Журнал обчисл. прикл. матем.
Вип. 1, 2009, стор. 57-69

J. Numer. Appl. Math.
No. 97, 2009, pp. 57-69

A POSTERIORI ERROR ESTIMATE FOR SOME TWO-DIMENSIONAL INTEGRAL EQUATIONS OF THE FIRST KIND ON UNCLOSED SURFACES IN POTENTIAL THEORY

UDC 518:517.948

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Abstract. In the article we consider the calculation of electrostatic field being formed by a number of charged electrodes which have to be represented as unclosed surfaces in $\mathbb{R}^3$. The effective algorithms for this problem solving in the essential space case can be obtained by integral equations method. The main object of our investigations is the construction of approximate schemes for solving some of two-dimensional integral equations of the first kind with weak singularities in kernels. Singular behavior of desired "charge distribution density" in the neighborhood of some isolated points on boundary surfaces was taken into account by various ways.

1. Introduction

The main object of our investigations is the construction of approximate schemes for solving some of two-dimensional integral equations of the first kind with weak singularities in kernels. Such equations simulate electrostatic field being formed by a number of charged electrodes which have to be represented as unclosed surfaces in $\mathbb{R}^3$. The efficiency of various approaches was verified by solving of some model tasks and boundary value problems in relation to practice.

2. Statement of the problem and appropriate approximate scheme

Let us assume that unclosed surface S is placed in Euclidean space $\mathbb{R}^3$, and also $\overline{S} := S \bigcup \partial S$, where ∂S is a piecewise smooth boundary of S. Further, assume that S belongs to the class of Lipschitz surfaces, and denote by M, P, N, etc. points of $\mathbb{R}^3$. Let us seek a solution of the boundary value problem considered in the domain $\Omega_s := \mathbb{R}^3 \backslash \overline{S}$ under double-sided condition of S. Therefore, it is necessary to seek the function $u \in H^1(\Omega_S, \Delta)$, which holds on

$$\begin{aligned} &\Delta u = 0 \quad \text{in} \quad \Omega_S, \\ &\delta^{\pm} u = g_{\pm} \quad \text{on} \quad S, \\ &\lim_{|P| \to \infty} u(P) = 0, \quad P \in \Omega_S, \end{aligned} \tag{2.1}$$

where $\delta^{\pm} : \ H^1(\Omega_S) \to H^{1/2}(S)$ are the trace maps [1], $g_{\pm}$ are the given values of desired function on S in accordance with positive or negative side of S, and

$$H^1(\Omega_S, \Delta) := \left\{ u \,\middle|\, u \in H^1(\Omega_S),\ \Delta u \in L_2(\Omega_S) \right\}.$$

† *Key words.* Electrostatic field in the essential space case, two-dimensional integral equations, numerical methods, singular behavior of desired solution.

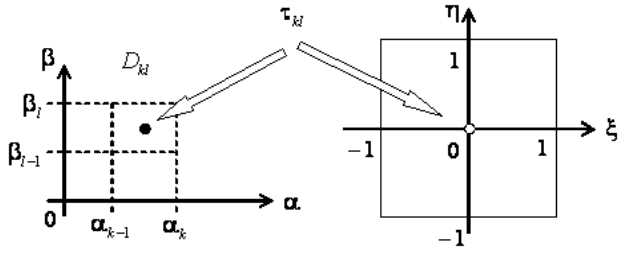

Fig. 1. Location of node in the case of piecewise-constant approximation.

As is well known [2] the solution of problem (2.1) can be written in such integral form as

$$u(P) = \iint\limits_S K(P,M)\,\tau(M)\,\mathrm{d}S_M - u_0(P), \quad P \in \Omega_S,$$

were $K(P,M) := 1/\mathrm{dist}(P,M)$,

$$u_0(P) := \iint\limits_S \frac{\partial K(P,M)}{\partial n_M}\,\big[g(M)\big]\,\mathrm{d}S_M$$

with $\big[g\big] := g_- - g_+ \in H_{00}^{1/2}(S)$, and also $\tau(M) \in H_{00}^{-1/2}(S)$ is a unique solution of integral equation (IE)

$$(K\tau)(P) := \iint\limits_S K(P,M)\,\tau(M)\,\mathrm{d}S_M = F(P), \quad P \in S, \tag{2.2}$$

provided that $F(P) := \frac{1}{2}\big[g_-(P) + g_+(P)\big] + u_0(P) \in H^{1/2}(S)$. Note that IE (2.2) and (2.1) are equivalent problems; moreover, the solvability of (2.2) can be expressed by such inequalities

$$m_1\|\tau\|_{H_{00}^{-1/2}(S)} \leqslant \|K\tau\|_{H^{1/2}(S)} \leqslant m_2\|\tau\|_{H_{00}^{-1/2}(S)} \quad (0 < m_1 \leqslant m_2).$$

The electrostatic treatment of problem (2.1) means that $g_+ = g_-$. In this case IE (2.2) can be rather simplified in the sense that $F(P)$ means boundary potential value on S only ($F(P) \equiv \mathrm{const}$).

Let S be represented parametrically:

$$M := \big\{x(\alpha,\beta), y(\alpha,\beta), z(\alpha,\beta);\ (\alpha,\beta) \in \overline{D} := [-1,1]^2\big\},$$

$$P := \big\{x(\alpha_0,\beta_0), y(\alpha_0,\beta_0), z(\alpha_0,\beta_0);\ (\alpha_0,\beta_0) \in D := (-1,1)^2\big\}.$$

Further, let us preserve the notations of K, τ, F. Then, according to the type of surface S, we can get for the sought "charge distribution density" two-dimensional IE in the form as

$$\iint\limits_{\overline{D}} \tau(\alpha,\beta)\,K(\alpha,\beta;\,\alpha_0,\beta_0)\,J(\alpha,\beta)\,\mathrm{d}\alpha\mathrm{d}\beta = F(\alpha_0,\beta_0), \quad (\alpha_0,\beta_0) \in D, \tag{2.3}$$

where

$$K(\alpha,\beta;\,\alpha_0,\beta_0) := \Big\{\big[x(\alpha,\beta) - x(\alpha_0,\beta_0)\big]^2 +$$

$$+\big[y(\alpha,\beta) - y(\alpha_0,\beta_0)\big]^2 + \big[z(\alpha,\beta) - z(\alpha_0,\beta_0)\big]^2\Big\}^{-1/2},$$

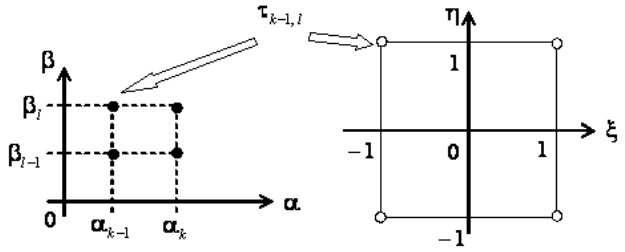

Fig. 2. Location of nodes in the case of piecewise-bilinear approximation.

$J(\alpha,\beta)\,\mathrm{d}\alpha\mathrm{d}\beta$ is a surface element expressed with the help of local coordinates (α,β). Note also that the functions $x(\alpha,\beta)$, $y(\alpha,\beta)$, $z(\alpha,\beta)$ which express Cartesian coordinates of point $M=(x,y,z)$ on S are continuously differentiable in $\overline{D}$ as far as $J(\alpha,\beta):=\left[E(\alpha,\beta)\,G(\alpha,\beta)-H^2(\alpha,\beta)\right]^{1/2}$ with $E(\alpha,\beta):=(x'_\alpha)^2+(y'_\alpha)^2+(z'_\alpha)^2$, $G(\alpha,\beta):=(x'_\beta)^2+(y'_\beta)^2+(z'_\beta)^2$, $H(\alpha,\beta):=x'_\alpha x'_\beta+y'_\alpha x'_\beta+z'_\alpha x'_\beta$.

Without loss of generality let us consider problem (2.1) in the case of S being a separate rectangular plate defined by its vertices A_j $(j=\overline{1,4})$ with corresponding coordinates (x_j,y_j,z_j). Then, corresponding parametric equations of S can be represented as

$$\begin{cases} x(\alpha,\beta):=0,5\sum\limits_{j=1}^{4}x_j\,\varphi_j(\alpha,\beta), \\ y(\alpha,\beta):=0,5\sum\limits_{j=1}^{4}y_j\,\varphi_j(\alpha,\beta), \\ z(\alpha,\beta):=0,5\sum\limits_{j=1}^{4}z_j\,\varphi_j(\alpha,\beta), \end{cases} \tag{2.4}$$

where $\varphi_j(\alpha,\beta):=\left(1+(-1)^k\alpha\right)\left(1+(-1)^l\beta\right)$ with $k:=\left\lfloor\frac{j}{2}\right\rfloor+1$, $l:=\left\lfloor\frac{j-1}{2}\right\rfloor+1$, $j=\overline{1,4}$.

Suppose that the rectangular plate S is located in $\mathbb{R}^3$ so that opposite edges of its one pair of edges, at least, are parallel to an arbitrary coordinate axis. Then, taking into account parametrization (2.4) and calculating the coefficients of corresponding quadratic form, equation (2.3) can be represented as

$$\sqrt{E\,G}\iint\limits_{\overline{D}}\frac{\tau(\alpha,\beta)\,\mathrm{d}\alpha\mathrm{d}\beta}{\sqrt{E\,(\alpha-\alpha_0)^2+G\,(\beta-\beta_0)^2}}=F(\alpha_0,\beta_0),\quad(\alpha_0,\beta_0)\in D. \tag{2.5}$$

We can also use parametric equations (2.4) for representation of an arbitrary quadrangular plate being determined by coordinates of its vertices placed in one plane. Taking into account the parallelism of a plate edge to an arbitrary coordinate axis we get an analogue of equation (2.5) with rather different kernel. In this case we can apply the numerically analytical approach to solve IE, as recommended by authors [3,4]. Moreover, to calculate certain two-dimensional proper and improper integrals it is possible to use specific changes of variables [3,4].

To construct the approximate scheme for IE (2.5) solution let us partition $\overline{D}$ into $N_\alpha\times N_\beta$ rectangular elements $D_{kl}:=[\alpha_{k-1},\alpha_k]\times[\beta_{l-1},\beta_l]$ $(k=\overline{1,N_\alpha};\ l=\overline{1,N_\beta})$, where

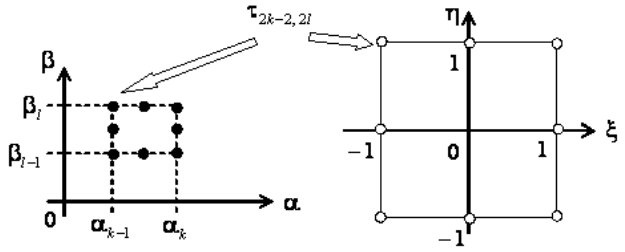

Fig. 3. Location of nodes in the case of piecewise-biquadratic approximation.

$\alpha_k := kh_\alpha - 1$ $(k = \overline{0, N_\alpha})$, $\beta_l := lh_\beta - 1$ $(l = \overline{0, N_\beta})$ with $h_\alpha := 2/N_\alpha$, $h_\beta := 2/N_\beta$. According to this partition of $\overline{D}$ IE (2.5) can be produced formally as

$$\sqrt{EG}\sum_{k=1}^{N_\alpha}\sum_{l=1}^{N_\beta}\iint\limits_{D_{kl}}\frac{\tau(\alpha,\beta)\,\mathrm{d}\alpha\mathrm{d}\beta}{\sqrt{E(\alpha-\alpha_0)^2+G(\beta-\beta_0)^2}} = F(\alpha_0,\beta_0), \quad (\alpha_0,\beta_0)\in D. \tag{2.6}$$

To seek the approximate solution of IE (2.6) let us consider its local approximation on every element D_{kl}. To this end let us realize such change of variables

$$\begin{cases} \alpha_k(\xi,\eta) = \dfrac{1}{2}h_\alpha(2k-1+\xi)-1, \quad k=\overline{1,N_\alpha}; \\ \\ \beta_l(\xi,\eta) = \dfrac{1}{2}h_\beta(2l-1+\eta)-1, \quad l=\overline{1,N_\beta}, \quad -1\leqslant\xi,\eta\leqslant 1, \end{cases}$$

and represent, in accordance with this, $\tau_{kl}(\xi,\eta) := \tau\big(\alpha_k(\xi,\eta),\beta_l(\xi,\eta)\big)$ as

$$\tau_{kl}(\xi,\eta) = \sum_{e=\sigma_n(k)}^{\sigma_n(k)+n}\sum_{f=\sigma_n(l)}^{\sigma_n(l)+n}\tau_{ef}\Psi_{nef}(\xi,\eta). \tag{2.7}$$

Here, τ_{ef} are the desired values of function $\tau(\alpha,\beta)$ at element D_{kl} nodal points; n is an order of used approximation $(n=\overline{0,3})$; $\sigma_n(p)$ $(p\in\{k,l\})$ is a function that realizes the connection of D_{kl} enumeration with nodal points enumeration on D_{kl}; then $\Psi_{nef}(\xi,\eta)$ are, in accordance with n, some basis functions.

Further, taking into account (2.7), we obtain a discrete analogue of (2.5)

$$\sum_{k=1}^{N_\alpha}\sum_{l=1}^{N_\beta}\sum_{e=\sigma_n(k)}^{\sigma_n(k)+n}\sum_{f=\sigma_n(l)}^{\sigma_n(l)+n}\tau_{ef}I_{kl}^{(nef)}(\alpha_0,\beta_0) = F(\alpha_0,\beta_0), \quad (\alpha_0,\beta_0)\in D,$$

where

$$I_{kl}^{(nef)}(\alpha_0,\beta_0) := \frac{1}{4}\,h_\alpha h_\beta\sqrt{EG}\times$$

$$\times\int_{-1}^{1}\int_{-1}^{1}\frac{\Psi_{nef}(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta}{\sqrt{\left(\dfrac{1}{2}\,h_\alpha\sqrt{E}\,\xi+\left(\dfrac{2k-1}{2}\,h_\alpha-1\right)\sqrt{E}-\alpha_0\sqrt{E}\right)^2+}}\cdot$$

$$\sqrt{\quad+\left(\dfrac{1}{2}\,h_\beta\sqrt{G}\,\eta+\left(\dfrac{2l-1}{2}\,h_\beta-1\right)\sqrt{G}-\beta_0\sqrt{G}\right)^2}$$

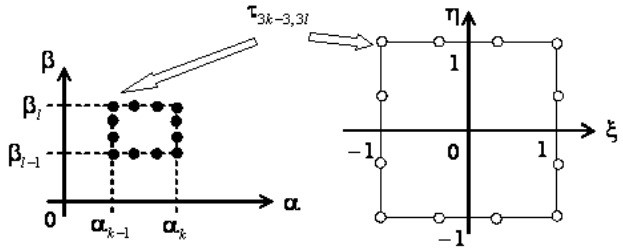

Fig. 4. Location of nodes in the case of piecewise-bicubic approximation.

Choosing (α_0, β_0) at the above mentioned nodal points on elements D_{kl}, we get a system of linear algebraic equations for unknown τ_{ef}. Note that the order of this system is $N_A := \left| \operatorname{sign}(n)\,(nN_\alpha + 1)\,(nN_\beta + 1) - (n-1)^2 N_\alpha N_\beta \right|$ and, in addition, it is possible to calculate integrals $I_{kl}^{(nef)}(\alpha_0, \beta_0)$ using specific formulas [3, 4].

Then, according to the order of $\tau(\alpha, \beta)$ approximation on every element D_{kl}, let us demonstrate concrete representations for $\tau_{kl}(\xi, \eta)$ under different values of parameter n.

In the case of piecewise-constant approximation (see fig. 1)

$$\tau_{kl}(\xi, \eta) \equiv \tau(\alpha_k - \frac{1}{2}\,h_\alpha, \beta_l - \frac{1}{2}\,h_\beta),$$

$$\Psi_{0kl}(\xi, \eta) \equiv 1, \quad -1 \leqslant \xi, \eta \leqslant 1, \quad k = \overline{1, N_\alpha}, \quad l = \overline{1, N_\beta},$$

$$\sigma_0(p) = p, \quad p \in \{\, k, l \,\}.$$

In the case of piecewise-bilinear approximation of function τ on D_{kl} $\tau_{kl}(\xi, \eta)$ may be represented as (see fig. 2)

$$\tau_{kl}(\xi, \eta) = \frac{1}{4}\,\tau_{k-1,\,l-1}(1-\xi)\,(1-\eta) + \frac{1}{4}\,\tau_{k,\,l-1}(1+\xi)\,(1-\eta) +$$

$$+\frac{1}{4}\,\tau_{k,\,l}(1+\xi)\,(1+\eta) + \frac{1}{4}\,\tau_{k-1,\,l}(1-\xi)\,(1+\eta),$$

and also

$$\Psi_{1kl}(\xi, \eta) = \frac{1}{4}(1 + (-1)^{k+e}\xi)\,(1 + (-1)^{l+f}\eta),$$

$$-1 \leqslant \xi, \eta \leqslant 1, \quad k = \overline{1, N_\alpha}, \quad l = \overline{1, N_\beta},$$

$$\sigma_1(p) = p - 1, \quad p \in \{\, k, l \,\}.$$

In the case of piecewise-biquadratic approximation of function τ on D_{kl} $\tau_{kl}(\xi, \eta)$ may be represented as (see fig. 3)

$$\tau_{kl}(\xi, \eta) = \frac{1}{4}\Big(\tau_{2k-2,\,2l-2}(1-\xi)\,(1-\eta)\,(-\xi-\eta-1) +$$

$$+\tau_{2k,\,2l-2}(1+\xi)\,(1-\eta)\,(\xi-\eta-1) + \tau_{2k,\,2l}(1+\xi)\,(1+\eta)\,(\xi+\eta-1) +$$

$$\tau_{2k-2,\,2l}(1-\xi)\,(1+\eta)\,(-\xi+\eta-1)\Big) +$$

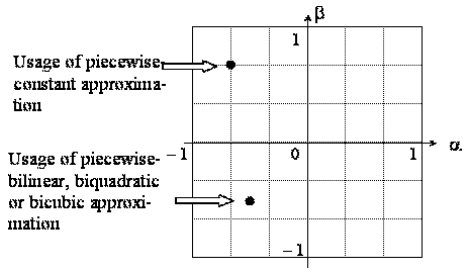

Fig. 5. Choice of "intermediate" points under various ways of "charge distribution density" approximation.

$$+\frac{1}{2}\Big(\tau_{2k-1,\,2l-2}(1-\xi^2)\,(1-\eta)+\tau_{2k,\,2l-1}(1+\xi)\,(1-\eta^2)+$$

$$\tau_{2k-1,\,2l}(1-\xi^2)\,(1+\eta)+\tau_{2k-2,\,2l-1}(1-\xi)\,(1-\eta^2)\Big),$$

$$\sigma_2(p)=2(p-1),\quad p\in\{\,k,l\,\},$$

$$-1\leqslant\xi,\eta\leqslant 1,\quad k=\overline{1,N_\alpha},\quad l=\overline{1,N_\beta}.$$

In the case of piecewise-bicubic approximation on D_{kl} we shall get 12 terms in representation of τ (see fig. 4)

$$\tau_{kl}(\xi,\eta)=\frac{1}{32}\Big(\tau_{3k-3,\,3l-3}(1-\xi)\,(1-\eta)\,(9\xi^2+9\eta^2-10)+$$

$$+\tau_{3k,\,3l-3}(1+\xi)\,(1-\eta)\,(9\xi^2+9\eta^2-10)+\tau_{3k,\,3l}(1+\xi)\,(1+\eta)\,(9\xi^2+9\eta^2-10)+$$

$$+\tau_{3k-3,\,3l}(1-\xi)\,(1+\eta)\,(9\xi^2+9\eta^2-10)\Big)+$$

$$+\frac{9}{32}\Big(\tau_{3k-2,\,3l-3}(1-\xi^2)\,(1-\eta)\,(1-3\xi)+\tau_{3k,\,3l-2}(1+\xi)\,(1-\eta^2)\,(1-3\eta)+$$

$$+\tau_{3k-1,\,3l}(1-\xi^2)\,(1+\eta)\,(1+3\xi)+\tau_{3k-3,\,3l-1}(1-\xi)\,(1-\eta^2)\,(1+3\eta)\Big)+$$

$$+\frac{9}{32}\Big(\tau_{3k-1,\,3l-3}(1-\xi^2)\,(1-\eta)\,(1+3\xi)+\tau_{3k,\,3l-1}(1+\xi)\,(1-\eta^2)\,(1+3\eta)+$$

$$+\tau_{3k-2,\,3l}(1-\xi^2)\,(1+\eta)\,(1-3\xi)+\tau_{3k-3,\,3l-2}(1-\xi)\,(1-\eta^2)\,(1-3\eta)\Big),$$

$$\sigma_3(p)=3(p-1),\quad p\in\{\,k,l\,\},$$

$$-1\leqslant\xi,\eta\leqslant 1,\quad k=\overline{1,N_\alpha},\quad l=\overline{1,N_\beta}.$$

If $\{\tau_{ef}\}$ is known, then in order to calculate the potential at an arbitrary point $P\in\Omega_S$ it is necessary to use such descrete analogue for the solution of problem (2.1)

$$u(P)=\sum_{k=1}^{N_\alpha}\sum_{l=1}^{N_\beta}\sum_{e=\sigma_n(k)}^{\sigma_n(k)+n}\sum_{f=\sigma_n(l)}^{\sigma_n(l)+n}\tau_{ef}J_{kl}^{nef}(P),$$

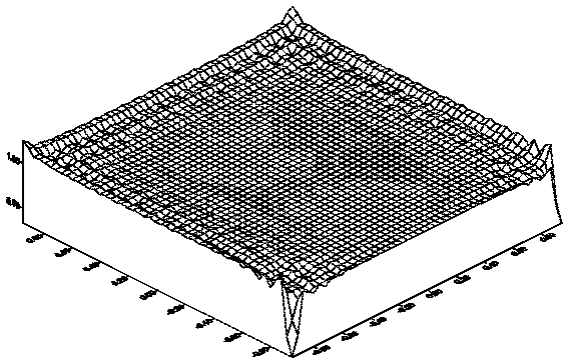

Fig. 6. Reproduction of the potential on the plate S at each "intermediate" grid points in the case of piecewise-biquadratic approximation under $N = 20$.

where

$$J_{kl}^{(nef)}(P) := \frac{1}{4} h_\alpha h_\beta \sqrt{EG} \times$$

$$\times \int_{-1}^{1} \int_{-1}^{1} \frac{\Psi_{nef}(\xi, \eta)\, \mathrm{d}\xi \mathrm{d}\eta}{\sqrt{\left(\frac{1}{2} h_\alpha \sqrt{E}\, \xi + \left(\frac{2k-1}{2} h_\alpha - 1\right)\sqrt{E} - \frac{a_0}{\sqrt{E}}\right)^2 + \left(\frac{1}{2} h_\beta \sqrt{G}\, \eta + \left(\frac{2l-1}{2} h_\beta - 1\right)\sqrt{G} - \frac{b_0}{\sqrt{G}}\right)^2 + c_0}},$$

and constants $a_0, b_0, c_0 > 0$ depend on the coordinates of point P. Note also that we can calculate analytically integrals $J_{kl}^{nef}(P)$ by specific formulas [3, 4].

3. Results obtained by approximate solution of some model problems

I. Let us consider the problem of calculation an electrostatic field being formed by electric charges, which are stationary in space and invariable in time, distributed on a quadrangular plate S. Suppose that S is defined by its vertecies $A_1 := (-1, -1, -1)$, $A_2 := (1, -1, -1)$, $A_3 := (1, 1, -1)$, $A_4 := (-1, 1, -1)$, and $F(P) \equiv 1$ is a potential value of S. Using the uniform grid, plate S has been partitioned into N^2 elements. The calculations have been realized under $N = 10$ and $n = 20$ using piecewise-constant, piecewise-bilinear, piecewise-biquadratic and piecewise-bicubic approximations of an unknown function. To control the accuracy of calculation it is necessary to find potential values on S at separate points which do not coincide with interpolational nodes of the desired function $\tau(\alpha, \beta)$ (see fig. 5). The numerical results are shown in tables 1-4 and fig. 6. Thus, we demonstrate the accuracy refinement of the numerical results being obtained in the neighborhood of surface corner points in accordance with the order of "charge distribution density".

II. In addition, let us consider a problem of approximate solution error estimate. To this end, a posteriori error estimate can be used as it is perfectly good in practice [5, 6]. By assumption as to solvability of IE (2.2) operator K is isomorphism of $H_{00}^{-1/2}(S)$ to $H^{1/2}(S)$. Let $\tau_h(\alpha, \beta)$ be approximate solution of IE (2.6), which belongs to a chosen approximation space ($n = \overline{0, 3}$). As is shown the solution of problem (2.1) at an arbitrary

Tabl. 1. Potential at certain "intermediate" points of grid on the plate S in the case of piecewise-constant approximation.

α	β	$n=0$, $N=10$	Error under $N=10$	α	β	$n=0$, $N=20$	Error under $N=20$
0.950	0.950	0.9029	0.0971	0.975	0.975	0.9209	0.0791
0.800	0.950	0.9650	0.0350	0.900	0.975	0.9717	0.0283
0.800	0.800	1.0338	0.0338	0.800	0.975	0.9637	0.0363
0.600	0.600	1.0000	0.0000	0.800	0.800	1.0000	0.0000

Tabl. 2. Potential at certain "intermediate" points of grid on the plate S in the case of piecewise-bilinear approximation.

α	β	$n=1$, $N=10$	Error under $N=10$	α	β	$n=1$, $N=20$	Error under $N=20$
0.900	0.900	1.1521	0.1521	0.950	0.950	1.0868	0.0868
0.900	0.700	1.0483	0.0483	0.950	0.850	1.0296	0.0296
0.700	0.700	0.9832	0.9832	0.850	0.850	0.9901	0.0099
0.500	0.500	1.0027	0.0027	0.750	0.750	1.0016	0.0016
0.300	0.300	0.9998	0.0002	0.650	0.650	1.0000	0.0000
0.300	0.100	1.0000	0.0000				

point $P \in \Omega_S$ is sought by means of integral form. Therefore, on the basis of $\tau_h(\alpha,\beta)$, the approximate value of problem (2.1) solution at P is as follows: $u_h(P) = (K\tau_h)(P)$. Taking into account the latter, error function $e_u := u - u_h$ can be obtained by a formula

$$e_u = K\tau - K\tau_h = K(\tau - \tau_h) = Ke_\tau, \tag{3.1}$$

where u is an exact solution of problem (2.1) and $e_\tau := \tau - \tau_h$ is an error of "charge distribution density" approximate value. It is easy to verify that e_τ is a solution of such IE as

$$Ke_\tau = F - K\tau_h, \quad P \in S. \tag{3.2}$$

Hence, to seek e_u on S it is necessary to solve (3.2) and to use (3.1). As is well known [7], the solution of IE (2.6) has singular behavior in the neighborhood of unclosed surface S contour. That is why, the function e_u is interesting for us only on so-called "extremal" elements D^e_{kl}, where it can take maximal values. So, the obtaining of preassigned given accuracy ε may be varified on D^e_{kl} by inequality

$$\begin{aligned} \varepsilon_e := \frac{\| e_u\|_{H^{1/2}(D^e_{kl})}}{\sqrt{\| u_h\|^2_{H^{1/2}(D^e_{kl})} + \| e_u\|^2_{H^{1/2}(D^e_{kl})}}} \leqslant \\ \leqslant \frac{\| e_\tau\|_{H^{-1/2}_{00}(D^e_{kl})}}{\sqrt{\| \tau_h\|^2_{H^{-1/2}_{00}(D^e_{kl})} + \| e_\tau\|^2_{H^{-1/2}_{00}(D^e_{kl})}}} < \varepsilon. \end{aligned} \tag{3.3}$$

If condition (3.3) is not realized on "extremal" element, it is necessary to partition this element into several ones and to solve the problem once more using a more precise grid. To verify inequality (3.3) it is necessary to calculate norms of some functions in the space $H^{-1/2}_{00}(D^e_{kl})$. We get functions e_τ and τ_h as polynomial approximations of solutions (2.6) and (3.2). So, from practical point of view it is suitable to use norms in the space $L_2(D^e_{kl})$ for objective estimate of obtained solution. In special cases it needs to use a

Tabl. 3. Potential at certain "intermediate" points of grid on the plate S in the case of piecewise-biquadratic approximation.

α	β	$n=2$, $N=10$	Error under $N=10$	α	β	$n=2$, $N=20$	Error under $N=20$
0.900	0.900	0.9979	0.0021	0.950	0.950	1.0050	0.0050
0.900	0.700	1.0030	0.0030	0.950	0.850	1.0026	0.0026
0.700	0.700	0.9998	0.0002	0.950	0.050	1.0020	0.0020
0.500	0.500	1.0027	0.0027	0.850	0.850	1.0000	0.0000
0.300	0.300	0.9998	0.0002	0.750	0.750	1.0000	0.0000
0.300	0.100	1.0000	0.0000				

Tabl. 4. Potential at certain "intermediate" points of grid on the plate S in the case of piecewise-bicubic approximation.

α	β	$n=3$, $N=10$	Error under $N=10$	α	β	$n=3$, $N=20$	Error under $N=20$
0.900	0.900	0.9799	0.0201	0.950	0.950	0.9942	0.0058
0.900	0.700	0.9959	0.0041	0.950	0.850	0.9982	0.0018
0.700	0.700	1.0003	0.0003	0.950	0.050	0.9987	0.0013
0.500	0.500	1.0000	0.0000	0.850	0.850	1.0002	0.0002
				0.750	0.750	1.0000	0.0000

weight analogue of $L_2(D^e_{kl})$-norm as $\|\,.\,\|_{L_2,\rho}$, where ρ is a function of distance from an arbitrary point on D^e_{kl} to the contour ∂S.

III. To seek the function e_u we need to solve a problem, similar to the initial one, but using an approximation of higher order and different output data. Certainly, this is a drawback of the approach described here. Therefore, let us consider an alternative version of such procedure which is to use piecewise-bilinear approximation in connection with (3.2) under $N = 10$ (see table 2). Here, the maximal error on corner element is 0,1521. Using for the solution of (3.2) piecewise-biquadratic approximation with the same number of elements we shall get, as to e_u, such improved values (see table 5).

Tabl. 5. Error e_u at some "intermediate" points of grid in the case of piecewise-biquadratic approximation.

α	β	$n=2$, under $N=10$	Error under $N=10$
0.900	0.900	0.1518	0.0003
0.900	0.700	0.1526	0.0005
0.700	0.700	0.1521	0.0000

IV. Suppose the function $\tau_h^{(1)}$ has been obtained using piecewise-bilinear approximation with respect to more precise grid. Then, to calculate e_u in (3.1) let us choose the function $\tau_h^{(2)}(\alpha,\beta)$, as τ, built under values of $\tau_h^{(1)}$ at grid nodes; here biquadratic basis is used, as is shown on fig. 7. Let us note that in this case $e_u^{(2)} = K\tau_h^{(2)} - K\tau_h^{(1)}$. The results of calculations are presented in table 6. They illustrate rather simple and effective way of e_u evaluation.

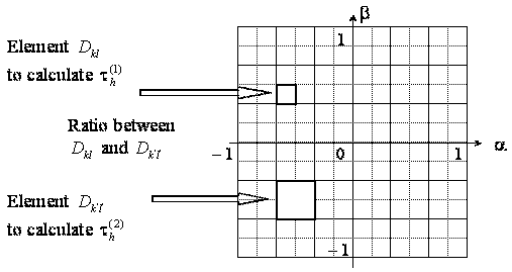

Fig. 7. Modification of grid density as calculation of $e_u^{(2)}$.

Tabl. 6. Potential at "intermediate" points of grid as calculation of error $e_u^{(2)}$.

α	β	$n = 1$, under $N = 10$	$n = 2$, under $N = 10$	Error $e_u^{(2)}$
0.900	0.900	1.1521	0.9979	0.1542
0.900	0.700	1.0483	1.0030	0.0453
0.700	0.700	0.9832	0.9998	0.0166
0.500	0.500	1.0027	1.0000	0.0027
0.300	0.300	0.9998	1.0000	0.0002
0.300	0.100	1.0000	1.0000	0.0000

4. Approximate solution of some problem in relation to practice

By solving one problem of electrostatics let us illustrate the above method of approximate scheme construction and its analysis. Besides, we are interested in assessing error in the obtained numerical results and possibilities of their optimization. Thus, let us consider a calculation of flat condenser field. Suppose that the geometric information of such electronic system can be given by condenser projections on coordinate planes XY and XZ (see fig. 8). Let us note that, as for the terminology suggested, surface S is an aggregate of two parallel identical plates S_1 and S_2 situated symmetricly with respect to a coordinate plane XY; the distance between them equals $2h$. Suppose that C_1 and C_2 are the given potential values on S_1 and S_2 respectively. The electrostatic treatment of problem (2.1) means that C_1 and C_2 are arbitrary constants.

In this case IE (2.5) may be represented by a system of two IEs as

$$\begin{cases} \iint\limits_{\Delta} \tau_1(x,y)\, K(x,y;x_0,y_0)\,\mathrm{d}x\mathrm{d}y + \\ \quad + \iint\limits_{\Delta} \tau_2(x,y)\, K_h(x,y;x_0,y_0)\,\mathrm{d}x\mathrm{d}y = C_1, \quad P := (x_0,y_0) \in S_1; \\ \iint\limits_{\Delta} \tau_1(x,y)\, K_h(x,y;x_0,y_0)\,\mathrm{d}x\mathrm{d}y + \\ \quad + \iint\limits_{\Delta} \tau_2(x,y)\, K(x,y;x_0,y_0)\,\mathrm{d}x\mathrm{d}y = C_2, \quad P := (x_0,y_0) \in S_2, \end{cases} \tag{4.1}$$

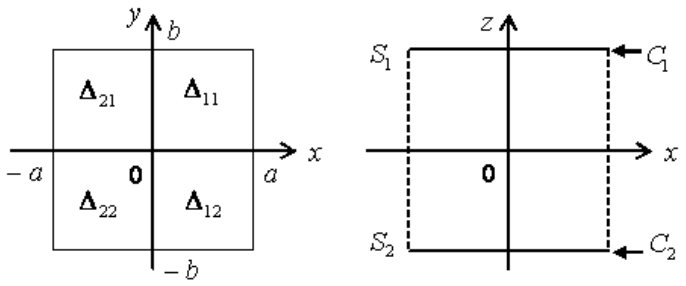

Fig. 8. Projections of plates S_1 and S_2 on coordinate planes.

where

$$\Delta := [-a, a] \times [-b, b],$$

$$K(x, y; x_0, y_0) := \left[(x - x_0)^2 + (y - y_0)^2\right]^{-1/2},$$

$$K_h(x, y; x_0, y_0) := \left[(x - x_0)^2 + (y - y_0)^2 + 4h^2\right]^{-1/2},$$

and $\{\tau_1(x, y);\, \tau_2(x, y),\, (x, y) \in \Delta\}$ – is a generalized "charge distribution density" on $S := S_1 \bigcup S_2$. Note also that $\Delta := \bigcup\limits_{i,j=1}^{2} \Delta_{ij}$, where Δ_{ij} is a congruent component of Δ (see fig. 8). Since initial problem (2.1) has an Abelian second order group of symmetry [8] then we can split (4.1) into two independent IEs

$$\iint\limits_{\Delta} \sigma_1(x, y)\,(K + K_h)(x, y; x_0, y_0)\,\mathrm{d}x\mathrm{d}y = F_1, \quad (x_0, y_0) \in S_1, \tag{4.2}$$

$$\iint\limits_{\Delta} \sigma_2(x, y)\,(K - K_h)(x, y; x_0, y_0)\,\mathrm{d}x\mathrm{d}y = F_2, \quad (x_0, y_0) \in S_1, \tag{4.3}$$

where $\sigma_1(x, y) := \tau_1(x, y) + \tau_2(x, y)$, $\sigma_2(x, y) := \tau_1(x, y) - \tau_2(x, y)$, $F_1 := C_1 + C_2$, $F_2 := C_1 - C_2$. The right sides of (4.2) and (4.3) are constants; therefore, desired solutions σ_1 and σ_2 are symmetric with respect to coordinate axises $0X$ and $0Y$. This makes it possible to represent IEs (4.2), (4.3), as follows:

$$\begin{aligned}(K_1\sigma_1)(P) &\equiv \iint\limits_{\Delta_{11}} \sigma_1(x, y)\times \\ &\times \sum_{i,j=1}^{2} (K + K_h)\big((-1)^{i-1}x, (-1)^{j-1}y; x_0, y_0\big)\,\mathrm{d}x\mathrm{d}y = F_1, \quad P \in \Delta_{11},\end{aligned} \tag{4.4}$$

$$\begin{aligned}(K_2\sigma_2)(P) &\equiv \iint\limits_{\Delta_{11}} \sigma_2(x, y)\times \\ &\times \sum_{i,j=1}^{2} (K - K_h)\big((-1)^{i-1}x, (-1)^{j-1}y; x_0, y_0\big)\,\mathrm{d}x\mathrm{d}y = F_2, \quad P \in \Delta_{11},\end{aligned} \tag{4.5}$$

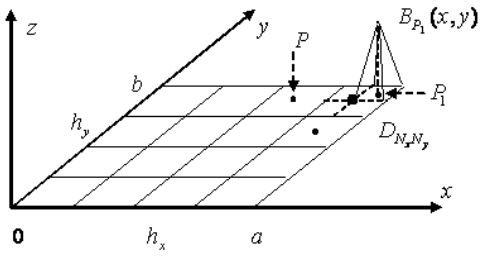

Fig. 9. Choice of P_1 in the neighborhood of S_1 corner point.

Tabl. 7. Potential at certain "intermediate" points of grid on the plate S_1 in the case of piecewise-bilinear bubble function and $F1 = 1$, $F2 = -10$.

N	x	y	u_h	λ_1	ε_e
4	0.9375	0.9375	0.6102	0.5244	0.4225
5	0.9375	0.9375	0.7451	0.6133	0.3867
	0.9500	0.9500	0.6353	0.6133	0.3867
10	0.9375	0.9375	1.0992	1.0171	0.2906
	0.9500	0.9500	1.0000	1.0171	0.2906
	0.9750	0.9750	0.6976	1.0171	0.2906
20	0.9375	0.9375	1.0061	1.7564	0.2198
	0.9500	0.9500	1.0856	1.7564	0.2198
	0.9750	0.9750	1.0000	1.7564	0.2198
	0.9875	0.9875	0.7517	1.7564	0.2198

where σ_1 in (4.4) and σ_2 in (4.5) are narrowing the solutions sought by Δ_{11}. Then, by solving (4.4) and (4.5), we get

$$\tau_1(x,y) = \frac{1}{2}\big(\sigma_1(x,y) + \sigma_2(x,y)\big),$$

$$\tau_2(x,y) = \frac{1}{2}\big(\sigma_1(x,y) - \sigma_2(x,y)\big).$$

Integral equations (4.4) and (4.5) were solved by collocation method using piecewise-constant basis functions. The domain $\Delta_{11} := (0,a) \times (0,b)$ has already been partitioned into some elements. The element $D_{N_xN_y}$ is considered as "extremal" one because it contains a vertex of the plate S_1 (see fig. 9). It is known that functions σ_1 and σ_2 have singularities in the neighborhood of S_1 corner point and near all points which belong to the contour of S_1. These singularities can be described, as a whole, with the help of a function

$$\Omega(x,y) = \frac{\big[(a-x)\,(b-y)\big]^{1/2}}{(a-x)^\gamma + (b-x)^\gamma},$$

where $\gamma \approx 1,7034$ [7]. Making specific changes of variables in appropriate two-dimensional integrals, we can get rid of these singularities. This approach is rather complicated. Hence, it does not allow us to use simple and effective numerically analytical scheme offered by authors. Therefore, ignoring the singularity $\Omega(x,y)$ at the representation of sought solution we must compensate its missing by grid concentration in the neighborhood of the plate corner point. That is why the desired accuracy of numerical results is obtained.

Thus, it is necessary to solve approximately IE (4.4) provided that the preassigned accuracy of sought solution ε is given. Suppose that by solving appropriate system of

linear equations we get $\sigma_{1\varepsilon}(x,y)$. Let P_1 be a point, placed in the neighborhood of S_1 corner point on "extremal" element $D_{N_xN_y} := [a-h_x,\, a] \times [b-h_y,\, b]$. Let us choose the coordinates of P_1 in the following way: $P_1 := \left\{a - \frac{h_x}{4},\, b - \frac{h_y}{4}\right\}$. Then, taking into account the representation of error function $e_{\sigma_1} := \sigma_1 - \sigma_{1\varepsilon}$ as $e_{\sigma_1}(x,y) = \lambda_1 B_{P_1}(x,y)$, we can calculate unknown parameter λ_1 by collocation of (4.4) at the point P_1:

$$\lambda_1 = \frac{F_1 - (K_1\sigma_{1\varepsilon})(P_1)}{(K_1 B_{P_1})(P_1)}.$$

The simplicity of the latter formula is explained by the fact that the support of piecewise-bilinear bubble function $B_{P_1}(x,y)$ is compact:

$$\operatorname{supp} B_{P_1}(x,y) = D^{1/4}_{N_xN_y} := \left[a - \frac{h_x}{2},\, a\right] \times \left[b - \frac{h_y}{2},\, b\right].$$

Now the condensation of grid is connected with verification of condition (3.3). To obtain approximate solution of (4.5) we use a similar procedure. In addition, numerical results of IE (4.5) solution are given in table 7.

5. Conclusion

Thus, we have considered some methods for taking into account "charge distribution density" singularities near the contours of unclosed boundary surfaces. By solving of some model tasks and problems in relation to practice we have shown that the most effective approach is the sequential condensation of grid for IE numerical solution.

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Received 15.01.2009; revised 24.03.2009