initial boundary value problem for parabolic equation in the planar double-connected domain with the Dirichlet and the Neumann boundary value conditions. This mixed problem is reduced by Rothe's method to the sequence of elliptic boundary value problems with first and second orders of the time approximation. Then the indirect boundary integral equation method is used. The boundary layer potentials are constructed using the fundamental solutions of the sequence of the elliptic equations. The obtained integral equations of the first kind contain the logarithmic and hypersingular kernels. They are solved by a discrete collocation method based on trigonometrical quadratures. The presented numerical experiments confirm a posterior error estimates and show the feasibility of the proposed method.

1. Introduction

The time depended problems that are modeled by initial boundary value problems for parabolic or hyperbolic partial differential equations have a large number of applications in engineering and various applied sciences like elastodynamics, fluid dynamics or acoustics. On the other hand the initial boundary value problem can arise as some partial sub-problem during numerical solution of more complicate applied problems, for example inverse problems. In this contest we refer to [6] for the inverse non-linear parabolic problem related to boundary reconstruction and to [12] for the inverse linear parabolic problem for the reconstruction of Cauchy data on the boundary. In the latter reference the iterative method is proposed which requires the solution of the mixed initial boundary value problem for parabolic equation on each step. Therefore to realize this approach numerically one needs the solver for the corresponding direct time-dependent problem. There are a large number of methods for the numerical solution for this kind of problems. We want to use the integral equations approach and the reason for this is the advantages of this method like decreasing of the dimension of the problem, reducing the problem in an unbounded domain to an integral equation on the boundary etc.

One can distinguish three approaches in the application of boundary integral methods on parabolic and hyperbolic initial boundary value problems: boundary-time integral equations, integral transform methods, and time-stepping methods [2, 8]. Boundary-time integral equations method use the fundamental solution of the parabolic or hyperbolic partial differential equations. By the direct or indirect approach the integral equations of the first or the second kind, which are of Volterra type in the time variable and Fredholm type in the space variables, are obtained. Of course, there is extensive literature available on this direct application of integral equation techniques for the full time dependent problems (see [1, 10, 13, 14] and the references therein).

In the second group of methods an integral transformation like the Laplace, Fourier or Laguerre transformation is used to achieve the reduction of a time dependent problem

† *Key words.* Heat equation, mixed problem, Rothe's method, boundary integral equation method, trigonometrical quadrature method.

to stationary boundary value problems (see [1, 5] and the references therein), which can be numerically solved by boundary integral equation techniques.

Time-stepping methods start from a time discretization of the original initial boundary value problem via an implicit scheme and then they use boundary integral equations to solve the resulting elliptic problems in each time step. A fundamental solution, which is also a discrete convolution operator, can be given explicitly for simple time discretization schemes like the backward Euler method ("Rothe's method" [4]). For a whole class of higher order one step or multistep methods, the fundamental solution can be constructed using Laplace transformation and the operational quadrature method [14]. In this paper we use the combination of Rothe's method with integral equations to a mixed initial boundary value problem for the heat equation in a double-connected domain D and investigate the variants of first and second orders of the time approximation.

We consider the mixed initial boundary value problem for the heat equation

$$\frac{\partial u}{\partial t} = \Delta u \quad \text{in } D \times (0, T). \tag{1.1}$$

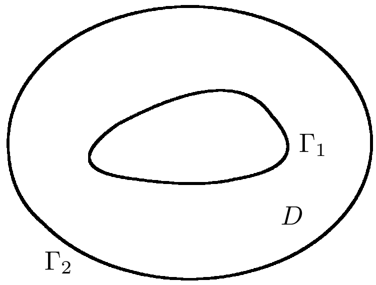


Fig. 1. Double-connected domain D

We assume that the boundaries Γ_1, Γ_2 are smooth enough and $T > 0$ is some constant. We are looking for a classical solution of (1.1) which is twice continuously differentiable with respect to the space variable and continuously differentiable with respect to the time variable on $D \times (0, T)$ and which satisfies the homogeneous initial condition

$$u(x, 0) = 0, \quad x \in D \tag{1.2}$$

and the boundary conditions

$$u = f_1 \quad \text{on } \Gamma_1 \times (0, T), \qquad \frac{\partial u}{\partial \nu} = f_2 \quad \text{on } \Gamma_2 \times (0, T), \tag{1.3}$$

where ν is the outward unit normal vector to the boundary Γ_2, f_1 and f_2 are given functions which satisfy compatibility conditions.

$$f_1(x, 0) = 0, \quad x \in \Gamma_1 \qquad \text{and} \qquad f_2(x, 0) = 0, \quad x \in \Gamma_2.$$

The plan of this paper is as follows. In Section 2 we describe the basic features of the Rothe method and the boundary integral equation method. The first and second order of time approximation are proposed. The following Section 3 briefly describes how well established numerical methods can be applied to solve these boundary integral equations. Also we proceed here with some convergence and error analysis, and in Section 4 we demonstrate the feasibility of our method through some numerical examples.

2. Rothe's methos and boundary integrals approach

Rothe's method combined with integral equations is described in details in [4]. This approach consists in replacing some initial boundary value problem with a sequence of stationary boundary value problems by semi-discretization in the time. The time derivative in the heat equation (1.1) is discretized by a finite difference approximation. Thus, on the equidistant mesh $\{t_n = (n+1)h,\ n = -1, \ldots, N-1,\ h = T/N,\ N \in \mathbb{N}\}$ we approximate the solution u of (1.1) by the sequence u_n, $n = 0, \ldots, N-1$ that solves the equations

$$\Delta u_n = \gamma^2 u_n - \gamma^2 u_{n-1}, \quad u_{-1} = 0, \quad \gamma^2 = \frac{1}{h}. \tag{2.1}$$

To obtain the second order of the approximation in the time we assume that u is two times continuosly differentiable with respect to time variable. So we can use the following differences

$$\frac{\partial u}{\partial t}(x, t_n) = \frac{u(x,t_n) - u(x,t_{n-1})}{h} + \frac{h}{2} u_{tt}(x, t_n - \theta_n h), \quad \theta_n \in [0,1],$$
$$\frac{\partial u}{\partial t}(x, t_{n-1}) = \frac{u(x,t_n) - u(x,t_{n-1})}{h} - \frac{h}{2} u_{tt}(x, t_n - \tilde{\theta}_n h), \quad \tilde{\theta}_n \in [0,1].$$

After substitution of these relationships in (1.1) and adding the obtained equations we get

$$\frac{2}{h} u_n - \frac{2}{h} u_{n-1} - \Delta u_n - \Delta u_{n-1} + O(h^2) = 0$$

and as result we receive the sequence

$$\Delta u_n = \sum_{m=0}^{n} \beta_{n-m} u_m, \quad \beta_0 = \frac{2}{h}, \quad \beta_i = (-1)^i \frac{4}{h},\ i = 1, \ldots, N-1. \tag{2.2}$$

Now we summarize both approaches in the following sequence of mixed stationary boundary value problems related with the problem (1.1)–(1.3)

$$\Delta u_n - \gamma^2 u_n = \sum_{m=0}^{n-1} \beta_{n-m} u_m \quad \text{in } D, \tag{2.3}$$

$$u_n = f_n^1 \quad \text{on } \Gamma_1, \qquad \frac{\partial u_n}{\partial \nu} = f_n^2 \quad \text{on } \Gamma_2, \tag{2.4}$$

where $n = 0, \ldots, N-1$, $f_n^\ell = f_\ell(\cdot, t_n)$, $\ell = 1, 2$, $\gamma^2 = \beta_0$ and the form of the known constants β_i depends on the used semi-discretization approach (2.1) or (2.2).

We proceed by noting the following results on uniqueness and stability.

Theorem 2.1 *The sequence* (2.3)–(2.4) *has at most one solution.*

Proof. By the Green's theorem any solution $v \in C^2(D) \cap C^1(\bar{D})$ of $\Delta v - \gamma^2 v = 0$ in D which has vanishing boundary values $v = 0$ on Γ_1 and $\partial v / \partial \nu = 0$ on Γ_2 must vanish identically in D. Then the statement of the theorem follows by induction. □

Now we introduce the fundamental solution to the sequence of elliptic differential equations (2.3).

Definition 2.2 *The functions sequence* (Φ_n), $n = 0, \ldots, N-1$ *is called a fundamental sequence for the equations* (2.3) *if*

$$\Delta_x \Phi_n(x, y) - \sum_{m=0}^{n} \beta_{n-m} \Phi_m(x, y) = \delta(x - y).$$

We consider the polynomials, which will be used for the representation of Φ_n

$$v_n(r) = \sum_{m=0}^{\left[\frac{n}{2}\right]} a_{n,2m} r^{2m}, \qquad w_n(r) = \sum_{m=0}^{\left[\frac{n-1}{2}\right]} a_{n,2m+1} r^{2m+1}$$

for $n = 0, 1, \ldots, N-1$ ($w_0 = 0$) with $a_{n,0} = 1$ and the remaining coefficients recursively defined through

$$a_{n,n} = -\frac{1}{2\gamma n}\, \beta_1 a_{n-1,n-1},$$

$$a_{n,k} = \frac{1}{2\gamma k}\left\{4\left[\frac{k+1}{2}\right]^2 a_{n,k+1} - \sum_{m=k-1}^{n-1} \beta_{n-m} a_{m,k-1}\right\}, \quad k = n-1, \ldots, 1.$$

The coefficients $a_{n,k}$ are chosen such that v_n, w_n solve the system of ordinary differential equations

$$v_n''(r) + \frac{1}{r} v_n'(r) - 2\gamma w_n'(r) = \sum_{m=0}^{n-1} \beta_{n-m} v_m(r),$$

$$-2\gamma v_n'(r) + w_n''(r) - \frac{1}{r} w_n'(r) + \frac{1}{r^2} w_n(r) = \sum_{m=0}^{n-1} \beta_{n-m} w_m(r).$$

Analogously to [5] we can find the explicit representation of the fundamental solutions.

Theorem 2.3 *The functions*

$$\Phi_n(x,y) = K_0(\gamma|x-y|) v_n(|x-y|) + K_1(\gamma|x-y|) w_n(|x-y|), \tag{2.5}$$

$n = 0, \ldots, N-1$ *are the fundamental solutions of* (2.3). *Here* K_0 *and* K_1 *are the modified Hankel functions of order zero and one, respectively.*

We note here, that in [9] it was found some different representation of the fundamental solutions Φ_n, which has the recurrence nature.
For the sequence of equations (2.3) we introduce both the single- and double layer potentials:

$$U_n(x) = \frac{1}{\pi} \sum_{m=0}^{n} \int_{\Gamma_i} q_m(y) \Phi_{n-m}(x,y)\, ds(y), \quad x \in D \tag{2.6}$$

and

$$V_n(x) = \frac{1}{\pi} \sum_{m=0}^{n} \int_{\Gamma_i} q_m(y)\, \frac{\partial}{\partial \nu(y)}\, \Phi_{n-m}(x,y)\, ds(y) \quad x \in D, \tag{2.7}$$

with continuous densities q_n for $n = 0, \ldots, N-1$ and $i = 1, 2$. Hence, by the classical jump- and regularity properties of the logarithmic potentials (see [13]) we reformulate the boundary value problem (2.3)-(2.4) as a sequence of boundary integral equations.

Theorem 2.4 *The combination of single- and double layer potentials*

$$u_n(x) = \frac{1}{\pi} \sum_{m=0}^{n} \int_{\Gamma_1} \varphi_m^1(y) \Phi_{n-m}(x,y) ds(y) + \frac{1}{\pi} \sum_{m=0}^{n} \int_{\Gamma_2} \varphi_m^2(y) \frac{\partial \Phi_{n-m}(x,y)}{\partial \nu(y)} ds(y), \; x \in D \tag{2.8}$$

solves the sequence of boundary value problems (2.3)–(2.4) provided the densities solve the sequence of integral equations of the first kind

$$\begin{cases} \dfrac{1}{\pi}\displaystyle\int_{\Gamma_1} \varphi_n^1(y)\Phi_0(x,y)ds(y) + \dfrac{1}{\pi}\int_{\Gamma_2} \varphi_n^2(y)\dfrac{\partial\Phi_0(x,y)}{\partial\nu(y)}ds(y) = G_n^1(x), \ x\in\Gamma_1, \\[2ex] \dfrac{1}{\pi}\displaystyle\int_{\Gamma_1} \varphi_n^1(y)\dfrac{\partial\Phi_0(x,y)}{\partial\nu(x)}ds(y) + \dfrac{1}{\pi}\dfrac{\partial}{\partial\nu(x)}\int_{\Gamma_2} \varphi_n^2(y)\dfrac{\partial\Phi_0(x,y)}{\partial\nu(y)}ds(y) = G_n^2(x), \ x\in\Gamma_2 \end{cases} \tag{2.9}$$

for $n = 0,\dots,N-1$ *with right hand sides*

$$G_n^1(x) := f_n^1(x) - \frac{1}{\pi}\sum_{m=0}^{n-1}\int_{\Gamma_1}\varphi_m^1(y)\Phi_{n-m}(x,y)ds(y) - \frac{1}{\pi}\sum_{m=0}^{n-1}\int_{\Gamma_2}\varphi_m^2(y)\frac{\partial\Phi_{n-m}(x,y)}{\partial\nu(y)}ds(y)$$

and

$$\begin{aligned} G_n^2(x) := \ & f_n^2(x) - \frac{1}{\pi}\sum_{m=0}^{n-1}\int_{\Gamma_1}\varphi_m^1(y)\frac{\partial\Phi_{n-m}(x,y)}{\partial\nu(x)}ds(y) \\ - \ & \frac{1}{\pi}\sum_{m=0}^{n-1}\frac{\partial}{\partial\nu(x)}\int_{\Gamma_2}\varphi_m^2(y)\frac{\partial\Phi_{n-m}(x,y)}{\partial\nu(y)}ds(y). \end{aligned}$$

In order to simplify the second integral equation we apply the following relationship between the normal derivate of double-layer potential (2.6) and the sigle-layer potential (2.7):

$$\begin{aligned} \sum_{m=0}^{n}\frac{\partial}{\partial\nu(x)}\int_{\Gamma_2} q_m(y)\frac{\partial\Phi_{n-m}(x,y)}{\partial\nu(y)}ds(y) &= \sum_{m=0}^{n}\frac{\partial}{\partial\theta(x)}\int_{\Gamma_2}\frac{\partial q_m}{\partial\theta}(y)\Phi_{n-m}(x,y)ds(y) \\ - \sum_{m=0}^{n}\int_{\Gamma_2} q_m(y)\sum_{k=0}^{n-m}\beta_{n-m-k}\Phi_k(x,y)&\left\langle\nu(x),\nu(y)\right\rangle ds(y), \quad x\in\Gamma_2, \end{aligned} \tag{2.10}$$

where θ is the unit tangent vector to Γ_2 and by $\langle\cdot,\cdot\rangle$ we denote the scalar product in $\mathbb{R}^2$.

Assume that our boundaries have the following parametric representations:

$$\Gamma_\ell = \{x_\ell(s) = (x_{\ell 1}(s), x_{\ell 2}(s)), s\in[0,2\pi]\}, \quad \ell = 1,2, \tag{2.11}$$

where $x_\ell : [0,2\pi) \to \mathbb{R}^2$ are injective and three times differentiable. Then after calculation with the use of (2.10) and extraction of hypersingularity the following parametrized systems are obtained

$$\begin{cases} \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\mu_n^1(\sigma)H_0^{11}(s,\sigma)d\sigma + \dfrac{1}{2\pi}\int_0^{2\pi}\mu_n^2(\sigma)H_0^{12}(s,\sigma)d\sigma = G_n^1(s), \\[2ex] \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\mu_n^1(\sigma)H_0^{21}(s,\sigma)d\sigma + \dfrac{1}{2\pi}\int_0^{2\pi}\left[\dot\mu_n^2(\sigma)\cot\dfrac{\sigma-s}{2} + \mu_n^2(\sigma)H_0^{22}(s,\sigma)\right]d\sigma = G_n^2(s), \end{cases} \tag{2.12}$$

where $s\in[0,2\pi]$, $\mu_n^1(s) := \varphi_n^1(x_1(s))$, $\mu_n^2(s) := \sum_{m=0}^{n}\varphi_n^2(x_2(s))$, $n = 0,\dots,N-1$ and

$$G_n^1(s) := f_n^1(s) - \frac{1}{2\pi}\sum_{m=0}^{n-1}\int_0^{2\pi}\left[\mu_m^1(\sigma)H_{n-m}^{11}(s,\sigma) + \psi_m^2(\sigma)H_{n-m}^{12}(s,\sigma)\right]d\sigma$$

and

$$G_n^2(s) := f_n^2(s) - \frac{1}{2\pi}\sum_{m=0}^{n-1}\int_0^{2\pi}\left[\mu_m^1(\sigma)H_{n-m}^{21}(s,\sigma) + \psi_m^2(\sigma)H_{n-m}^{22}(s,\sigma)\right]d\sigma,$$

where $f_n^1(s) = f_n^1(x_1(s))$, $f_n^2(s) = f_n^2(x_2(s))|x_2'(s)|$, $\psi_n^2(s) = \varphi_n^2(x_2(s))$. Here we have the kernels with logarithmic singularity

$$H_n^{11}(s,\sigma) := 2|x_1'(\sigma)|\Phi_n(x_1(s), x_1(\sigma)), \quad n = 0, \dots, N-1,$$

$$\begin{aligned} H_0^{22}(s,\sigma) :=\ & 2\gamma\frac{K_1(\gamma|r_{22}(s,\sigma)|)}{|r_{22}(s,\sigma)|}\left[2h_2(s,\sigma) - h_1(s,\sigma)\right] \\ & + 2\gamma^2 K_0(\gamma|r_{22}(s,\sigma)|)\left[h_2(s,\sigma) - h_1(s,\sigma)\right] - \frac{1}{2\sin^2\frac{s-\sigma}{2}}, \end{aligned}$$

and

$$\begin{aligned} H_n^{22}(s,\sigma) :=\ & 2K_0(\gamma|r_{22}(s,\sigma)|)\left[(v_n^1(|r_{22}(s,\sigma)|) + \gamma^2)h_1(s,\sigma) + v_n^2(|r_{22}(s,\sigma)|)h_2(s,\sigma)\right] \\ & + 2K_1(\gamma|r_{22}(s,\sigma)|)\left[w_n^1(|r_{22}(s,\sigma)|)h_1(s,\sigma) + w_n^2(|r_{22}(s,\sigma)|)h_2(s,\sigma)\right] \\ & - 2h_1(s,\sigma)\sum_{k=0}^{n}\beta_{n-k}\Phi_k(x_2(s), x_2(\sigma)), \end{aligned}$$

for $n = 1, \dots, N-1$. In the computation of the kernels we use the following formulas for the derivatives of the modified Hankel functions

$$K_0'(z) = -K_1(z), \qquad K_1'(z) = -K_0(z) - \frac{1}{z}K_1(z).$$

The regular kernels have the form

$$H_0^{12}(s,\sigma) := 2\gamma\frac{K_1(\gamma|r_{12}(s,\sigma)|)}{|r_{12}(s,\sigma)|}\left\langle r_{12}(s,\sigma), \nu(x_2(\sigma))\right\rangle |x_2'(\sigma)|,$$

$$H_0^{21}(s,\sigma) := -2\gamma\frac{K_1(\gamma|r_{21}(s,\sigma)|)}{|r_{21}(s,\sigma)|}\left\langle r_{21}(s,\sigma), \nu(x_2(s))\right\rangle |x_1'(\sigma)|,$$

$$\begin{aligned} H_n^{12}(s,\sigma) := -2\left\langle r_{12}(s,\sigma), \nu(x_2(\sigma))\right\rangle |x_2'(\sigma)| \{ & K_0(\gamma|r_{12}(s,\sigma)|)v_n^1(|r_{12}(s,\sigma)|) \\ & + K_1(\gamma|r_{12}(s,\sigma)|)w_n^1(|r_{12}(s,\sigma)|)\} \end{aligned}$$

and

$$\begin{aligned} H_n^{21}(s,\sigma) := 2\left\langle r_{21}(s,\sigma), \nu(x_2(s))\right\rangle |x_1'(\sigma)| \{ & K_0(\gamma|r_{21}(s,\sigma)|)v_n^1(|r_{21}(s,\sigma)|) \\ & + K_1(\gamma|r_{21}(s,\sigma)|)\left[w_n^1(|r_{21}(s,\sigma)|) - \frac{\gamma}{|r_{21}(s,\sigma)|}\right]\}. \end{aligned}$$

Here we introduced the following functions and notations

$$r_{ij}(s,\sigma) := x_i(s) - x_j(\sigma), \quad i,j = 1,2,$$

$$h_1(s,\sigma) = \left\langle x_2'(s), x_2'(\sigma)\right\rangle, \quad h_2(s,\sigma) = \frac{\left\langle r_{22}(s,\sigma), x_2'(s)\right\rangle \left\langle r_{22}(s,\sigma), x_2'(\sigma)\right\rangle}{|r_{22}(s,\sigma)|^2},$$

$$v_n^1(r) = -\gamma \sum_{k=0}^{\left[\frac{n-1}{2}\right]} a_{n,2k+1} r^{2k} + 2\sum_{k=1}^{\left[\frac{n}{2}\right]} k a_{n,2k} r^{2k-2},$$

$$w_n^1(r) = -\gamma \sum_{k=1}^{\left[\frac{n}{2}\right]} a_{n,2k} r^{2k-1} + 2\sum_{k=1}^{\left[\frac{n-1}{2}\right]} k a_{n,2k+1} r^{2k-1},$$

$$v_n^2(r) = \gamma^2 \sum_{k=1}^{\left[\frac{n}{2}\right]} a_{n,2k} r^{2k} - 4\gamma \sum_{k=1}^{\left[\frac{n-1}{2}\right]} k a_{n,2k+1} r^{2k} - 4\sum_{k=2}^{\left[\frac{n}{2}\right]} (k-k^2) a_{n,2k-2} r^{2k},$$

$$w_n^2(r) = \gamma^2 \sum_{k=0}^{\left[\frac{n-1}{2}\right]} a_{n,2k+1} r^{2k+1} - 2\gamma \sum_{k=1}^{\left[\frac{n}{2}\right]} (2k-1) a_{n,2k} r^{2k-1} - 4\sum_{k=2}^{\left[\frac{n-1}{2}\right]} (k-k^2) a_{n,2k+1} r^{2k-1}.$$

We transform the kernels with logarithmic singularity to the form

$$H_n^{\ell\ell}(s,\sigma) = H_{n0}^{\ell\ell}(s,\sigma) \ln\frac{4}{e}\sin^2\frac{s-\sigma}{2} + H_{n1}^{\ell\ell}(s,\sigma), \quad \ell = 1,2,$$

where

$$H_{n0}^{11}(s,\sigma) := \left[-I_0(\gamma|r_{11}(s,\sigma)|) v_n(|r_{11}(s,\sigma)|) + I_1(\gamma|r_{11}(s,\sigma)|) w_n(|r_{11}(s,\sigma)|)\right] |x_1'(\sigma)|$$

for $n = 0,\dots,N-1$,

$$H_{00}^{22}(s,\sigma) := \gamma\frac{I_1(\gamma|r_{22}(s,\sigma)|)}{|r_{22}(s,\sigma)|}\left[2h_2(s,\sigma) - h_1(s,\sigma)\right] - \gamma^2 I_0(\gamma|r_{22}(s,\sigma)|)\left[h_2(s,\sigma) - h_1(s,\sigma)\right],$$

$$H_{n0}^{22}(s,\sigma) := -I_0(\gamma|r_{22}(s,\sigma)|)\left[v_n^1(|r_{22}(s,\sigma)|) h_1(s,\sigma) + v_n^2(|r_{22}(s,\sigma)|) h_2(s,\sigma) + \gamma^2 h_1(s,\sigma)\right]$$

$$+ I_1(\gamma|r_{22}(s,\sigma)|)\left[w_n^1(|r_{22}(s,\sigma)|) h_1(s,\sigma) + w_n^2(|r_{22}(s,\sigma)|) h_2(s,\sigma)\right]$$

$$+ h_1(s,\sigma)\sum_{k=0}^{n} \beta_{n-k}\left[I_0(\gamma|r_{22}(s,\sigma)|) v_k(|r_{22}(s,\sigma)|) - I_1(\gamma|r_{22}(s,\sigma)|) w_k(|r_{22}(s,\sigma)|)\right]$$

for $n = 1,\dots,N-1$ and

$$H_{n1}^{\ell\ell}(s,\sigma) = H_n^{\ell\ell}(s,\sigma) - H_{n0}^{\ell\ell}(s,\sigma) \ln\frac{4}{e}\sin^2\frac{s-\sigma}{2}$$

with diagonal terms

$$H_{00}^{22}(s,s) = \frac{\gamma^2}{2}|x_2'(s)|^2, \quad H_{n0}^{22}(s,s) = (-\gamma^2 + \gamma a_{n,1} - 2a_{n,2})|x_2'(s)|^2 + |x_2'(s)|^2 \sum_{k=0}^{n} \beta_{n-k},$$

$$H_{n1}^{11}(s,s) = (-\tilde{C}_1(s) - 2C_E + \frac{2}{\gamma} a_{n,1})|x_1'(s)|,$$

$$H_{01}^{22}(s,s) = -\frac{1}{6} + \frac{1}{3}\frac{\langle x_2'(s), x_2'''(s)\rangle}{|x_2'(s)|^2} + \frac{1}{2}\frac{\langle x_2''(s), x_2''(s)\rangle}{|x_2'(s)|^2} - \frac{\langle x_2'(s), x_2''(s)\rangle^2}{|x_2'(s)|^4}$$
$$+ \frac{\gamma^2}{2}|x_2'(s)|^2 (2C_E + \tilde{C}_2(s) - 1),$$

and

$$H_{n1}^{22}(s,s) = (\tilde{C}_2(s)+2C_E)(\gamma a_{n,1} - 2a_{n,2} - \gamma^2)|x_2'(s)|^2+\frac{2}{\gamma}(\gamma^2 a_{n,1} - 3\gamma a_{n,2}+2a_{n,3})|x_2'(s)|^2$$
$$+ 2|x_2'(s)|^2 \sum_{k=0}^{n} \beta_{n-k} \left(\frac{\tilde{C}_2(s)}{2} + C_E - \frac{a_{k,1}}{\gamma} \right).$$

Here $\tilde{C}_\ell(s) = \ln \frac{e\gamma^2 |x_\ell'(s)|^2}{4}$ and C_E is the Euler constant. Also we assume that $a_{n,k} = 0$ for $n < k$.

To rewrite the sequence (2.12) in the operator form we introduce integral operators

$$(S_m^\ell \mu)(s) := \frac{1}{2\pi} \int_0^{2\pi} \mu(\sigma) H_{m0}^{\ell\ell}(s,\sigma) \ln \frac{4}{e} \sin^2 \frac{s-\sigma}{2} d\sigma,$$

$$(A_m^\ell \mu)(s) := \frac{1}{2\pi} \int_0^{2\pi} \mu(\sigma) H_{m1}^{\ell\ell}(s,\sigma) d\sigma,$$

$$(T\mu)(s) := \frac{1}{2\pi} \int_0^{2\pi} \dot{\mu}(\sigma) \cot \frac{\sigma - s}{2} d\sigma,$$

$$(S\mu)(s) := \frac{1}{2\pi} \int_0^{2\pi} \mu(\sigma) \ln \frac{4}{e} \sin^2 \frac{s-\sigma}{2} d\sigma$$

and

$$(B_m^{\ell k}\mu)(s) := \frac{1}{2\pi} \int_0^{2\pi} \mu(\sigma) H_m^{\ell k}(s,\sigma) d\sigma.$$

According to the smooth properties of the kernels we have that the operators A are compact in the corresponding Hölder spaces. Thus the systems (2.12) has the following equivalent operator representation

$$(\mathcal{U} + \mathcal{B}_0)\tilde{\mu}_n = \tilde{f}_n - \sum_{m=0}^{n-1} \mathcal{B}_{n-m}\tilde{\psi}_m, \quad n = 0, \dots, N-1, \tag{2.13}$$

where we introduced the vectors $\tilde{\mu}_n := \left(\mu_n^1, \mu_n^2\right)^\top$, $\tilde{\psi}_n := \left(\mu_n^1, \psi_n^2\right)^\top$, $\tilde{f}_n := (f_n^1, f_n^2)^\top$ and operator matrices

$$\mathcal{U} := \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}, \ \mathcal{B}_0 := \begin{pmatrix} S_0^1 + A_0^1 - S & B_0^{12} \\ B_0^{21} & S_0^2 + A_0^2 \end{pmatrix}, \ \mathcal{B}_m := \begin{pmatrix} S_m^1 + A_m^1 & B_m^{12} \\ B_m^{21} & S_m^2 + A_m^2 \end{pmatrix}$$

for $m = 1, \dots, N-1$.

Theorem 2.5 *For any sequences f_n^1 in $C^{1,\alpha}[0,2\pi]$ and f_n^2 in $C^{0,\alpha}[0,2\pi]$ the system (2.12) possesses a unique solution μ_n^1 in $C^{0,\alpha}[0,2\pi]$ and μ_n^2 in $C^{1,\alpha}[0,2\pi]$.*

Proof. Since the operators $S : C^{0,\alpha}[0,2\pi] \to C^{1,\alpha}[0,2\pi]$ and $T : C^{1,\alpha}[0,2\pi] \to C^{0,\alpha}[0,2\pi]$ are bounded and have bounded inverses [13], we can reduce the first equation in (2.13) to the form $(I + D_0)\tilde{\mu}_0 = \tilde{g}_0$, where $D_0 = \{D_0^{k\ell}\}$, $D_0^{0\ell} : C^{0,\alpha}[0,2\pi] \to C^{0,\alpha}[0,2\pi]$, $D_0^{1\ell} : C^{1,\alpha}[0,2\pi] \to C^{1,\alpha}[0,2\pi]$, $k = 1,2$, $\ell = 1,2$ are compact. Then from the theorem 2.1 and the Riesz-Schauder theory [13] lead us to the existence of the solution $\tilde{\mu}_0 \in C^{0,\alpha}[0,2\pi] \times C^{1,\alpha}[0,2\pi]$. The statement of the theorem follows by induction. □

3. Numerical solution of integral equations

For integral equations of the form (2.12) we use a discrete collocation method based on trigonometric interpolation with equidistant grid points [4]. For this method, we choose $M \in \mathbb{N}$ and an equidistant mesh by setting

$$s_k := \frac{k\pi}{M}, \quad k = 0, \dots, 2M-1,$$

and use the following quadrature rules

$$\frac{1}{2\pi}\int_0^{2\pi} g(\sigma)\ln\left(\frac{4}{e}\sin^2\frac{s_j-\sigma}{2}\right)d\sigma \approx \sum_{k=0}^{2M-1} R_{|j-k|}\, g(s_k), \tag{3.1}$$

$$\frac{1}{2\pi}\int_0^{2\pi} g'(\sigma)\cot\frac{\sigma-s_j}{2}d\sigma \approx \sum_{k=0}^{2n-1} T_{|j-k|}\, g(s_k), \tag{3.2}$$

$$\frac{1}{2\pi}\int_0^{2\pi} g(\sigma)\,d\sigma \approx \frac{1}{2M}\sum_{k=0}^{2M-1} g(s_k) \tag{3.3}$$

with the weights

$$R_j := -\frac{1}{M}\sum_{m=1}^{M-1}\frac{1}{m}\cos\frac{mj\pi}{M} - \frac{(-1)^j}{2M^2},$$

$$T_j := \begin{cases} \dfrac{1}{2M}\sin^{-2}\dfrac{j\pi}{2M}, & j \text{ odd}, \\[2mm] 0, & j \text{ even},\ j \neq 0, \\[2mm] -\dfrac{M}{2}, & j = 0, \end{cases}$$

These quadratures are obtained by replacing the integrand g by its trigonometric interpolation polynomial of degree M with respect to the grid points s_k, $k = 0, \dots, 2M-1$.

We use the quadrature rules (3.1)–(3.3) to approximate the three types of integrals in (2.12) and collocate the approximated equation at the nodal points to obtain the linear system

$$\begin{cases} \displaystyle\sum_{k=0}^{2M-1}\left[\mu^1_{n,k}\left\{R_{|j-k|}H^{11}_{00}(s_j,s_k) + \frac{1}{2M}H^{11}_{01}(s_j,s_k)\right\} + \mu^2_{n,k}\frac{1}{2M}H^{12}_0(s_j,s_k)\right] = G^1_{n,j}, \\ \displaystyle\sum_{k=0}^{2M-1}\left[\frac{\mu^1_{n,k}}{2M}H^{21}_0(s_j,s_k) + \mu^2_{n,k}\left\{T_{|j-k|} + R_{|j-k|}H^{22}_{00}(s_j,s_k) + \frac{1}{2M}H^{22}_{01}(s_j,s_k)\right\}\right] = G^2_{n,j}, \end{cases} \tag{3.4}$$

for $j = 0, \dots, 2M-1$, $n = 0, \dots, N-1$, which we have to solve for the nodal values $\mu^\ell_{n,j}$ of the approximating trigonometric polynomial $\mu^\ell_{n,M}$. Of course, the approximate values $G^\ell_{n,j}$ for the right hand side are also obtained via using (3.1) and (3.3). To write the linear systems in operator form we consider the interpolation operators $P_M : C[0,2\pi] \to \mathcal{T}_M$, where $\mathcal{T}_M$ is the space of trigonometrical polynomials of the degrees M. Then we can rewrite the system (3.4) in the equivalent operator form

$$(\mathcal{U} + P_M\mathcal{B}_{0,M})\,\tilde{\mu}_{n,M} = P_M\tilde{f}_n - \sum_{m=0}^{n-1} P_M\mathcal{B}_{n-m,M}\tilde{\psi}_{m,M}, \quad n = 0, \dots, N-1. \tag{3.5}$$

The convergence and error analysis for this quadrature method can be established on the basis of the collectively compact operators theory (see [4]) or on the basis of some estimate of trigonometric interpolation in Hölder spaces (see for example [7]). In the latter case this analysis is based on the estimate

$$\|P_M\mu - \mu\|_{m,\alpha} \le c\,\frac{\ln M}{M^{\ell-m+\beta-\alpha}}\,\|\mu\|_{\ell,\beta} \tag{3.6}$$

for the trigonometric interpolation which is valid for $0 \le m \le \ell$, $0 < \alpha \le \beta < 1$, and some constant c depending only on m, ℓ, α and β.

Theorem 3.1 *For $\Gamma_1,\ \Gamma_2 \in C^{\ell+2}$, $\ell \ge 1$, $f_n^1 \in C^{\ell,\beta}[0,2\pi]$, $f_n^2 \in C^{\ell-1,\beta}[0,2\pi]$ and for sufficiently large M the system of approximate equations (3.5) for every $n = 0,\dots,N-1$ has a unique solution $\tilde{\mu}_{n,M} \in \mathcal{T}_M$. For the exact solution $\tilde{\mu}_n$ to (2.13) we have the error estimates*

$$\|\tilde{\mu}_n - \tilde{\mu}_{n,M}\|_{m,\alpha} \le C_n\frac{\ln M}{M^{\ell-m+\beta-\alpha}}\|\tilde{\mu}_n\|_{\ell,\beta} \tag{3.7}$$

for $0 \le m \le \ell$, $0 < \alpha \le \beta < 1$ and some constants C_k depending only on α, β, m, ℓ.

Proof. Let $X = C^{m-1,\alpha}[0,2\pi] \times C^{m,\alpha}[0,2\pi]$ and $Y = C^{m,\alpha}[0,2\pi] \times C^{m-1,\alpha}[0,2\pi]$. By the smoothness properties of the kernels in the operators B_n^{ik} and from the estimate (3.6) it can be shown that

$$\|P_M B_{n,M}^{ik}\mu - B_n^{ik}\mu\|_{m,\alpha} \le c\,\frac{\ln M}{M^{\ell-m+\beta-\alpha}}\,\|\mu\|_{\ell,\beta}, \quad k, i = 1, 2.$$

Analogous estimates hold for other operators in the matrix $\mathcal{B}_n$ (for the case of the operators with logarithmic singularity see [15]). This implies, in particular, for $\ell = m$ the norm convergence

$$\|P_M\mathcal{B}_{0,M} - \mathcal{B}_0\|_{X\to Y} \to 0, \quad M \to \infty.$$

Therefore, from the Neumann series, we can conclude that, for sufficiently large n_1 and n_2, the operators $\mathcal{U} + \mathcal{P}_{n_1,n_2}\mathcal{B}_{n_1,n_2} : X \to Y$ are invertible and the inverse operators are uniformly bounded. Then the error estimate (3.7) for $n = 0$ follows from the identity

$$\tilde{\mu}_{0,M} - \tilde{\mu}_0 = (\mathcal{U} + P_M\mathcal{B}_{0,M})^{-1}\{(P_M\tilde{f}_0 - \tilde{f}_0) + (\mathcal{B}_0 - P_M\mathcal{B}_{0,M})\tilde{\mu}_0\}.$$

The statement of the theorem follows by induction. □

4. Numerical experiments

Example 1. To test our method we will use the double connected domain with the boundaries (see Fig. 2)

$$\Gamma_1 = \left\{(0.06\cos(t), 0.1 + 0.12\sin(t) - 0.1\sin^2(t)), s \in [0,2\pi]\right\},$$
$$\Gamma_2 = \left\{(0.2\cos(t), 0.2 + 0.2\cos(2t) + 0.3\sin(t)), s \in [0,2\pi]\right\}.$$

Let us choose the boundary conditions of elliptic problems sequence as

$$f_n^1(x,t) = \Phi_n(x,\tilde{y}), \qquad x \in \Gamma_1,$$
$$f_n^2(x,t) = \frac{\partial}{\partial\nu(x)}\Phi_n(x,\tilde{y}), \qquad x \in \Gamma_2,$$

where Φ_n is the fundamental solution and $\tilde{y} \notin D$. It is obvious that the exact solution of this sequence will be

$$u_n(x) = \Phi_n(x,\tilde{y}).$$

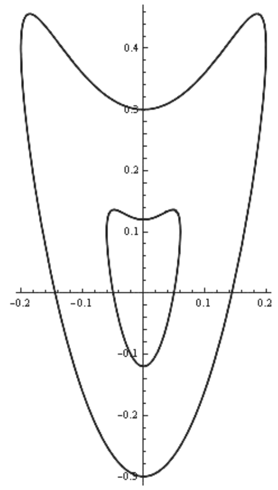


Fig. 2. The domain D for numerical experiments.

Tabl. 1. Numerical results for the first order approximation.

	$n = 0$	$n = 5$	$n = 10$
$M = 16$	$1.505 \cdot 10^{-5}$	$3.858 \cdot 10^{-4}$	$8.211 \cdot 10^{-4}$
$M = 32$	$5.521 \cdot 10^{-8}$	$1.199 \cdot 10^{-6}$	$2.456 \cdot 10^{-6}$
$M = 64$	$5.549 \cdot 10^{-14}$	$4.768 \cdot 10^{-12}$	$1.074 \cdot 10^{-11}$

Let us choose $\tilde{y} = (0, 0.8)$ and the test point $x = (0, -0.2)$. In the Table 1 we show absolute errors of first order approximation for different amount of points in quadratures. The absolute errors of the elliptic problems sequence in the case of second order approximation of the time derivate are given in the Table 2. These results empirically proof correctness of the developed methods and their consistency with the obtained error estimation.

Tabl. 2. Numerical results for the second order approximation.

	$n = 0$	$n = 5$	$n = 10$
$M = 16$	$1.303 \cdot 10^{-6}$	$3.392 \cdot 10^{-4}$	$8.011 \cdot 10^{-4}$
$M = 32$	$6.723 \cdot 10^{-9}$	$1.072 \cdot 10^{-6}$	$2.403 \cdot 10^{-6}$
$M = 64$	$3.310 \cdot 10^{-14}$	$4.126 \cdot 10^{-12}$	$1.048 \cdot 10^{-11}$

Example 2. For testing the nonstationary problem we define a boundary function as the restriction of the fundamental solution

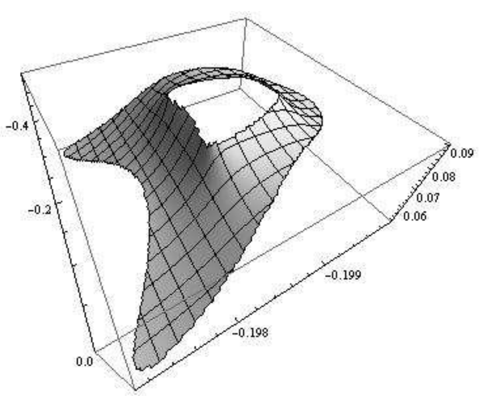
$$u(x,t) = \frac{1}{4\pi t} \exp\left(\frac{-|x-\tilde{y}|}{4t}\right), \qquad \tilde{y} \notin D, \quad t > 0.$$

Table 3 gives absolute errors of the solution of nonstationary problem in different moments of time (the parameters x and $\tilde{y}$ are as in the previous example). There we can observe linear and quadratic convergence with respect to the time stepsize h in the case of first- and second-order approximation respectively. We have also solved mixed initial boundary value problem with $f_1(x,t) = t(1-t)$ and $f_2(x,t) = tx_1^2 - t(1-t)x_2^2$. The numerical solution in different time moments are illustrated in the figures below.

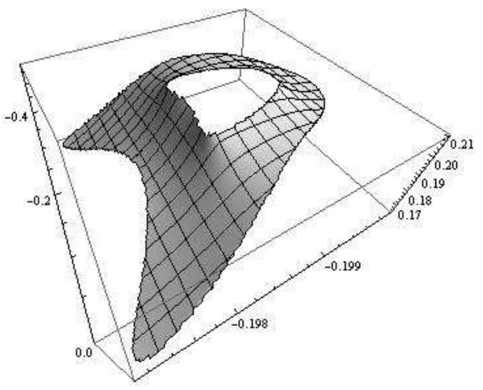
Tabl. 3. Numerical results of the nonstationary problem.

		$O(h)$			$O(h^2)$		
t	M	$N=10$	$N=20$	$N=40$	$N=10$	$N=20$	$N=40$
0.2	16	$3.6363 \cdot 10^{-3}$	$1.8222 \cdot 10^{-3}$	$7.8118 \cdot 10^{-4}$	$4.0537 \cdot 10^{-3}$	$1.4497 \cdot 10^{-4}$	$6.6200 \cdot 10^{-4}$
	32	$3.7517 \cdot 10^{-3}$	$1.9317 \cdot 10^{-3}$	$8.8993 \cdot 10^{-4}$	$3.9243 \cdot 10^{-3}$	$2.4923 \cdot 10^{-4}$	$4.7892 \cdot 10^{-5}$
	64	$3.7521 \cdot 10^{-3}$	$1.9320 \cdot 10^{-3}$	$8.9029 \cdot 10^{-4}$	$3.9239 \cdot 10^{-3}$	$2.4958 \cdot 10^{-4}$	$4.7533 \cdot 10^{-5}$
0.4	16	$7.3118 \cdot 10^{-5}$	$1.1243 \cdot 10^{-4}$	$1.5682 \cdot 10^{-4}$	$1.1377 \cdot 10^{-3}$	$2.0642 \cdot 10^{-4}$	$8.4101 \cdot 10^{-2}$
	32	$2.4404 \cdot 10^{-4}$	$5.9221 \cdot 10^{-5}$	$1.8372 \cdot 10^{-5}$	$9.6896 \cdot 10^{-4}$	$4.2692 \cdot 10^{-5}$	$3.3700 \cdot 10^{-6}$
	64	$2.4455 \cdot 10^{-4}$	$5.9736 \cdot 10^{-5}$	$1.8899 \cdot 10^{-5}$	$9.6846 \cdot 10^{-4}$	$4.2176 \cdot 10^{-5}$	$2.8315 \cdot 10^{-6}$
0.6	16	$2.4393 \cdot 10^{-4}$	$2.1034 \cdot 10^{-4}$	$1.9306 \cdot 10^{-4}$	$3.4879 \cdot 10^{-4}$	$1.0880 \cdot 10^{-4}$	—
	32	$7.9836 \cdot 10^{-5}$	$4.3238 \cdot 10^{-5}$	$2.2137 \cdot 10^{-5}$	$1.8731 \cdot 10^{-4}$	$6.0893 \cdot 10^{-6}$	$1.5064 \cdot 10^{-6}$
	64	$7.9360 \cdot 10^{-5}$	$4.2751 \cdot 10^{-5}$	$2.1639 \cdot 10^{-5}$	$1.8684 \cdot 10^{-4}$	$5.6016 \cdot 10^{-6}$	$7.6298 \cdot 10^{-7}$
0.8	16	$2.1890 \cdot 10^{-4}$	$1.8603 \cdot 10^{-4}$	$1.7209 \cdot 10^{-4}$	$8.9494 \cdot 10^{-5}$	$1.6334 \cdot 10^{-4}$	—
	32	$7.0743 \cdot 10^{-5}$	$3.4614 \cdot 10^{-5}$	$1.7267 \cdot 10^{-5}$	$5.6398 \cdot 10^{-5}$	$2.1477 \cdot 10^{-6}$	$1.1390 \cdot 10^{-6}$
	64	$7.0319 \cdot 10^{-5}$	$3.4180 \cdot 10^{-5}$	$1.6822 \cdot 10^{-5}$	$5.6815 \cdot 10^{-5}$	$2.5832 \cdot 10^{-6}$	$4.4396 \cdot 10^{-7}$
1.0	16	$1.8085 \cdot 10^{-4}$	$1.5914 \cdot 10^{-4}$	$1.5038 \cdot 10^{-4}$	$1.5191 \cdot 10^{-5}$	$1.0136 \cdot 10^{-3}$	—
	32	$4.8266 \cdot 10^{-5}$	$2.3492 \cdot 10^{-5}$	$1.1746 \cdot 10^{-5}$	$1.1533 \cdot 10^{-4}$	$2.7563 \cdot 10^{-6}$	$1.2450 \cdot 10^{-4}$
	64	$4.7890 \cdot 10^{-5}$	$2.3106 \cdot 10^{-5}$	$1.1351 \cdot 10^{-5}$	$1.1570 \cdot 10^{-4}$	$3.1433 \cdot 10^{-6}$	$2.5307 \cdot 10^{-7}$

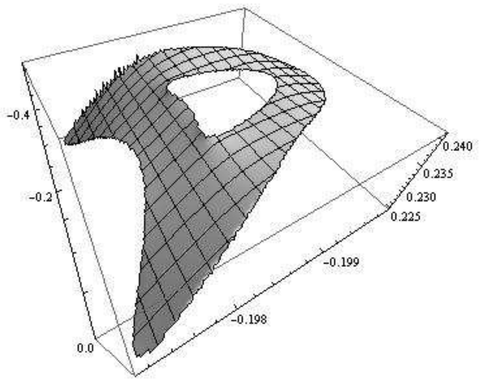
$t = 0.1$



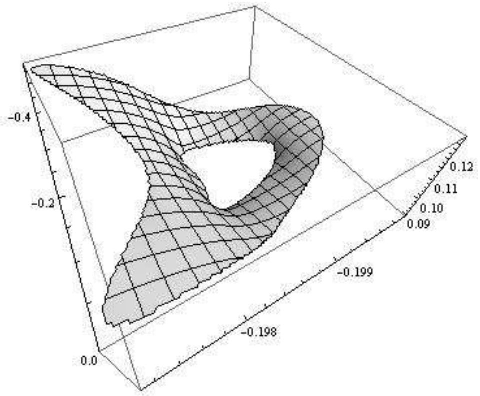
$t = 0.3$



$t = 0.6$



$t = 0.9$



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