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CONSIDERATION OF BOUNDARY EFFECT FOR WEIGHT UNIFORM ACCURACY ESTIMATES OF MESH METHOD FOR POISSON EQUATION

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Abstract. Poisson equation in polyhedral domain $\Omega \subset \mathrm{R}^n$, $n = 2, 3$ with boundary Γ, when Dirichlet conditions are given on all faces or on all but one where Neimann conditions are given, is considered.

Traditional difference schemes with semi-constant steps along axes precisely approximate Dirichlet conditions hence it is expected that their accuracy order increases approaching to corresponding part of boundary γ. This paper is dedicated to quantitative investigation of this boundary effect. It is also shown that analogous boundary effect in the mesh knots takes place also for finite-element method (super convergence). The shorten version of this paper is published in [9].

1. Introduction

Let $\Omega \subset \mathrm{R}^2$ be bounded polygonal convex domain with boundary Γ. Consider Dirichlet problem for Poisson equation

$$\begin{aligned} -\Delta u(x) &= f(x),\ x \in \Omega, \\ u(x) &= u_0(x),\ x \in \Gamma, \end{aligned} \tag{1.1}$$

where $x = (x_1, x_2)$, $f(x), u_0(x)$ are given functions. Suppose, that the domain Ω may be covered by the mesh $\hat{\omega}$ with semi-constant step by each direction along axes in such a way that if a polygon side crosses a mesh cell then it necessarily goes through it's two knots. Then domain γ of the mesh polygon $\hat{\omega}$ totally coincides with the boundary Γ (boundary compliance).

Approximate problem (1.1) with the following difference scheme

$$\begin{aligned} -\Delta_h y(x) &= -y(x)_{\bar{x}_1\hat{x}_1} - y(x)_{\bar{x}_2\hat{x}_2} = f(x),\ x \in \hat{\omega}, \\ y(x) &= u_0(x),\ x \in \gamma, \end{aligned} \tag{1.2}$$

where $y(x)_{\bar{x}_1\hat{x}_1} = \frac{2}{h_1^+ + h_1^-}\left(\frac{y(x_1+h_1^+,x_2)-y(x_1,x_2)}{h_1^+} - \frac{y(x_1,x_2)-y(x_1-h_1^-,x_2)}{h_1^-}\right)$ is an ordinary three point approximation of the second derivative on x_1 on nonuniform mesh, $y(x)_{\bar{x}_2\hat{x}_2}$ is determined analogously. Let us write difference scheme for the error function

$$z(x) = y(x) - u(x).$$

We obtain

$$\begin{aligned} -\Delta_h z(x) &= \psi(x),\ x \in \hat{\omega}, \quad z(x) = 0,\ x \in \gamma, \\ \psi(x) &= \Delta_h u(x) - \Delta u(x), \end{aligned} \tag{1.3}$$

[†] *Key words.* Difference scheme, accuracy estimates, boundary problem, Poisson equation, approximation.

where $\psi(x)$ is approximation error, which has $O(|h|^2)$-th order, if the point $x \in \hat{\omega}$ is center of regular template. (i.e. $h_1^+ = h_1^-, h_2^+ = h_2^-$) i $u(x) \in C^4(\bar{\Omega})$ and it has $O(|h|)$-th order , if template is nonregular or $u(x) \in C^3(\bar{\Omega})$. Here $|h| = \max\{h_1^+, h_1^-, h_2^+, h_2^-\}$.

Difference operator of the scheme (1.2) fulfills maximum principle and hence well known estimate (see for. ex. [1]) is valid here

$$\|z\|_\infty \leq \ C\left\{ \begin{array}{ll} |h|^2, & u(x) \in C^4(\bar{\Omega}) \\ |h| \ , & u(x) \in C^3(\bar{\Omega}) \end{array} \right.$$

where constant C depends only on corresponding norm of $u(x)$ and does not depend on $|h|$. But adduced estimate has one defect: it does not take into account the fact that the error $z(x)$ on the mesh boundary γ equals zero. Hence, one expects, that when point $x \in \hat{\omega}$ approaches to domain γ accuracy order of the difference scheme increases. This fact was noticed in [2], where Poisson equation in parallelepiped was considered. On the one side Neumann condition was given and on the other Dirichlet one. One proved that when point $x \in \hat{\omega}$ approaches to the boundary opposite to the one with Neumann conditions, accuracy order of the corresponding difference scheme in uniform metrics increases by one order. More precisely this result will be given in section 4.

In the paper [3] for more general elliptic type equation of divergent form the following accuracy estimate

$$\max_{x\in\bar{\omega}} \left|\rho^{-1/2}(x)\, z(x)\right| \leq Mh^2 \, \|u\|_{W_4^2(\Omega)}, \quad x \in \bar{\omega}$$

was obtained, where

$$\rho(x) = \rho(x_1, x_2) = min\left[x_1x_2, x_1(1-x_2), (1-x_1)x_2, (1-x_1)(1-x_2)\right].$$

Similar estimate was obtained in [4] for quasilinear elliptic type equations with main non-linear part of non-divergent form.

Current paper is devoted to strengthening of the results mentioned above. In the s.2 for the case when Ω is unit square and $u(x) \in C^4\left(\bar{\Omega}\right)$ it is shown that approximation order 2 of the difference scheme (1.3) approaching to the square side increases by one and approaching to square vertexes by two with respect to $\ln 1$. In s.3 the case when domain Ω is rectangular trapezoid is considered. It is proved that accuracy order of corresponding difference scheme with semi-constant step along abscissa when $u(x) \in C^4\left(\bar{\Omega}\right)$ increases by one approaching to vertical and horizontal sides of trapezoid and increases by two (with respect to $\ln\frac{1}{|h|}$) approaching to vertexes of the angles less then $\frac{\pi}{2}$.

It is shown that results of the s.2 and 3 are valid both for finite element method. Hereby estimates from [8] are improved. Section 4 is dedicated to strengthening of the results from [2] for the case when Ω is unit cube.

2. Unit square domain case

Consider the case when the domain Ω is unit square, $f(x_1, x_2) \equiv 1$ and the mesh ω is chosen to be uniform with equal steps along the axes Ox_1, Ox_2, $h_1 = h = 1/N$. Then difference scheme (1.2) accuracy is characterized by the estimate

$$|z(x)| \leq Ch^2 \, |u|_{W^4_\infty(\Omega)}\, v(x),$$

where function $v(x)$ is solution of the problem

$$-\Delta_h v(x) = 1, \ x \in \omega, \quad v(x) = 0, \ x \in \gamma$$

which can be obtained in the analytical form:

$$v(x_{1,s},x_{2,t}) = \\ = \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\frac{(1-(-1)^i)(1-(-1)^j)}{2N^4(1-\cos(\frac{i\pi}{N}))(1-\cos(\frac{j\pi}{N}))}\frac{\sin(\frac{i\pi}{N})\sin(\frac{j\pi}{N})\sin(\frac{si\pi}{N})\sin(\frac{tj\pi}{N})}{(2-\cos(\frac{i\pi}{N})-\cos(\frac{j\pi}{N}))}. \tag{2.1}$$

Note that function $2v(x_{1,s},x_{2,t})$ is finite difference approximation of the stress function during torsion of the square profile shank. Norm notification used is this paper is same as in [7].

Estimate the function (2.1) at point $(x_{1,1},x_{2,1})$, which enables to define the behavior of the error $z(x_1,x_2)$ when the point $(x_1,x_2)\in\omega$ approaches to the vertex of the square. We have

$$v(x_{1,1},x_{2,1}) = \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\frac{4}{16N^4\sin^2(\frac{i\pi}{2N})\sin^2(\frac{j\pi}{2N})}\frac{\sin^2(\frac{i\pi}{N})\sin^2(\frac{j\pi}{N})}{(\sin^2(\frac{i\pi}{2N})+\sin^2(\frac{j\pi}{2N}))} = \\ = \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\frac{4}{N^4}\frac{\cos^2(\frac{i\pi}{2N})\cos^2(\frac{j\pi}{2N})}{(\sin^2(\frac{i\pi}{2N})+\sin^2(\frac{j\pi}{2N}))} \le \frac{4}{N^2}\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\frac{1}{i^2+j^2} \le \frac{4}{N^2}\ln(N). \tag{2.2}$$

Hence we deduce that approaching to the vertex of the square error order is improving almost by two degree (accurate within logarithm).

Not diminishing generality, we suppose that N is even number. Estimate the function (2.1) in the point $(x_{1,1},x_{2,N/2})$, which enables to define the behavior of the error $z(x_1,x_2)$ when the point $(x_1,x_2)\in\omega$ approaches to middle of the square side. We have

$$v(x_{1,1},x_{2,N/2}) = \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\frac{4}{16N^4\sin^2(\frac{i\pi}{2N})\sin^2(\frac{j\pi}{2N})}\frac{\sin^2(\frac{i\pi}{N})\sin(\frac{j\pi}{N})\sin(\frac{j\pi}{2})}{(\sin^2(\frac{i\pi}{2N})+\sin^2(\frac{j\pi}{2N}))} = \\ = \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\frac{2}{N^4}\frac{\cos^2(\frac{i\pi}{2N})\cos(\frac{j\pi}{2N})\sin(\frac{j\pi}{2})}{\sin(\frac{j\pi}{2N})(\sin^2(\frac{i\pi}{2N})+\sin^2(\frac{j\pi}{2N}))} \le \frac{2}{N}\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\frac{1}{j(i^2+j^2)} \le \\ \le \frac{1}{N}\sum_{j=1}^{\infty}\frac{\pi\coth(\pi j)j-1}{j^3} \le \frac{1}{N}(\frac{\pi^3}{6}\coth(\pi)-\varsigma(3)) \le \frac{1}{N}1.9925, \tag{2.3}$$

where $\varsigma(n)$ is Riemann zeta function. Hence we conclude that approaching to the middle of the square side error order improves by one degree.

Estimate function (2.1) in the point $(x_{1,N/2},x_{2,N/2})$, which enables to define the behavior of the error $z(x_1,x_2)$ when the point $(x_1,x_2)\in\omega$ approaches to the middle of the square. We have

$$v(x_{1,N/2},x_{2,N/2}) = \\ = \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\frac{4}{16N^4\sin^2(\frac{i\pi}{2N})\sin^2(\frac{j\pi}{2N})}\frac{\sin(\frac{i\pi}{N})\sin(\frac{j\pi}{N})\sin(\frac{i\pi}{2})\sin(\frac{j\pi}{2})}{(\sin^2(\frac{i\pi}{2N})+\sin^2(\frac{j\pi}{2N}))} = \\ = \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\frac{1}{N^4}\frac{\cos(\frac{i\pi}{2N})\cos(\frac{j\pi}{2N})\sin(\frac{i\pi}{2})\sin(\frac{j\pi}{2})}{\sin(\frac{i\pi}{2N})\sin(\frac{j\pi}{2N})(\sin^2(\frac{i\pi}{2N})+\sin^2(\frac{j\pi}{2N}))} \le \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\frac{1}{ij(i^2+j^2)} = \\ = -\frac{1}{2}\sum_{j=1}^{\infty}\frac{1}{j^4}(-2\gamma j-2\Psi(jI)j+\pi\coth(\pi j)jI+I) = \\ = \sum_{j=1}^{\infty}\frac{1}{j^3}(\gamma+Re(\Psi(jI))) \le 1.73007... < 2, \tag{2.4}$$

where $\Psi(z)$ is digamma function (see. [5]).

From here we conclude, that approaching to the center of the square error order remains constant. Therefore the following theorem is proved

Theorem 2.1 *Let solution of the problem* (1.1) *belong to the space* $C^4(\bar{\Omega})$, *then accuracy of the difference scheme* (1.2) *is characterized by the estimate*

$$|z(x)| \le Ch^2 v(x)\, |u|_{C^{(4)}(\bar{\Omega})}\,, \tag{2.5}$$

where constant C does not depend on $h, u(x)$, and function $v(x)$ approaching ti the sides of the mesh square ω behaves itself as $O(h)$ and approaching to its vertex - as $O(h^2\ln(\frac{1}{h}))$.

Remark 2.2 *Obtained results allows to derive new estimates of the finite elements method which adduce to traditional difference schemes. Let us illustrate this by the example of the problem* (2.1), *when domain Ω is a square and $u_0(x) \equiv 0$. Let for each $h \in (0,1)$, T_h implies triangulation Ω as it is shown on the figure 1.1. Let $S_1^h(\Omega) = S_1^h$ imply finite-measurable space of continuous functions on $\bar{\Omega}$, confinement of which on each triangular $\tau \in T_h$ is polynomial of the first degree and $\overset{0}{S}{}_1^h(\Omega)$ implies subspace $S_1^h(\Omega)$, which contains functions which vanish on Γ. It is obvious that $S_1^h(\Omega) \subset W_\infty^1(\Omega)$.*

System of functions (see. [7])

$$\omega_{k,l}(x_1,x_2) = \begin{cases} 1-\frac{1}{h}(x_{1,k}-x_1)-\frac{1}{h}(x_{2,l}-x_2), & (x_1,x_2)\in\Omega^h_{k,l,1} \\ 1-\frac{1}{h}(x_{1,k}-x_1), & (x_1,x_2)\in\Omega^h_{k,l,2} \\ 1+\frac{1}{h}(x_{2,k}-x_2), & (x_1,x_2)\in\Omega^h_{k,l,3} \\ 1+\frac{1}{h}(x_{1,k}-x_1)+\frac{1}{h}(x_{2,l}-x_2), & (x_1,x_2)\in\Omega^h_{k,l,4} \\ 1+\frac{1}{h}(x_{1,k}-x_1), & (x_1,x_2)\in\Omega^h_{k,l,5} \\ 1-\frac{1}{h}(x_{2,k}-x_2), & (x_1,x_2)\in\Omega^h_{k,l,6} \end{cases} \tag{2.6}$$
$$k,l = 1,2,...,N-1$$

is a basis in $\overset{0}{S}{}_1^h(\Omega)$.

Finite-element approximation of $u(x)$ is

$$y(x) = \sum_{k,l=1}^{N-1} y(x_{k,l})\omega_{k,l}(x), \tag{2.7}$$

where $y(x_{k,l})$ is defined as solution of difference scheme

$$\begin{aligned} &-\Delta_h y(x) = -y(x)_{\bar{x}_1x_1} - y(x)_{\bar{x}_2x_2} = \varphi(x),\ x\in\omega, \\ &y(x) = 0,\ x\in\gamma, \\ &\varphi(x_{1,k},x_{2,l}) = \frac{1}{h^2}\iint_{\Omega_{k,l}} f(x)\omega_{k,l}(x)dx_1dx_2,\ P_{k,l} = (x_{1,k},x_{2,l})\in\omega. \end{aligned} \tag{2.8}$$

We use a technique from [7] for investigation of accuracy of difference scheme (2.8) and write down the integral consequence of Poisson equation. We obtain

$$\begin{aligned} &-\frac{1}{h^2}\iint_{\Omega_{k,l}} \Delta u(x)\omega_{k,l}(x)dx_1dx_2 = \varphi(x_{1,k},x_{2,l}) = \\ &= \frac{1}{h^2}\iint_{\Omega_{k,l}} f(x)\omega_{k,l}(x)dx_1dx_2. \end{aligned} \tag{2.9}$$

Now write down the difference scheme for the error

$$-\Delta_h z(x) = -z(x)_{\bar{x}_1 x_1} - z(x)_{\bar{x}_2 x_2} = \psi(x), \ x \in \omega, \qquad z(x) = 0, \ x \in \gamma, \tag{2.10}$$

where local truncation error $\psi(x)$ has the following form

$$\psi(x_{1,k}, x_{2,l}) = l_{k,l}(u(\cdot)) = \Delta_h u(x) - \frac{1}{h^2} \iint_{\Omega_{k,l}} \Delta u(x) \omega_{k,l}(x) dx_1 dx_2. \tag{2.11}$$

For the solution of the problem (2.10) the following estimate holds true

$$|z(x)| \leq \|\psi\|_\infty v(x), \tag{2.12}$$

where function $v(x)$ has the same meaning as in theorem 2.1.

Linear functional $l_{k,l}(u(\cdot))$ which is defined and bounded on the space of functions $W^2_\infty(\Omega^h_{k,l})$ and vanishes on polynomials of the third degree. Using the Bremble-Hilbert lemma (see. [7]), we obtain

$$|\psi(x_{1,k}, x_{2,l})| \leq Ch^2 \, |u|_{W^4_\infty(\Omega)} ,$$

which together with (2.12) proves the following statement

Theorem 2.3 *Let solution of the problem* (2.1) *belong to the space* $W^4_\infty(\Omega)$*, then accuracy of the finite-element method in the knots of the mesh* ω *will be characterized by the estimate*

$$|z(x)| \leq Ch^2 v(x) \, |u|_{W^4_\infty(\Omega)} , \ x \in \omega, \tag{2.13}$$

where constant C *does not depend on* h *and* $u(x)$*, and function* $v(x)$ *approaching to the sides of the mesh square* ω *behaves itself as* $O(h)$*, and approaching to its vertexes - as* $O(h^2 \ln(\frac{1}{h}))$.

3. Rectangular trapezoid domain case

Consider the domain $ABCD$, which is represented in the figure 3.2a). Its left part is a unit square $ABCE$, which we designate as Ω_1. Let its mesh step along the axis Ox_1 be h_1. And the right side of the domain is a triangle ECD which we designate as Ω_2. Let its mesh step along the axis Ox_1 be $h_2 = (a-1)/N_2$. and let the mesh step along the axis Ox_2 be h_1 . Introduce the following notations $\omega_1 = \Omega_1 \cap \hat{\omega}, \ \omega_2 = \Omega_2 \cap \hat{\omega}, \ \gamma_1 = (\bar{\Omega}_1 \cap \bar{\Omega}_2) \cap \omega$. Then for the local truncation error $\psi(x)$ з (3) we obtain

$$\psi(x) = \begin{cases} O(h_1^2), \ x \in \omega_1, \\ O(h_1^2 + h_2^2), \ x \in \omega_2, \\ O(|h| + h_1^2), \ x \in \gamma_1, \quad |h| = \max(h_1, h_2) \end{cases} \tag{3.1}$$

if $u(x) \in C^4(\bar{\Omega})$ and $\psi(x) = O(|h|), \ x \in \omega$, if $u(x) \in C^3(\bar{\Omega})$.

Technique of investigation of the difference scheme accuracy in Tchebyshev norm for Dirichlet problem for Poisson equation in the complex form domain which is based on the maximum principle is adduced in monograph [1]. Thus the unevenness of mesh is concentrated only near boundary of domain. Some other situation is examined here. An unevenness of mesh is in the middle of area (around line CE) and although the technique of receipt of a priori estimates of exactness is also based on principle of maximum, obtained a priori estimates take into account boundary effect.

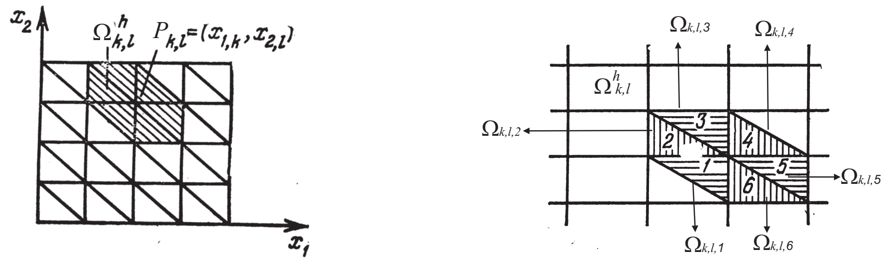
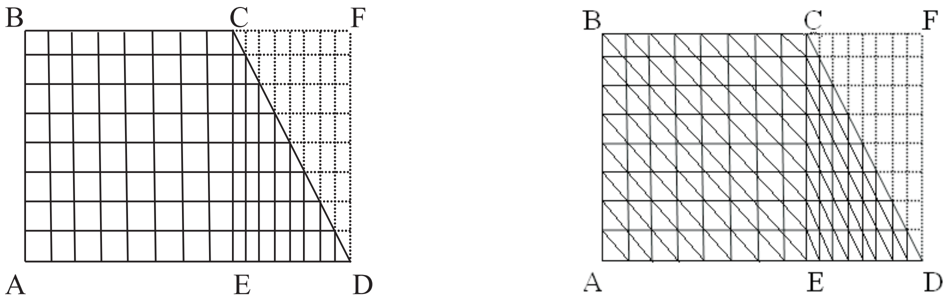


Fig. 1. Triangulation Ω



a) b)

Fig. 2. Trapezoid domain

For the case $u(x) \in C^4(\bar{\Omega})$ we present the solution of the problem (1.3) in the form

$$z(x) = z_1(x) + z_2(x),$$

where $z_i(x),\ i = 1, 2$ are solutions of the following problems

$$\hat{\Delta}_h z_i(x) = -z(x)_{\bar{x}_1 \hat{x}_1} - z(x)_{\bar{x}_2 x_2} = \psi_i(x),\ x \in \hat{\omega}, z_i(x) = 0, x \in \gamma \tag{3.2}$$

and

$$\psi_1(x) = \begin{cases} \psi(x),\ x \in \omega \backslash \gamma_1 \\ 0,\ x \in \gamma_1 \end{cases} \qquad \psi_2(x) = \begin{cases} 0,\ x \in \omega \backslash \gamma_1 \\ \psi(x),\ x \in \gamma_1 \end{cases}.$$

For the estimation of $z_2(x)$ we use the following majorant function

$$v_2(x) = \hbar \begin{cases} x_1,\ 0 \le x_1 \le 1 \\ 2 - x_1,\ 1 \le x_1 \le a \end{cases}, \qquad \hbar = \frac{h_1 + h_2}{2}.$$

It fulfills the conditions

$$\Delta_h v_2(x) = \begin{cases} -2,\ x \in \gamma_1, \\ 0,\ \ x \in \omega \backslash \gamma_1 \end{cases}$$

$$v_2(x) > 0, x \in \gamma.$$

Then the following estimate

$$z_2(x) \le \max_{x \in \gamma_1}(|\psi(x)|) v_2(x) = O(|h| + h^2)\hbar \le C\,|h|^2 \tag{3.3}$$

is valid. Further we obtain

$$\begin{aligned}
&|z_1(x)| \le \max_{x\in\hat{\omega}\backslash\gamma_1} (|\psi(x)|\,)\times \\
&\times \max(x_1(a-x_1), x_2(1-x_2), x_1(a-x_1-(a-1)x_2)\,) \le \\
&\le C\,|h|^2 \max(x_1(a-x_1), x_2(1-x_2), x_1(a-x_1-(a-1)x_2)\,).
\end{aligned} \tag{3.4}$$

Thus, taking into account definitions and inequalities (3.3),(3.4) we obtain

$$\max_{x\in\gamma_1}(|z(x)|) \le C\,|h|^2\,. \tag{3.5}$$

To estimate the behavior of $z(x)$ more precise, when x approaches to vertexes A, B, D, present $z(x)$ through the solutions of the two following problems

$$\begin{aligned}
&\Delta_h w_1(x) = -\psi(x),\ x\in\hat{\omega}_1, \quad w_1(x)=0, x\in(\bar{\omega}_1\backslash\omega_1)\backslash\gamma_1, \quad w_1(x)=z(x), x\in\ \gamma_1,\\
&\Delta_h w_2(x) = -\psi(x),\ x\in\hat{\omega}_2, \quad w_2(x)=0, x\in(\bar{\omega}_2\backslash\omega_2)\backslash\gamma_1, \quad w_2(x)=z(x), x\in\ \gamma_1
\end{aligned}$$

namely

$$z(x) = \begin{cases} w_1(x), x\in\omega_1,\\ w_2(x), x\in\omega_2.\end{cases}$$

For estimation of functions $w_1(x), w_2(x)$ consider the following auxiliary problems

$$\begin{aligned}
&\Delta_h w_{1,1}(x) = -2,\ x\in\omega_1, \quad w_1(x)=0, x\in\bar{\omega}_1\backslash\omega_1,\\
&\Delta_h w_{1,2}(x) = 0,\ x\in\hat{\omega}_1, \quad w_{1,2}(x)=0, x\in(\bar{\omega}_1\backslash\omega_1)\backslash\gamma_1,\ w_{1,2}(x)=1, x\in\gamma_1,\\
&\Delta_h w_{2,1}(x) = -2,\ x\in\omega_2, \quad w_{2,1}(x)=0, x\in\bar{\omega}_2\backslash\omega_2,\\
&\Delta_h w_{2,2}(x) = 0,\ x\in\omega_2, \quad w_{2,2}(x)=0, x\in(\bar{\omega}_2\backslash\omega_2)\backslash\gamma_1, \quad w_{2,2}(x)=1, x\in\ \gamma_1.
\end{aligned}$$

It is not difficult to see that taking into account (3.5) the following estimate

$$|z(x)| \le C\,|h|^2 \begin{cases} w_{1,1}(x)+w_{1,2}(x), x\in\omega_1\\ w_{2,1}(x)+w_{2,2}(x), x\in\omega_2\end{cases} \tag{3.6}$$

is valid. It is obvious that $w_{1,1}(x)\equiv v(x)$, and function of discrete arguments $w_{1,2}(x)$ has the following form

$$w_{1,2}(x_{1,i}, x_{2,k}) = \frac{1}{N}\sum_{n=1}^{N-1}\frac{1-(-1)^n}{1-\cos(\frac{n\pi}{N})}\sin(\frac{n\pi}{N})\sin(\frac{nk\pi}{N})\frac{U_{i-1}(2-\cos(\frac{n\pi}{N}))}{U_{N-1}(2-\cos(\frac{n\pi}{N}))}. \tag{3.7}$$

The following inequalities

$$\begin{aligned}
&w_{1,2}(x_{1,1}, x_{2,1}) = w_{1,2}(x_{1,1}, x_{2,N-1}) \le \frac{2\pi^2}{N^2},\ w_{1,1}(x_{1,1}, x_{2,1}) = \\
&= w_{1,1}(x_{1,1}, x_{2,N-1}) \le \frac{\ln(N)}{N^2}\,,\\
&w_{1,2}(x_{1,1}, x_{2,N/2}) \le \frac{\pi^2}{N},\ w_{1,1}(x_{1,1}, x_{2,N/2}) = w_{1,1}(x_{1,N-1}, x_{2,N/2}) \le \frac{2}{N}
\end{aligned} \tag{3.8}$$

are valid. Among the inequalities mentioned above we prove only that which concern to the function $w_{1,2}(x)$, because inequalities, that concern to the function $w_{1,1}(x)$ were proved earlier. We obtain

$$\begin{aligned} N^2 w_{1,2}(x_{1,1},x_{2,1}) &= N\sum_{n=1}^{N-1}\frac{1-(-1)^n}{1-\cos(\frac{n\pi}{N})}\sin^2(\frac{n\pi}{N})\frac{1}{U_{N-1}(2-\cos(\frac{n\pi}{N}))} = \\ &= 2N\sum_{n=1}^{N-1}(1-(-1)^n)\cos^2(\frac{n\pi}{2N})\frac{1}{U_{N-1}(2-\cos(\frac{n\pi}{N}))} \le \\ &\le 4N\sum_{n=1}^{N-1}\frac{1}{U_{N-1}(2-\cos(\frac{n\pi}{N}))} \le 4N\sum_{n=1}^{N-1}\frac{1}{U_{N-1}(1+2(\frac{n}{N})^2)} \le \\ &\le 4N\sum_{n=1}^{N-1}\frac{\sinh(\frac{n\ln(3)}{N})}{\sinh(n\ln(3))}. \end{aligned} \tag{3.9}$$

Here we used the fact that function $U_{N-1}(x)$ is monotonously increasing with respect to x on the interval $[1,\infty)$ and inequality $(1+x)^{1/x}<3$. The following inequality

$$\frac{N\sinh(\frac{j\ln(3)}{N})}{\sinh(j\ln(3))}-\frac{3}{j^2}\le\frac{j\ln(3)\sinh(\ln(3))}{\sinh(j\ln(3))}-\frac{3}{j^2}<0$$

is valid. At $j=1,2$ it can be testified by direct calculations, and when $j\ge 3$ by common extremum investigation of the function $g(j)=j^3\ln(3)\sinh(\ln(3))-3\sinh(j\ln(3))$. Turning back to (0.12_1) taking into account previous inequality we obtain

$$N^2 w_{1,2}(x_{1,1},x_{2,1})\le 12\sum_{n=1}^{N-1}\frac{1}{j^2}\le 2\pi^2,$$

which leads to the first estimate (3.8). Further we obtain

$$\begin{aligned} N w_{1,2}(x_{1,1},x_{2,N/2}) &= \sum_{n=1}^{N-1}\frac{1-(-1)^n}{1-\cos(\frac{n\pi}{N})}\sin(\frac{n\pi}{N})\sin(\frac{n\pi}{2})\frac{1}{U_{N-1}(2-\cos(\frac{n\pi}{N}))} \le \\ &\le 2\sum_{n=1}^{N-1}\frac{1}{\sin(\frac{n\pi}{2N})}\cos(\frac{n\pi}{2N})\frac{1}{U_{N-1}(2-\cos(\frac{n\pi}{N}))} \le 2\sum_{n=1}^{N-1}\frac{1}{\frac{n}{N}}\frac{1}{U_{N-1}(2-\cos(\frac{n\pi}{N}))} \le \\ &\le 2N\sum_{n=1}^{N-1}\frac{1}{U_{N-1}(2-\cos(\frac{n\pi}{N}))} \le \pi^2, \end{aligned}$$

where previous computations were used. This proves justice of the third estimate (3.8). Following conclusions result from the estimates (3.6), (3.8). Approaching to the knots of the mesh boundary γ which are situated near middle of the intervals AB, BC, AE accuracy order of difference scheme (1.2) increases to three and approaching to vertexes A,B it increases almost to four (with respect to logarithm).

At last one must specify the behavior of the error approaching to vertex D. According to (3.6) one must obtain estimates for $w_{2,1}(x), w_{2,2}(x)$ for this. We have

$$w_{2,1}(x)\le\min\left\{(a-x_1)(x_1-1),\ x_2(a-x_1-(a-1)x_2)\right\}.$$

The function $w_{2,2}(x)$ we estimate through solution of the problem

$$\Delta_h\bar w_{2,2}(x)=0,\ x\in\omega_2^{\rightarrow},\qquad w_{2,2}(x)=0, x\in(\bar\omega_2^{\rightarrow}\backslash\omega_2^{\rightarrow})\backslash\gamma_1,\quad \bar w_{2,2}(x)=1, x\in\gamma_1,$$

where $\omega_2^{\rightarrow}$ is mesh domain, which covers the rectangle $CFDE$. The last one has following form

$$\bar{w}_{2,2}(1+ih_2,x_{2,k})=\frac{1}{N}\sum_{n=1}^{N-1}\frac{1-(-1)^n}{1-\cos(\frac{n\pi}{N})}\sin(\frac{n\pi}{N})\sin(\frac{nk\pi}{N})\frac{U_{N_1-i-1}(2-\cos(\frac{n\pi}{N}))}{U_{N_1-1}(2-\cos(\frac{n\pi}{N}))}$$

and is estimated similar to function $w_{1,2}(x)$

Thus we proved

Theorem 3.1 *Let solution of the problem* (1.1) *belong to the space* $C^4(\bar{\Omega})$, *then accuracy of the difference scheme* (1.2) *is characterized by the estimate*

$$|z(x)|\leq C\,|h|^2\,\rho(x),$$

where function $\rho(x)$ *approaching to the side* AB *of the quadrangle* $ABCD$ *and to the middle of the intervals* BC, AE, ED *behaves itself as* $O(|h|)$ *and approaching to vertexes* A, B, D *- as* $O(|h|^2\ln(\frac{1}{|h|}))$.

A little easier is to prove

Theorem 3.2 *Let solution of the problem* (1.1) *belongs to space* $C^3(\bar{\Omega})$, *then accuracy of the difference scheme* (1.2) *is characterized by the estimate*

$$|z(x)|\leq C\,|h|\left\{\begin{array}{l}\min\left\{x_1(a-x_1),x_2(1-x_2),w_{1,2}(x)\right\},x\in\omega_1\\ \min\left\{x_2(1-x_2),x_1(a-x_1-(a-1)x_2)\right\},x\in\omega_2\end{array}\right.$$

i.e. accuracy of the difference scheme (1.2) *when the point* $x\in\hat{\omega}$ *approaches to sides* AB, BC, CD, AD *behaves itself as* $O(|h|^2)$ *and approaching to the vertex* D *- as* $O(|h|^3)$ *and approaching to vertexes* A, B *- as* $O(|h|^3\ln(\frac{1}{|h|}))$.

Theorems 3.1, 3.2 in common way can be generalized on the case when polygon Ω is such that it can be covered with semi-even mesh and its sides goes through either sides or diagonals of the mesh cells.

Remark 3.3 *We show that analogue of the theorems 3.1, 3.2 are valid for finite elements method also. Let* T_h *be triangulation* Ω *as it is shown in the fig.3.2b). Let, same as previously,* $S_1^h(\Omega)=S_1^h$ *be finite-dimensional space of continuous on* $\bar{\Omega}$ *functions which narrowing on every triangle* $\tau\in T_h$ *is polynomial of the first degree and* $\overset{0}{S}{}_1^h(\Omega)$ *defines subspace* $S_1^h(\Omega)$ *which contains the functions which vanishes on* Γ. *Base in* $\overset{0}{S}{}_1^h(\Omega)$ *are functions* $\omega_{k,l}(x_1,x_2)$ *which for* $(x_1,x_2)\in\omega_1$ *are defined by formula* (2.6) *and for* $(x_1,x_2)\in\gamma_1\cup\omega_2$ *by formula*

$$\omega_{k,l}(x_1,x_2)=\left\{\begin{array}{ll}1-\frac{1}{h_1}(x_{1,k}-x_1)-\frac{1}{h_1}(x_{2,l}-x_2), & (x_1,x_2)\in\Omega_{k,l,1}^h\\ 1-\frac{1}{h_1}(x_{1,k}-x_1), & (x_1,x_2)\in\Omega_{k,l,2}^h\\ 1+\frac{1}{h_1}(x_{2,k}-x_2), & (x_1,x_2)\in\Omega_{k,l,3}^h\\ 1+\frac{1}{h_2}(x_{1,k}-x_1)+\frac{1}{h_1}(x_{2,l}-x_2), & (x_1,x_2)\in\Omega_{k,l,4}^h\\ 1+\frac{1}{h_2}(x_{1,k}-x_1), & (x_1,x_2)\in\Omega_{k,l,5}^h\\ 1-\frac{1}{h_1}(x_{2,k}-x_2), & (x_1,x_2)\in\Omega_{k,l,6}^h\end{array}\right. \qquad (3.10)$$

$$k,l=1,2,...,N-1,\ (x_1,x_2)\in\gamma_1\,,$$

$$\omega_{k,l}(x_1,x_2)=\begin{cases} 1-\frac{1}{h_2}(x_{1,k}-x_1)-\frac{1}{h_1}(x_{2,l}-x_2), & (x_1,x_2)\in\Omega^h_{k,l,1}\\ 1-\frac{1}{h_2}(x_{1,k}-x_1), & (x_1,x_2)\in\Omega^h_{k,l,2}\\ 1+\frac{1}{h_1}(x_{2,k}-x_2), & (x_1,x_2)\in\Omega^h_{k,l,3}\\ 1+\frac{1}{h_2}(x_{1,k}-x_1)+\frac{1}{h_1}(x_{2,l}-x_2), & (x_1,x_2)\in\Omega^h_{k,l,4}\\ 1+\frac{1}{h_2}(x_{1,k}-x_1), & (x_1,x_2)\in\Omega^h_{k,l,5}\\ 1-\frac{1}{h_1}(x_{2,k}-x_2), & (x_1,x_2)\in\Omega^h_{k,l,6}\end{cases} \tag{3.11}$$
$$k,l=1,2,...,N-1,\ (x_1,x_2)\in\omega_2\,.$$

Finite-element approximation of $u(x)$ is

$$y(x)=\sum_{(x_{1,k},x_{2,l})\in\hat{\omega}} y(x_{1,k},x_{2,l})\omega_{k,l}(x), \tag{3.12}$$

where $y(x_{k,l})=y(x_{1,k},x_{2,l})$ is defined as solution of the difference scheme

$$\begin{aligned} &-\hat{\Delta}_h y(x)=-y(x)_{\bar{x}_1\hat{x}_1}-y(x)_{\bar{x}_2x_2}=\varphi(x),\ x\in\hat{\omega},\\ &y(x)=0,\ x\in\gamma,\\ &\varphi(x_{1,k},x_{2,l})=\frac{1}{h_1\hbar_1}\iint_{\Omega_{k,l}} f(x)\omega_{k,l}(x)dx_1dx_2,\ (x_{1,k},x_{2,l})\in\hat{\omega}.\end{aligned} \tag{3.13}$$

Here expression for $y(x)_{\bar{x}_1\hat{x}_1}$ i $\hbar_1$ are resulted in s. 1.

For investigation of accuracy of the difference scheme (3.13) we again use technique from [7] and write down integral conclusion of Poisson equation. We obtain

$$\begin{aligned} &-\frac{1}{h_1\hbar_1}\iint_{\Omega_{k,l}}\Delta u(x)\omega_{k,l}(x)dx_1dx_2=\\ &=\varphi(x_{1,k},x_{2,l})=h_1\hbar_1\iint_{\Omega_{k,l}} f(x)\omega_{k,l}(x)dx_1dx_2.\end{aligned} \tag{3.14}$$

Write down difference scheme for the error

$$\begin{aligned} &-\hat{\Delta}_h z(x)=-z(x)_{\bar{x}_1\hat{x}_1}-z(x)_{\bar{x}_2x_2}=\psi(x),\ x\in\hat{\omega},\\ &z(x)=0,\ x\in\gamma,\end{aligned} \tag{3.15}$$

where $\psi(x)$ is local truncation error of the following form

$$\psi(x_{1,k},x_{2,l})=l_{k,l}(x_{k,l},u(\cdot))=\hat{\Delta}_h u(x)-\frac{1}{h_1\hbar_1}\iint_{\Omega_{k,l}}\Delta u(x)\omega_{k,l}(x)dx_1dx_2. \tag{3.16}$$

Linear functional $l_{k,l}(x_{k,l},u(\cdot))$ for $x_{k,l}\in\omega\backslash\gamma_1$ is defined and bounded in the space of functions $W^4_\infty(\Omega^h_{k,l})$ and vanishes on polynomials of the third degree, and for $x_{k,l}\in\gamma_1$ it is defined and bounded in the space of functions $W^3_\infty(\Omega^h_{k,l})$ and vanishes on polynomials of the second degree. Using Bremble-Hilbert lemma (see. [7]) one obtains

$$|\psi(x_{1,k},x_{2,l})|\le C\begin{cases}|h|^2\,|u|_{W^4_\infty(\Omega)}\,,(x_{1,k},x_{2,l})\in\hat{\omega}\backslash\gamma_1\\ |h|\ \ |u|_{W^3_\infty(\Omega)}\,,(x_{1,k},x_{2,l})\in\gamma_1\end{cases}. \tag{3.17}$$

Further, repeating reasonings which were used for proving of the theorem 3.1 we convince in correctness of the statement

Theorem 3.4 *Let solution of the problem* (1.1) *belong to the space* $W^4_\infty(\Omega)$, *then accuracy of the finite-element method in the knots of the mesh* $\hat{\omega}$ *is characterized by the estimate*

$$|z(x)| \le C\,|h|^2\,\rho(x)\,\|u\|_{W^4_\infty(\Omega)}\,,\ x \in \hat{\omega},$$

where constant C *does not depend on* h_1, h_2 *and* $u(x)$, *besides function* $\rho(x)$ *has the same meaning as in theorem 3.1.*

Analogously to the theorem 3.2 one proves

Theorem 3.5 *Let solution of the problem* (1.1) *belong to the space* $W^3_\infty(\Omega)$, *then accuracy of the finite-element method in the knots of the mesh* $\hat{\omega}$ *is characterized by the estimate*

$$|z(x)| \le C\,|h|\,\|u\|_{W^3_\infty(\Omega)} \left\{ \begin{array}{l} \min\left\{x_1(a-x_1), x_2(1-x_2), w_{1,2}(x)\right\}, x \in \omega_1 \\ \min\left\{x_2(1-x_2), x_1(a-x_1-(a-1)x_2)\right\}, x \in \omega_2 \end{array} \right. ,$$

where constant C *does not depend on* h_1, h_2, $u(x)$, *i.e. when point* $x \in \hat{\omega}$ *approaches to the sides* AB, BC, CD, AD *accuracy of the finite-element method in the knots of the mesh behaves itself as* $O(|h|^2)$ *and when this point approaches to vertex* D *it behaves as* $O(|h|^3 \ln(\frac{1}{|h|}))$.

4. Unit cube domain case

Consider the case when the domain Ω is unit cube, $f(x_1, x_2, x_3) \equiv 1$ and mesh ω is chosen to be even with equal steps along the axes Ox_1, Ox_2, Ox_3, $h_1 = h_2 = h_3 = 1/N$. Then fir error of the difference scheme (1.2) при $u(x) \in C^{(4)}(\bar{\Omega})$ the following estimate holds true

$$|z(x)| \le C\,|u|_{C^{(4)}(\bar{\Omega})}\, v(x),$$

where $v(x)$ is solution of the problem

$$-\Delta_h v(x) = 1\ , x \in \omega,\ \ v(x) = 0, x \in \gamma,$$

which can be written in analytical form. It can be presented as follows

$$v(x_{1,s}, x_{2,t}, x_{3,l}) = \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\sum_{k=1}^{N=1} \frac{(1-(-1)^i)(1-(-1)^j)(1-(-1))^k}{2N^5(1-\cos(\frac{i\pi}{N}))(1-\cos(\frac{j\pi}{N}))(1-\cos(\frac{k\pi}{N}))} \times$$

$$\times \frac{\sin(\frac{i\pi}{N})\sin(\frac{j\pi}{N})\sin(\frac{k\pi}{N})\sin(\frac{si\pi}{N})\sin(\frac{tj\pi}{N})\sin(\frac{lk\pi}{N})}{(3-\cos(\frac{i\pi}{N})-\cos(\frac{j\pi}{N})-\cos(\frac{k\pi}{N}))}. \qquad (4.1)$$

Estimate behavior of the function $v(x)$ when point $x \in \omega$ approaches to the vertexes of cube. We obtain

$$v(x_{1,1}, x_{2,1}, x_{3,1}) = \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\sum_{k=1}^{N-1} \frac{(1-(-1)^i)(1-(-1)^j)(1-(-1))^k}{2N^5(1-\cos(\frac{i\pi}{N}))(1-\cos(\frac{j\pi}{N}))(1-\cos(\frac{k\pi}{N}))} \times$$

$$\times \frac{\sin^2(\frac{i\pi}{N})\sin^2(\frac{j\pi}{N})\sin^2(\frac{k\pi}{N})}{(3-\cos(\frac{i\pi}{N})-\cos(\frac{j\pi}{N})-\cos(\frac{k\pi}{N}))} =$$

$$= \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\sum_{k=1}^{N-1} \frac{2(1-(-1)^i)(1-(-1)^j)(1-(-1))^k}{N^5(\sin^2(\frac{i\pi}{2N})+\sin^2(\frac{j\pi}{2N})+\sin^2(\frac{k\pi}{2N}))} \le$$

$$\le \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\sum_{k=1}^{N-1}\frac{16}{N^3(i^2+j^2+k^2)} \le$$

$$\le \frac{16}{N^3}\,(\sum_{k=1}^{N-1}\frac{1}{2+k^2}+2\int_1^{N-1}\frac{dj}{2+j^2}+3\int_1^{N-1}\int_1^{N-1}\frac{dk\,dj}{1+j^2+k^2}+$$

$$+\int_1^{N-1}\int_1^{N-1}\int_1^{N-1}\frac{dk\,dj}{1+i^2+j^2+k^2}\,)\le \frac{16}{N^3}\,(\,\frac{\pi^2}{6}+2(1-\frac{1}{N-1})+3\frac{\pi}{4}\ln(N-1)+$$

$$+\frac{\pi}{4}(N-1)\ln(N-1)\,)\le \frac{16}{N^3}\,(\,\frac{\pi^2}{6}+2+\frac{\pi}{4}(N+2)\ln(N-1)\,). \qquad (4.2)$$

Thus accuracy of the difference scheme (1.2) increases by two orders, with respect to logarithm, when the point $x\in\omega$ approaches to the vertexes of cube.

Estimate the behavior of the function $v(x)$ when point $x\in\omega$ approaches to the sides of cube. We have

$$v(x_{1,1},x_{2,1},x_{3,N/2})=\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\sum_{k=1}^{N-1}\frac{(1-(-1)^i)(1-(-1)^j)(1-(-1))^k}{2N^5(1-\cos(\frac{i\pi}{N}))(1-\cos(\frac{j\pi}{N}))(1-\cos(\frac{k\pi}{N}))}\times$$

$$\times\frac{\sin^2(\frac{i\pi}{N})\sin^2(\frac{j\pi}{N})\sin(\frac{k\pi}{N})\sin(\frac{k\pi}{2})}{(3-\cos(\frac{i\pi}{N})-\cos(\frac{j\pi}{N})-\cos(\frac{k\pi}{N}))}=$$

$$=\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\sum_{k=1}^{N-1}\frac{(1-(-1)^i)(1-(-1)^j)(1-(-1))^k}{N^5\sin(\frac{k\pi}{2N})(\sin^2(\frac{i\pi}{2N})+\sin^2(\frac{j\pi}{2N})+\sin^2(\frac{k\pi}{2N}))}\times$$

$$\times\cos^2(\frac{i\pi}{2N})\cos^2(\frac{j\pi}{2N})\cos(\frac{k\pi}{2N})\sin(\frac{k\pi}{2})\le$$

$$\le\frac{8}{N^2}\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\sum_{k=1}^{N-1}\frac{1}{k(i^2+j^2+k^2)}\le$$

$$\le\frac{8}{N^2}\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\frac{1}{i^2+j^2+1}+\frac{8}{N^2}\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\int_1^{N-1}\frac{1}{k(i^2+j^2+k^2)}dk\le$$

$$\le\frac{8}{N^2}\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\frac{1}{i^2+j^2+1}+\frac{8}{N^2}\sum_{i=1}^{N-1}\int_1^{N-1}\frac{1}{k(i^2+1+k^2)}dk+ \qquad (4.3)$$

$$+\frac{8}{N^2}\sum_{i=1}^{N-1}\int_1^{N-1}\int_1^{N-1}\frac{dj\,dk}{k(i^2+j^2+k^2)}\le\frac{8}{N^2}\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\frac{1}{i^2+j^2+1}+$$

$$+\frac{8}{N^2}\sum_{i=1}^{N-1}\int_1^{N-1}\frac{1}{k(i^2+1+k^2)}dk+\frac{8}{N^2}\int_1^{N-1}\int_1^{N-1}\frac{dj\,dk}{k(1+j^2+k^2)}+$$

$$+\frac{8}{N^2}\int_1^{N-1}\int_1^{N-1}\int_1^{N-1}\frac{di\,dj\,dk}{k(i^2+j^2+k^2)}\le\frac{8}{N^2}\sum_{i=1}^{N-1}\frac{1}{i^2+2}+$$

$$+\frac{8}{N^2}\int_1^{N-1}\int_1^{N-1}\frac{di\,dj}{i^2+j^2+1}+\frac{16}{N^2}\int_1^{N-1}\frac{1}{2+k^2}dk+$$

$$+\frac{16}{N^2}\int_1^{N-1}\int_1^{N-1}\frac{di\,dk}{k(i^2+1+k^2)}+\frac{8}{N^2}\int_1^{N-1}\int_1^{N-1}\int_1^{N-1}\frac{di\,dj\,dk}{k(i^2+j^2+k^2)}\le$$

$$\le\frac{8}{N^2}(\frac{\pi^2}{6}+\frac{\pi}{2}\ln(N)+2+\pi+\frac{\pi}{2}(\ln(N))^2).$$

Thus accuracy of the difference scheme (1.2) increases by two orders, with respect to logarithm, when point $x \in \omega$ approaches to side of the cube.

At last estimate the behavior of the function $v(x)$ when point $x \in \omega$ approaches to face of cube. We obtain

$$\begin{aligned}
&v(x_{1,1}, x_{2,N/2}, x_{3,N/2}) = \\
&= \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\sum_{k=1}^{N-1} \frac{(1-(-1)^i)(1-(-1)^j)(1-(-1))^k}{2N^5(1-\cos(\frac{i\pi}{N}))(1-\cos(\frac{j\pi}{N}))(1-\cos(\frac{k\pi}{N}))} \times \\
&\times \frac{\sin^2(\frac{i\pi}{N})\sin(\frac{j\pi}{N})\sin(\frac{j\pi}{2})\sin(\frac{k\pi}{N})\sin(\frac{k\pi}{2})}{(3-\cos(\frac{i\pi}{N})-\cos(\frac{j\pi}{N})-\cos(\frac{k\pi}{N}))} \leq \\
&\leq \frac{4}{N}\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\sum_{k=1}^{N-1} \frac{1}{jk(i^2+j^2+k^2)} \leq \\
&\leq \frac{4}{N}\sum_{j=1}^{N-1}\sum_{k=1}^{N-1} \frac{1}{jk(1+j^2+k^2)} + \frac{4}{N}\sum_{j=1}^{N-1}\sum_{k=1}^{N-1}\int_1^{N-1} \frac{di}{jk(i^2+j^2+k^2)} \leq \\
&\leq \frac{\pi^2}{18N} + \frac{2\pi}{N}\sum_{j=1}^{N-1}\sum_{k=1}^{N-1} \frac{1}{jk\sqrt{j^2+k^2}} \leq \\
&\leq \frac{\pi^2}{18N} + \frac{\sqrt{2}\pi}{N}\Big(\sum_{j=1}^{N-1}\frac{1}{j^{3/2}}\Big)^2 \leq \frac{1}{N}\Big(\frac{\pi^2}{18} + 9\sqrt{2}\pi\Big).
\end{aligned} \tag{4.4}$$

From here one can see that accuracy of the difference scheme (1.2) increases by one order when point $x \in \omega$ approaches to face of cube.

Theorem 4.1 *Let solution of the problem* (1.1) *belong to space* $C^{(4)}(\bar{\Omega})$, *then accuracy of the difference scheme* (1.2) *is characterized by the estimate*

$$|z(x)| \leq Ch^2 v(x)\, |u|_{C^{(4)}(\bar{\Omega})},$$

where constant C *does not depend on* $h, u(x)$ *and function* $v(x)$ *behaves as* $O(h)$ *approaching to faces of cube and approaching to sides as* $O(h^2\ln^2(1/h))$, *approaching to vertexes as* $O(h^2\ln(1/h))$.

Now consider mixed boundary problem

$$\begin{aligned}
&-\Delta u(x) = -\frac{\partial^2 u(x)}{\partial x_1^2} - \frac{\partial^2 u(x)}{\partial x_2^2} - \frac{\partial^2 u(x)}{\partial x_3^2} = f(x),\ x \in \Omega, \\
&u(x) = u_0(x),\ x \in \Gamma_1,\ \frac{\partial u(x)}{\partial n} = g(x),\ x \in \Gamma_2,
\end{aligned} \tag{4.5}$$

where $\Gamma = \Gamma_1 \cup \Gamma_2$ is boundary of the cube Ω,
$\Gamma_2 = \{x = (x_1, x_2, x_3) : x_3 = 0, 0 < x_\alpha < 1, \alpha = 1, 2\}$, n is external normal line to Γ. Introduce in $\bar{\Omega}$ the following cube mesh with the step h:

$$\bar{\omega} = \left\{x = (x_{1,i_1}, x_{2,i_2}, x_{3,i_3}) \in \bar{\Omega} :\ x_{\alpha,i_\alpha} = i_\alpha h,\ i_\alpha = 0, 1, ..., N, \alpha = 1, 2, 3,\ h = 1/N\right\}.$$

Designate the set of internal knots of the mesh $\bar{\omega}$ as
$\omega = \{x = (x_1, x_2, x_3) \in \Omega\}$ and the set of limiting points as $\gamma = \bar{\omega}\backslash\omega$. Let

$$\gamma_2 = \bar{\omega} \cap \Gamma_2,\ \gamma_1 = \gamma\backslash\gamma_2,\ \omega^* = \omega \cup \gamma_2.$$

Approximate the problem (4.5) on the mesh $\bar{\omega}$ with a difference scheme

$$-\Lambda y = -\sum_{\alpha=1}^{3} \Lambda_\alpha y = \varphi(x), \;\; x \in \omega^*, \;\; y(x) = u_0(x), x \in \gamma_1, \tag{4.6}$$

where

$$\Lambda_\alpha y(x) = \begin{cases} y(x)_{\bar{x}_\alpha x_\alpha}, \;\; x \in \omega, \\ \frac{2}{h} y(x)_{x_3}, \;\; x \in \gamma_2, \end{cases}$$

$$\varphi(x) = \begin{cases} f(x), \;\; x \in \omega, \\ f(x) + \frac{2}{h} g(x)\,, x \in \gamma_2. \end{cases}$$

It has been shown in [2] that under condition $u(x) \in C^{(4)}(\bar{\Omega})$ for the error $z(x) = y(x) - u(x)$ of this difference scheme the following estimate is valid

$$|z(x)| \le C(1 - x_{3,i_3})h^2 \;, x = (x_{1,i_1}, x_{2,i_2}, x_{3,i_3}) \in \omega^*. \tag{4.7}$$

The estimate testifies that approaching to the cube face which lies in the plane $x_3 = 1$ accuracy order increases by one order.

Further we present substantial strengthening of the obtained result.

Present $z(x)$ in the form

$$z(x) = z_1(x) + z_2(x),$$

where functions $z_i(x), \; i = 1, 2$ are solutions of the following Dirichlet difference problems

$$-\Lambda z_1(x) = -\sum_{\alpha=1}^{3} \Lambda_\alpha z_1(x) = \psi(x), \;\; x \in \omega, \;\; y(x) = 0, x \in \gamma,$$
$$\psi(x) = \Lambda u(x) - \Delta u(x) = O(h^2), \;\; u(x) \in C^{(4)}(\bar{\Omega}),$$

$$-\Lambda z_2(x) = -\sum_{\alpha=1}^{3} \Lambda_\alpha z_2(x) = 0, \;\; x \in \omega, \;\; z_2(x) = 0, x \in \gamma_1, \;\; z_2(x) = z(x), x \in \gamma_2.$$

It is obvious that to estimate $z_1(x)$ one can use theorem 4.1 while taking for the function $z_2(x)$ taking into account (4.7) the following estimate is valid

$$|z_2(x)| \le Ch^2 v_2(x),$$

where $v_2(x)$ is solution of the following problem

$$-\Lambda v_2(x) = -\sum_{\alpha=1}^{3} \Lambda_\alpha v_2(x) = 0, \;\; x \in \omega, \;\; v_2(x) = 0, x \in \gamma_1, \;\; v_2(x) = 1, x \in \gamma_2,$$

which has the following form

$$\begin{aligned} & v_2(x_{1,s}, x_{2,k}, x_{3,i}) = \\ & = \frac{1}{N^2} \sum_{n=1}^{N-1} \sum_{j=1}^{N-1} \frac{(1 - (-1)^n)(1 - (-1)^j)}{(1 - \cos(\frac{n\pi}{N}))(1 - \cos(\frac{j\pi}{N}))} \sin(\frac{n\pi}{N}) \sin(\frac{nk\pi}{N}) \times \\ & \times \sin(\frac{j\pi}{N}) \sin(\frac{sj\pi}{N}) \frac{U_{N-i-1}(3 - \cos(\frac{n\pi}{N}) - \cos(\frac{j\pi}{N}))}{U_{N-1}(3 - \cos(\frac{n\pi}{N}) - \cos(\frac{j\pi}{N}))}. \end{aligned} \tag{4.8}$$

Estimate the behavior of the function $v_2(x)$ when point $x \in \omega$ approaches to the vertex $(0,0,1)$ of cube. We obtain

$$
\begin{aligned}
&N^3 v_2(x_{1,1}, x_{2,1}, x_{3,N-1}) = \\
&= N \sum_{n=1}^{N-1}\sum_{j=1}^{N-1} \frac{(1-(-1)^n)(1-(-1)^j)}{(1-\cos(\frac{n\pi}{N}))(1-\cos(\frac{j\pi}{N}))} \frac{\sin^2(\frac{n\pi}{N})\sin^2(\frac{j\pi}{N})}{U_{N-1}(3-\cos(\frac{n\pi}{N})-\cos(\frac{j\pi}{N}))} \le \\
&\le 16N \sum_{n=1}^{N-1}\sum_{j=1}^{N-1} \frac{1}{U_{N-1}(3-\cos(\frac{n\pi}{N})-\cos(\frac{j\pi}{N}))} \le \\
&\le 16N \sum_{n=1}^{N-1}\sum_{j=1}^{N-1} \frac{1}{U_{N-1}(1+2(\frac{n}{N})^2+2(\frac{j}{N})^2)} \le \\
&\le 16N \sum_{n=1}^{N-1}\sum_{j=1}^{N-1} \frac{\sinh(\sqrt{(\frac{n}{N})^2+(\frac{j}{N})^2}\ln(3))}{\sinh(\sqrt{n^2+j^2}\ln(3))} \le 48 \sum_{n=1}^{N-1}\sum_{j=1}^{N-1} \frac{1}{n^2+j^2} \le \\
&\le 48 \sum_{n=1}^{N-1} \frac{1}{1+n^2} + 48 \sum_{n=1}^{N-1} \int_1^{N-1} \frac{dj}{n^2+j^2} \le \\
&\le 8\pi^2 + \sum_{n=1}^{N-1} \frac{48}{n} \arctan(\frac{n(N-2)}{n^2+N-1}) \le \\
&\le 8\pi^2 + 24\pi \sum_{n=1}^{N-1} \frac{1}{n} \le 8\pi^2 + 24\pi(1+\ln(N-1)).
\end{aligned}
\tag{4.9}
$$

Further we define behavior of the function $v_2(x)$ when point $x \in \omega$ approaches to the middle of the cube side, to the point $(x_{1,1}, x_{2,N/2}, x_{3,N-1})$. We have

$$
\begin{aligned}
&N^2 v_2(x_{1,1}, x_{2,N/2}, x_{3,N-1}) = \\
&= \sum_{n=1}^{N-1}\sum_{j=1}^{N-1} \frac{(1-(-1)^n)(1-(-1)^j)}{(1-\cos(\frac{n\pi}{N}))(1-\cos(\frac{j\pi}{N}))} \frac{\sin(\frac{n\pi}{N})\sin(\frac{n\pi}{2})\sin^2(\frac{j\pi}{N})}{U_{N-1}(3-\cos(\frac{n\pi}{N})-\cos(\frac{j\pi}{N}))} \le \\
&\le 8 \sum_{n=1}^{N-1}\sum_{j=1}^{N-1} \frac{1}{\sin(\frac{n\pi}{2N})U_{N-1}(3-\cos(\frac{n\pi}{N})-\cos(\frac{j\pi}{N}))} \le \\
&\le 8\,N \sum_{n=1}^{N-1}\sum_{j=1}^{N-1} \frac{1}{nU_{N-1}(1+2(\frac{n}{N})^2+2(\frac{j}{N})^2)} \le \\
&\le 8N \sum_{n=1}^{N-1}\sum_{j=1}^{N-1} \frac{\sinh(\sqrt{(\frac{n}{N})^2+(\frac{j}{N})^2}\ln(3))}{n\sinh(\sqrt{n^2+j^2}\ln(3))} \le 24 \sum_{n=1}^{N-1}\sum_{j=1}^{N-1} \frac{1}{n(n^2+j^2)} \le \\
&\le 24 \sum_{n=1}^{N-1} \frac{1}{n(1+n^2)} + 24 \sum_{n=1}^{N-1} \int_1^{N-1} \frac{dj}{n(n^2+j^2)} \le \\
&\le 2\pi^2 + \sum_{n=1}^{N-1} \frac{24}{n^2} \arctan(\frac{n(N-2)}{n^2+N-1}) \le 2\pi^2 + 12\pi \sum_{n=1}^{N-1} \frac{1}{n^2} \le 2\pi^2 + 2\pi^3.
\end{aligned}
\tag{4.10}
$$

Now define behavior of the function $v_2(x)$ when point $x \in \omega$ approaches to the face of cube, to the point $(x_{1,N/2}, x_{2,N/2}, x_{3,N-1})$. We have

$$
\begin{aligned}
&Nv_2(x_{1,1}, x_{2,N/2}, x_{3,N-1}) = \\
&= \sum_{n=1}^{N-1}\sum_{j=1}^{N-1} \frac{(1-(-1)^n)(1-(-1)^j)}{N(1-\cos(\frac{n\pi}{N}))(1-\cos(\frac{j\pi}{N}))} \frac{\sin(\frac{n\pi}{N})\sin(\frac{n\pi}{2})\sin(\frac{j\pi}{N})\sin(\frac{j\pi}{2})}{U_{N-1}(3-\cos(\frac{n\pi}{N})-\cos(\frac{j\pi}{N}))} \le \\
&\le 4\sum_{n=1}^{N-1}\sum_{j=1}^{N-1} \frac{1}{N\sin(\frac{n\pi}{2N})\sin(\frac{j\pi}{2N})U_{N-1}(3-\cos(\frac{n\pi}{N})-\cos(\frac{j\pi}{N}))} \le \\
&\le 4N\sum_{n=1}^{N-1}\sum_{j=1}^{N-1} \frac{1}{njU_{N-1}(1+2(\frac{n}{N})^2+2(\frac{j}{N})^2)} \le \\
&\le 4N\sum_{n=1}^{N-1}\sum_{j=1}^{N-1} \frac{\sinh(\sqrt{(\frac{n}{N})^2+(\frac{j}{N})^2}\ln(3))}{nj\sinh(\sqrt{n^2+j^2}\ln(3))} \le 12\sum_{n=1}^{N-1}\sum_{j=1}^{N-1}\frac{1}{nj(n^2+j^2)} \le \\
&\le 6\,(\sum_{n=1}^{N-1}\frac{1}{n^2})^2 \le \frac{\pi^4}{6}.
\end{aligned}
\tag{4.11}
$$

The only thing left is to define the behavior of the function $v_2(x)$ when the point $x \in \omega$ approaches to the middle of the vertical face of cube, to the point $(x_{1,1}, x_{2,N/2}, x_{3,N/2})$ and when point $x \in \omega$ approaches to the middle of the vertical side of cube, to the point $(x_{1,1}, x_{2,1}, x_{3,N/2})$. We have

$$
\begin{aligned}
&v_2(x_{1,1}, x_{2,1}, x_{3,N/2}) = \sum_{n=1}^{N-1}\sum_{j=1}^{N-1} \frac{(1-(-1)^n)(1-(-1)^j)}{N^2(1-\cos(\frac{n\pi}{N}))(1-\cos(\frac{j\pi}{N}))} \times \\
&\times \frac{\sin^2(\frac{n\pi}{N})\sin^2(\frac{j\pi}{N})U_{N/2-1}(3-\cos(\frac{n\pi}{N})-\cos(\frac{j\pi}{N}))}{U_{N-1}(3-\cos(\frac{n\pi}{N})-\cos(\frac{j\pi}{N}))} \le \\
&\le 16\sum_{n=1}^{N-1}\sum_{j=1}^{N-1} \frac{U_{N/2-1}(3-\cos(\frac{n\pi}{N})-\cos(\frac{j\pi}{N}))}{N^2U_{N-1}(3-\cos(\frac{n\pi}{N})-\cos(\frac{j\pi}{N}))} \le \\
&\le 16\sum_{n=1}^{N-1}\sum_{j=1}^{N-1} \frac{U_{N/2-1}(1+2(\frac{n}{N})^2+2(\frac{j}{N})^2)}{N^2U_{N-1}(1+2(\frac{n}{N})^2+2(\frac{j}{N})^2)} \le \\
&\le 16\sum_{n=1}^{N-1}\sum_{j=1}^{N-1} \frac{\sinh(\frac{\ln(3)}{2}\sqrt{n^2+j^2})}{N^2\sinh(\ln(3)\sqrt{n^2+j^2})} \le \\
&\le 8\sum_{n=1}^{N-1}\sum_{j=1}^{N-1} \frac{1}{N^2\cosh(\frac{\ln(3)}{2}\sqrt{n^2+j^2})} \le 64\sum_{n=1}^{N-1}\sum_{j=1}^{N-1}\frac{1}{N^2(n^2+j^2)^{5/4}} \le \\
&\le \frac{64}{N^2}\left(\sum_{n=1}^{N-1}\frac{1}{(n^2+1)^{5/4}} + \sum_{n=1}^{N-1}\int_1^{N-1}\frac{dj}{(n^2+j^2)^{5/4}}\right) \le \\
&\le \frac{64}{N^2}\,(\,\frac{1}{2^{5/4}} + 2\int_1^{N-1}\frac{dj}{(1+j^2)^{5/4}} + \int_1^{N-1}\int_1^{N-1}\frac{dn\,dj}{(n^2+j^2)^{5/4}}\,) \le \\
&\le \frac{64}{N^2}\,(\,\frac{1}{2^{5/4}} + \ 2/3\ \mathrm{hypergeom}([3/4,\ 5/4],\ [7/4],\ -1) - 1/3\sqrt{2}\times \\
&\times(\pi\sqrt{2}\mathrm{hypergeom}([1/2,\ 3/4,\ 5/4],[3/2,\ 7/4],-1) - 6\Gamma(3/4)^2\sqrt{\pi})/\pi\ ) \le \\
&\le \frac{64}{N^2}\,(\,\frac{1}{2^{5/4}} + \ 4/3 + 2).
\end{aligned}
\tag{4.12}
$$

where hypergeom([a_1, a_2, ...], [b_1, b_2, ...], z) is hypergeometrical function (see. [8]).

Here one used inequality $\cosh(\frac{\ln(3)}{2}z) \geq \frac{1}{8}z^{5/2}$, $z \geq \sqrt{2}$ which justice can be testified by common facilities of mathematical analysis.

At last we have

$$
\begin{aligned}
&v_2(x_{1,1}, x_{2,N/2}, x_{3,N/2}) = \\
&= \sum_{n=1}^{N-1}\sum_{j=1}^{N-1} \frac{(1-(-1)^n)(1-(-1)^j)}{N^2(1-\cos(\frac{n\pi}{N}))(1-\cos(\frac{j\pi}{N}))} \times \\
&\times \frac{\sin^2(\frac{n\pi}{N})\sin(\frac{j\pi}{N})\sin(\frac{j\pi}{2})U_{N/2-1}(3-\cos(\frac{n\pi}{N})-\cos(\frac{j\pi}{N}))}{U_{N-1}(3-\cos(\frac{n\pi}{N})-\cos(\frac{j\pi}{N}))} \leq \\
&\leq \frac{8}{N^2}\sum_{n=1}^{N-1}\sum_{j=1}^{N-1} \frac{1}{\sin(\frac{j\pi}{2N})} \frac{U_{N/2-1}(3-\cos(\frac{n\pi}{N})-\cos(\frac{j\pi}{N}))}{U_{N-1}(3-\cos(\frac{n\pi}{N})-\cos(\frac{j\pi}{N}))} \leq \\
&\leq \frac{8}{N}\sum_{n=1}^{N-1}\sum_{j=1}^{N-1} \frac{\sinh(\frac{\ln(3)}{2}\sqrt{n^2+j^2})}{j\ \sinh(\ln(3)\sqrt{n^2+j^2})} \leq \qquad (4.13) \\
&\leq \frac{4}{N}\sum_{n=1}^{N-1}\sum_{j=1}^{N-1} \frac{1}{j\ \cosh(\frac{\ln(3)}{2}\sqrt{n^2+j^2})} \leq \frac{32}{N}\sum_{n=1}^{N-1}\sum_{j=1}^{N-1} \frac{1}{j(n^2+j^2)^{5/4}} \leq \\
&\leq \frac{32}{2^{5/4}N}\sum_{n=1}^{N-1}\frac{1}{n^{5/4}}\sum_{j=1}^{N-1}\frac{1}{j^{9/4}} \leq \frac{32}{2^{5/4}N}\left(1+\int_1^{N-1}\frac{dn}{n^{5/4}}\right) \\
&\left(1+\int_1^{N-1}\frac{dj}{j^{9/4}}\right) \leq \frac{288}{2^{5/4}N}.
\end{aligned}
$$

Taking into account (4.9)-(4.13) we convince in justice of the following statement

Theorem 4.2 *Let solutions of the problem* (4.5) *belong to space* $C^{(4)}(\bar{\Omega})$ *then accuracy of the difference scheme* (4.6) *is characterized by the estimate*

$$|z(x)| \leq Ch^2(v(x)+v_2(x))\,|u|_{C^{(4)}(\bar{\Omega})}\,,$$

where function $v(x)+v_2(x)$ *approaching to faces of cube behaves itself as* $O(h)$, *approaching to sides - as* $O(h^2\ln^2(1/h))$ *and approaching to vertexes - as* $O(h^2\ln(1/h))$. *Here constant* C *does not depend on* $h, u(x)$.

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