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VARIATION-GRADIENT METHOD OF THE SOLUTION OF ONE CLASS OF NONLINEAR MULTIPARAMETER EIGENVALUE PROBLEMS

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Abstract. In the finite-dimensional real Euclidean space the nonlinear generalized spectral problem is put in accordance to the variation problem on the minimum of some functional. The equivalence of spectral and variation problems is proved. On the base of gradient procedure the numerical algorithm of finding of its eigenvalues and eigenvectors is offered. Under certain conditions over the operators the local convergence of method is proved.

1. Introduction

The generalized eigenvalue problems $T(\lambda)x = 0$ with the operator-function $T(\lambda)$: $R^m \to X(H)$ ($X(H)$ is a set of the linear bounded operators in the finite-dimensional real Hilbert space), which linearly or nonlinearly depends on a several spectral parameters $\lambda_1, \lambda_2, \dots, \lambda_m$ arise in many fields of analysis and mathematical physics. In particular, such problems play important role when one investigates the stableness of mechanical and electrodynamic systems, researches the branching and the bifurcation of solutions of the nonlinear integral equations of Hammerstein type. So, when it is necessary to find the branching points of nonlinear integral equations, which we got as a result of solving of the synthesis problem [6], two-parameter eigenvalue problems with nonlinear spectral parameters which are included analytically in the kernel of a linear operator, are arisen. These problems are a not adequately explored both from the theoretical point of view, and from the point of view of the construction of numerical methods of their solution. This, to a considerable degree explains the interest both to different aspects of spectral theory of nonlinear multiparameter eigenvalue problems (see, for example, [1]-[5]) and to the numeral methods of the solution of such problems (see, for example, [7]-[10]).

This paper is generalization of results of the work [11] in the case of the nonlinear eigenvalue problems. The feature of nonlinearity consists in the fact that this problem can generally have not the solutions or, opposite, it have them as a continuum set.

On the given time there are many opened questions related to this problem, for example, such as the existence of solutions and their quantity, the development of numeral methods for solving of such spectral problems for the algebraic, differential and integral equations.

Generally speaking, the nonlinear occurence of spectral parameters in equation $T(\lambda)x = 0$ leads to the continuum solutions of the problem. In this case it can be reduced to the sequential decision of one-parameter problems prescribing definite value to other $n-1$ parameters. So, in particular, for the two-parameter problem the calculation of eigenpairs (λ_1, λ_2) and corresponding eigenvectors can be reduced to the sequential solution of one-parameter nonlinear problems, when one of parameters is set, similarly as it is carried out for a linear two-parameter eigenvalue problem in the paper [12]. On this way we obtain the dependence, for example,$\lambda_1(\lambda_2)$, i.e. the proper curve, that corresponds to the eigenvector.

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†*Key words.* Multiparameter eigenvalue problem, numerical algorithm, gradient method.

If the operator-function $T(\lambda)$ is presented by a block-diagonal matrix obtained as a result of discretization of linked systems of the differential equations, then such problem can also have the finite or the enumerable number of solutions. In this case the numerical computing of the equation $T(\lambda)x = 0$ by the method when for $n-1$ parameters some definite values are prescribed, and one-parameter eigenvalue problem are solved, is not effective. Instead of this, it is expediently to do iterations regarding to all variables simultaneously assuming $(\lambda_1, \lambda_2, \dots, \lambda_m)$ as a vector.

In this paper exactly such class of problems, for which the multiparameter eigenvalue problem is substituted by an equivalent variational problem of minimization of some quadratic functional, is considered. In the basis of numerical algorithm of minimization of functional the variant of gradient procedure lies as the method of the numerical finding of eigenvector. The set of eigenvalues of the problem is determined from the system of nonlinear algebraic equations, constructing by the found approximation to the eigenvector.

Let $H = R^n$ be the real Euclidean space with the scalar product $(\cdot,\cdot)$ and the norm $\|\cdot\|$ and let $T(\lambda): R^m \to X(H)$ be the matrix with the elements nonlinearly depending on $\lambda_1, \lambda_2, \dots, \lambda_m$. Assume that the matrix $T(\lambda)$ is twice differentiable by Frechet, i.e. for any $\lambda_k \in R$, $k = 1, 2, \dots, m$, there are the partial derivatives $\frac{\partial T(\lambda)}{\partial \lambda_k}$, $k = 1, 2, \dots, m$, and $\frac{\partial^2 T(\lambda)}{\partial \lambda_k \partial \lambda_l}$, $k, l = 1, 2, \dots, m$.

Nonlinear multiparameter eigenvalue problems consist in finding of such set of spectral parameters $\lambda^* = \{\lambda_1^*, \dots, \lambda_m^*\}$, for which the nontrivial solution $x \neq 0$ of equation

$$T(\lambda)x = 0 \tag{1.1}$$

exists. We will name such set of spectral parameters $\lambda^* = \{\lambda_1^*, \dots, \lambda_m^*\}$ by generalized eigenvalue or eigen set, and the corresponding vector $x \in H$ we will name the generalized eigenvector of the problem (1.1).

2. Eigenvectors as the points of minimum

We consider now the problem of finding of such set of parameters $\lambda(x) = \{\lambda_1(x), \dots, \lambda_m(x)\}$ and such vectors x on which functional

$$F(x) = \frac{1}{2}\| T(\lambda(x))x \|^2, \quad \forall x \in H\backslash\{0\} \tag{2.1}$$

reaches its minimum value, i.e.

$$F(x) \to \min, \quad x \in U \subset H = E^n, \tag{2.2}$$

where U is a some convex set in E^n.

A points set of minimum of $F(x)$ on U we will denote as

$$U_* = \{x: \ x \in U, \quad F(x) = 0\}.$$

We will prove the equivalence of the problems (1.1) and (2.2).

Consider the increment of functional $F(x+h) - F(x)$ for arbitrary x, $x+h \in U$ where U is some convex set in H. After simple transformations we obtain that

$$F(x+h) - F(x) = (\,T(\lambda)x\,,\ [T(\lambda)h + dT(\lambda,h)x])\, +$$

$$+\ \frac{1}{2}\{(T(\lambda)h, T(\lambda)h) + 2(T(\lambda)h, dT(\lambda,h)x) + (dT(\lambda,h)x, dT(\lambda,h)x) +$$

$$+\,2\,(T(\lambda)x, dT(\lambda,h)h) + (T(\lambda)x, d^2T(\lambda,h)x)\,\} + o(\|h\|^2), \tag{2.3}$$

where

$$dT(\lambda,h)=\frac{\partial T(\lambda)}{\partial\lambda_1}d\lambda_1(x,h)+\frac{\partial T(\lambda)}{\partial\lambda_2}d\lambda_2(x,h)+\ \dots\ +\frac{\partial T(\lambda)}{\partial\lambda_m}d\lambda_m(x,h). \qquad (2.4)$$

Since the expression for the second differential $d^2T(\lambda,h)$ is bulky and it will not be used below, it is not pointed here. Consequently, the first differential of $F(x)$ will be written as

$$dF(x,h)=(\,T(\lambda)x\,,\ [T(\lambda)h\ +dT(\lambda,h)x\,])\ =$$
$$=(\,T(\lambda)x\,,\ T(\lambda)h\,)+(T(\lambda)x\,,\ dT(\lambda,h)x\,). \qquad (2.5)$$

Taking into account (2.4), the second component in (2.5) is present as

$$(T(\lambda)x\,,\ dT(\lambda,h)x\,)=$$
$$=(T(\lambda)x,\ \sum_{i=1}^{m}d\lambda_i(x,h)\frac{\partial T(\lambda)}{\partial\lambda_i}x)=(T(\lambda)x,\ \sum_{i=1}^{m}d\lambda_i(x,h)B_ix),$$

where $B_i=\frac{\partial T(\lambda)}{\partial\lambda_i}$, $i=1,\,2,\,\dots,\ m$, and we will consider it as the system of the nonlinear algebraic equations

$$f_i(\lambda)\equiv(T(\lambda)x,\ B_i(\lambda)x)=0,\quad i=1,\,2,\ \dots\ ,\ m \qquad (2.6)$$

for determination $\lambda(x)=\{\lambda_1(x),\lambda_2(x),\ \dots\ ,\lambda_m(x)\}$ for any fixed value of the vector x.

It follows that

$$dF(x,h)=(T(\lambda)x\,,\ T(\lambda)h\,).$$

Therefore, for the gradient of functional (2.1) we get

$$grad\,F(x)\ \equiv\ \nabla F(x)=T^*(\lambda)T(\lambda)x.$$

Thus,

$$(\nabla F(x),x)\ =\ 2F(x),$$

so, such assertion is satisfied.

Lemma 2.1 *Let $\lambda(x)\ =\{\,\lambda_1(x),\ \dots\ ,\ \lambda_m(x)\,\}$ be the solution of the system of nonlinear equations (2.6). Then every eigenvector of problem (1.1) is the stationary point of functional (2.1) and conversely, every stationary point of functional (2.1) is the eigenvector of problem (1.1).*

Lemma 2.2 *The functional (2.1) is a convex on a stationary set.*

Proof. In accordance with definition (see, for example, [13, p.88]) from the formula (2.3) for the second differential of functional (2.1) we have

$$d^2F(x,h)\equiv(F''(x)h,\ h)=\|\ T(\lambda)h\ +\ dT(x,h)x\|^2+$$
$$+(T(\lambda)x,dT(\lambda,h)h)+(T(\lambda)x,d^2T(\lambda,h)x), \qquad (2.7)$$

where $F''(x)$ is a matrix of second derivatives. Since on the set of the stationary points $T(\lambda)x=0$ then from (2.7) it follows

$$d^2F(x,h)\equiv(F''(x)h,\ h)=\|\ T(\lambda)h\ +\ dT(x,h)x\|^2\geq 0, \qquad (2.8)$$

i.e. the functional $F(x)$ is a convex one [13, p.173]. Lemma is proved.□

Now on the basis of the lemma 2.1 and lemma 2.2 such assertion is confirmed.

Theorem 2.3 *Every eigenvector of problem (1.1) is the point of minimum of functional (2.1) and conversely, every point of minimum of functional (2.1) is the eigenvector of problem (1.1).*

Proof. From lemma 2.1 follows, that every eigenvector of the problem (1.1) is the stationary point of functional (2.1) and conversely. We will show now, that the stationary point is the minimum of functional (2.1). Indeed, let $\lambda^* = \{\lambda_1^*, ..., \lambda_m^*\}$ and x^* are the eigen set and eigenvector of the problem (1.1). Then $T(\lambda^*)x^* = 0$ and from the formula of the finite increments for $F(x)$

$$F(x^* + h) - F(x^*) = (\nabla F(x^*),\, h) + \frac{1}{2}(F''(x^*)h,\, h) + \alpha(h, x),$$

where $\alpha(h,x)/\|h\|^2 \to 0$ for $\|h\| \to 0$, taking into account the equality $\nabla F(x^*) = 0$ (lemma 2.1) and inequality (2.8) (lemma 2.2) we get that $F(x^* + h) - F(x^*) \geq 0$, i.e.

$$F(x^* + h) \geq F(x^*).$$

It means that x^* is the point of minimum of $F(x)$. Theorem is proved.□

Thus, solving of problem (1.1) is equivalent to finding the stationary points of functional (2.1) which are its points of minimum.

3. Numerical algorithm

This result allows to construct the gradient procedure as a method of the numerical solution of the problem (1.1), when for the given value of vector x_k the corresponding value $\lambda_{(k)} \equiv \lambda(x_k) = col\{\lambda_i(x_k)\}_{i=1}^m$ is finding as the solution of the system of nonlinear algebraic equations (2.6), and a next approximation to the eigenvector is searched as

$$x_{k+1} = x_k - \gamma(x_k)\nabla F(x_k)\ , \quad k = 0,\ 1,\ 2,\ .... \tag{3.1}$$

Here a constant $\gamma_k = \gamma(x_k)$ at every step is determined from the condition of minimum of functional (2.1) in direction of its gradient $\nabla F(x_k)$. As long as

$$F(x_{k+1}) = \frac{1}{2}\left\|T(\lambda_{(k)})x_{k+1}\right\|^2 = \frac{1}{2}(T(\lambda_{(k)})x_k, T(\lambda_{(k)})x_k) -$$

$$-\gamma_k(T(\lambda_{(k)})x_k, T(\lambda_{(k)})\nabla F(x_k)) + \frac{1}{2}\gamma_k^2(T(\lambda_{(k)})\nabla F(x_k), T(\lambda_{(k)})\nabla F(x_k))$$

from the necessary condition of minimum of functional $\frac{\partial F}{\partial \gamma} = 0$ we find that

$$\gamma(x_k) = \frac{(\nabla F(x_k), \nabla F(x_k))}{(T(\lambda_{(k)})\nabla F(x_k), T(\lambda_{(k)})\nabla F(x_k))} = \frac{\|\nabla F(x_k)\|^2}{\left\|T(\lambda_{(k)})\nabla F(x_k)\right\|^2}.$$

Consequently, the iterative process will be realized by means the formulas

$$y_{k+1} = x_k - \gamma(x_k)\nabla F(x_k)\ , \quad k = 0,\ 1,\ 2,\ ... \tag{3.2}$$

$$x_{k+1} = y_{k+1}/||\, y_{k+1}\, || \tag{3.3}$$

$$\gamma(x_k) = \begin{cases} \frac{\|\nabla F(x_k)\|^2}{\|T(\lambda_{(k)})\nabla F(x_k)\|^2}, & if\ \nabla F(x_k) \neq 0, \\ 0\ , & otherwise\,, \end{cases} \tag{3.4}$$

i.e. the solution is seached in the class of the normalized vectors, and the system of nonlinear equation (2.6) is solved by one of the known methods. In particular, the iterative process regarding to the eigenvalue can be realized by means of Newton's method, i.e. for each $k = 0, 1, ...$

$$\lambda^{(l)}(x_k) = \lambda^{(l-1)}(x_k) - \Big[J(\lambda^{(l-1)}(x_k), x_k)\Big]^{-1} f(\lambda^{(l-1)}(x_k), x_k), \ l = 1, 2, ..., \tag{3.5}$$

where the superscript (l) is the number of iteration of Newton's method,

$$J(\lambda, x) = matr\left\{\frac{\partial f_i(\lambda, x)}{\partial \lambda_j}\right\}_{i,j=1}^{m}, \tag{3.6}$$

$$f(\lambda, x) = col\{f_i(\lambda, x)\}_{i=1}^{m}.$$

Since $f_i(\lambda, x) = (T(\lambda)x,\ B_i(\lambda)x)$, $i = 1,\ 2,\ \dots,\ m$, for the elements of matrix (3.6) we get such expressions

$$\frac{\partial f_i(\lambda, x)}{\partial \lambda_j} = (T(\lambda)x, \frac{\partial^2 T(\lambda)}{\partial \lambda_i \partial \lambda_j} x) + (\frac{\partial T(\lambda)}{\partial \lambda_j} x, \frac{\partial T(\lambda)}{\partial \lambda_i} x) \quad i,\ j = 1,\ \dots,\ m \tag{3.7}$$

or

$$\frac{\partial f_i(\lambda, x)}{\partial \lambda_j} = (T(\lambda)x, \frac{\partial^2 T(\lambda)}{\partial \lambda_i \partial \lambda_j} x) + (B_j(\lambda)x, B_i(\lambda)x), \quad i,\ j = 1,\ \dots,\ m.$$

Consequently, the algorithm of finding the eigenvector x and eigen set $\lambda(x) = \{\lambda_1(x),\ \dots,\ \lambda_m(x)\}$ of problem (1.1) can be construct in a form

Algorithm.

1. Start with the initial approximations $\lambda^{(0)} = (\lambda_1^{(0)}, ..., \lambda_m^{(0)})$ and x_0 to the eigenvalue and eigenvector of the problem (1.1)
2. **for** k=1,2 … until convergence **do**
3. **for** l=1,2 … until convergence **do**
4. Calculate the elements of matrix (3.6)

$$J_{ij} = (T(\lambda^{(l-1)})x_{k-1}, \frac{\partial^2 T(\lambda^{(l-1)})}{\partial \lambda_i^{(l-1)} \partial \lambda_j^{(l-1)}} x_{k-1}) + (\frac{\partial T(\lambda^{(l-1)})}{\partial \lambda_j^{(l-1)}} x_{k-1}, \frac{\partial T(\lambda^{(l-1)})}{\partial \lambda_i^{(l-1)}} x_{k-1}),$$

$$i, j = 1,\ \dots,\ m$$

and the elements of vector $f(\lambda^{(l-1)}, x_{k-1}) = (f_1,\ \dots, f_m)$:

$$f_i = (T_n(\lambda^{(l-1)})x_{k-1}, \frac{\partial T_n(\lambda^{(l-1)})}{\partial \lambda_i^{(l-1)}} x_{k-1}), \quad i, j = 1,\ \dots,\ m,$$

5. By elements J_{ij} and f_i we build a matrix (3.6) $J(\lambda^{(l-1)}, x_{k-1})$ and the vector $f(\lambda^{(l-1)}, x_{k-1})$ and then calculate the next approximation for $\lambda^{(l)}(x_{k-1})$ by the formula (3.5)
6. **end for** l
7. Calculate the gradient $\nabla F(\lambda(x_{k-1}), x_{k-1}) = T^*(\lambda(x_{k-1}))T(\lambda(x_{k-1}))x_{k-1}$
8. **if** $\nabla F(\lambda(x_{k-1}), x_{k-1}) = 0$ **go to 10**
9. Calculate the next approximation x_k by the formulas (3.2)-(3.4)
10. **end for** k.

When we choice of the initial approximation in definite sense of nearness to the eigenvector, iterative process (3.2)-(3.4) converges to the stationary points of functional (2.1), in which its minimum is achieved that is to the eigenvector x^* of the problem (1.1).

Thus, for the proposed above iterative process such theorem of local convergence is satisfied.

Theorem 3.1 *Let U be some closed neighborhood of eigenspace N_λ of the problem (1.1) ($U \supset N_\lambda$) and for some initial approximation $x_0 \in U$ the Lebesgue set*

$$M(x_0) = \{x : \; x \in U, \quad F(x) \le F(x_0)\} \tag{3.8}$$

is a bounded set, moreover, let the gradient of functional $F(x)$ satisfy the condition

$$\|F'(x) - F'(y)\| \le L\|x - y\|, \quad \forall x, y \in U, \;\; L > 0 \tag{3.9}$$

and the matrix (3.6), with the elements calculated by the formula (3.7), is nonsingular one. Then for the sequence $\{x_k\}$ obtained by the iterative process (3.2)-(3.4) the correlations

$$\lim_{k\to\infty} F(x_k) = 0, \quad \lim_{k\to\infty} \rho(x_k, U_*) = \lim_{k\to\infty} \rho(x_k, N_\lambda) = \lim_{k\to\infty} \rho(x_k, x^*) = 0$$

are satisfied. It means that iterative process (3.2)-(3.4) converges to the point of minimum of functional (2.1), i.e. it converges to the eigenvector x^* of the problem (1.1).

Proof. Assume that $\nabla F(x_k) \ne 0$, for $k = 0,\ 1,\ \dots$. In other case, if for some $k \ge 0$ will appear that $\nabla F(x_k) = 0$ then from correlations (3.2)-(3.4) we get nominally that

$$x_k = x_{k+1} = \dots$$

and assertion of theorem is satisfied. Consequently, as

$$F(x_{k+1}) = F(x_k - \gamma_k F'(x_k)) \le \inf_{\gamma \ge 0} (F(x_k - \gamma F'(x_k))) \le F(x_k - \gamma F'(x_k))$$

for any $\gamma \ge 0$ then

$$F(x_k) - F(x_{k+1}) \ge F(x_k) - F(x_k - \gamma F'(x_k)).$$

Now, on the basis of inequality

$$|F(x) - F(y) - (F'(y), x - y)| \le L\|x - y\|^2/2$$

which follows from condition (3.9) for the gradient of functional $F(x)$, for $y = x_k$, $x = x_{k+1} = x_k - \gamma F'(x_k)$ we obtain that

$$F(x_k) - F(x_{k+1}) \ge \gamma(1 - L\gamma/2)\|F'(x_k)\|^2$$

for any γ, $k = 0,\ 1,\ \dots$. Thus,

$$F(x_k) - F(x_{k+1}) \ge \max\left(\gamma(1 - L\gamma/2)\right) \cdot \|F'(x_k)\|^2 = \frac{1}{2L}\|F'(x_k)\|^2,$$

from where

$$F(x_{k+1}) \le F(x_k) - \|F'(x_k)\|^2/2L,$$

that is

$$F(x_{k+1}) \le F(x_k). \tag{3.10}$$

Summing the inequality (3.10) over k from 0 to $n-1$, we obtain

$$F(x_n) \le F(x_0), \quad n = 1,\ 2,\ \dots,$$

i.e. the sequence $\{x_k\}$ belongs to the Lebesgue set

$$M(x_0) = \{\, x \in U \;:\; F(x) \le F(x_0)\},$$

which under the condition of theorem is the bounded set, and consequently, the points set of minimum of functional $F(x)$ is not empty and consists of only one point x^*, to which the sequence obtained by (3.2)-(3.4) with the initial approximation $x_0 \in U$ converges (see theorem 1 from [13, p.186]). The theorem is proved.□

4. Some remarks and conclusions

1. If the matrix $T(\lambda)$ linearly depends on the parameters $\lambda_1, \dots, \lambda_m$, i.e. $T(\lambda) = A + \sum_{i=1}^{m} \lambda_i B_i$, where $A, B_i : H \to H$, $i = 1, \dots, m$, are some matrices then $\frac{\partial^2 T(\lambda)}{\partial\lambda_i \partial\lambda_j} x = 0$, matrix (3.6), and the elements of vector $f(\lambda, x) = col\{f_i(\lambda, x)\}_{i=1}^m$ will take the form

$$J(\lambda, x) = matr\{(B_j x, B_i x)\}_{i,j=1}^m, \tag{4.1}$$

$$f_i(\lambda, x) = (Ax,\ B_i x) + \sum_{j=1}^{m} \lambda_j (B_j x, B_i x), \quad i = 1, 2, \dots, m. \tag{4.2}$$

In this case the iterative process (3.5) will be reduced to the solution of the linear equations system. Indeed, substituting the matrix (4.1) and vector (4.2) in (3.5), we obtain

$$\lambda^{(l)}(x_k) = \lambda^{(l-1)}(x_k) - \Big[matr\{(B_j x_k, B_i x_k)\}_{i,j=1}^m\Big]^{-1} \alpha(x_k) -$$

$$-\lambda^{(l-1)}(x_k)\Big[matr\{(B_j x_k, B_i x_k)\}_{i,j=1}^m\Big]^{-1} matr\{(B_j x_k, B_i x_k)\}_{i,j=1}^m,$$

i.e.

$$\lambda(x_k) = -\Big[matr\{(B_j x_k, B_i x_k)\}_{i,j=1}^m\Big]^{-1} \alpha(x_k), \quad k = 0, 1, \dots,$$

where $\alpha(x_k) = col\{(Ax_k, B_i x_k)\}_{i=1}^m$. It means that $\lambda(x_k) = col\{\lambda_i(x_k)\}_{i=1}^m$ is the solution of the linear system

$$matr\{(B_j x_k, B_i x_k)\}_{i,j=1}^m \lambda(x_k) = -\alpha(x_k), \tag{4.3}$$

for each x_k, $k = 0, 1, \dots$ [11].

If matrix $T(\lambda)$ linearly depends of one parameter λ ($m = 1$), i.e. $T(\lambda) = A + \lambda B$, then from (4.3), and also directly from (2.6) follows, that for calculation of $\lambda(x)$ we get the classic Rayleight ratio

$$\lambda(x) = -(Ax, x)/(Bx, x).$$

From this follows that the nonlinear system of equations (2.6) can be considered as generalization of classic Rayleight functional over multiparameter spectral problems.

2. It is possible the other statement of the variation problem, when the set of spectral parameters $\lambda = \{\lambda_1, \dots, \lambda_m\}$ and vector x are considered as independent variables on which functional

$$F(u) = \frac{1}{2}\| T(\lambda)x \|^2, \quad \forall\, u = \{x, \lambda\} \in H = R^n \backslash \{0\} \oplus R^m, \tag{4.4}$$

acquires the minimum value, i.e.

$$F(u) \to \min, \quad u \in U \subset H$$

where U is a convex set from H in which scalar product and norm are determined by a standard way.

If one consider the increment of the functional $F(u + \Delta u) - F(u) = F(x + h, \lambda + q) - F(x, \lambda)$, for any u, $u + \Delta u \in U$, then simple transformations we obtain that

$$F(u + \Delta u) - F(u) = F(x + h, \lambda + q) - F(x, \lambda) =$$

$$= (\ T(\lambda)x\,,\ T(\lambda)h\,) + (T(\lambda)x, \sum_{i=1}^{m} \frac{\partial T(\lambda)}{\partial\lambda_i} x q_i) \ + \frac{1}{2}\{\ (T(\lambda)h, T(\lambda)h) +$$

$$+2(T(\lambda)h, \sum_{i=1}^{m} \frac{\partial T(\lambda)}{\partial \lambda_i} x q_i) + (\sum_{i=1}^{m} \frac{\partial T(\lambda)}{\partial \lambda_i} x q_i, \sum_{i=1}^{m} \frac{\partial T(\lambda)}{\partial \lambda_i} x q_i) +$$

$$+2(T(\lambda)x, \sum_{i=1}^{m} \frac{\partial T(\lambda)}{\partial \lambda_i} h q_i) + (T(\lambda)x, d^2 T(\lambda, q)x) \Bigg\} + o(\|\Delta u\|^2),$$

Thus, the first differential of functional (4.4) will acquire a form

$$dF\{(x,\lambda);(h,q)\} = (T(\lambda)x, T(\lambda)h) + \sum_{i=1}^{m} (T(\lambda)x, B_i(\lambda)x) q_i =$$

$$= (T(\lambda)x, T(\lambda)h) + (f(\lambda,x), q),$$

where $f(\lambda, x)$ is the vector with components $\{(T(\lambda)x, B_1(\lambda)x), \dots, (T(\lambda)x, B_m(\lambda)x)\}$, from this for the gradient of functional (4.4) we get the form

$$grad\, F(x,\lambda) \equiv \nabla F(x,\lambda) = \{T^*(\lambda)T(\lambda)x, f(\lambda,x)\}. \tag{4.5}$$

Consequently, these two approaches are differed by construction of algorithm of finding of the eigen set and the eigenvector. For the first approach considered in this paper, it is needed the solution of nonlinear with regard to λ system of equations (2.6), i.e. in the algorithm it is needed to apply the iterative Newton's procedure (3.5) with the initial approximation x_0 for the eigenvector. For the second approach, the solution of the system (2.6) is not needed. Therefore, the iterative process (3.5) is not needed, but, from other side, it is needed to have not only the initial approximation x_0 for the eigenvector, but also the initial approximation λ_0 for the corresponding eigenvalue. Construction and justification of the algorithm, using the gradient in a form (4.5) are the objects of the separate consideration and research.

3. In the offered algorithm the constant γ_k at every step is determined from the condition of minimum of functional in direction of gradient. There are possible other methods of choice of the value, as by correlations of type (10)

$$x_{k+1} = x_k - \gamma_k p_k \quad k = 0, 1, \dots \tag{4.6}$$

the whole class of methods differing by the choice of direction of descent p_k and the value of step γ_k is set. It is marked in work [14, p. 253], that in the case when the some functional $g(x): R^n \to R^1$ satisfies the condition $g(x) \geq 0$, for all $x \in R^n$ the value of γ_k can be calculated by one step of Newton's iteration for the scalar equation $g(x_k - \gamma_k p_k) = 0$, i.e.

$$\gamma_k = g(x_k)/g'(x_k)p_k.$$

For $p_k = \nabla g(x_k)^T$ the iterative process (4.6) will take a form

$$x_{k+1} = x_k - [g(x_k)/\|\nabla g(x_k)\|^2]\nabla g(x_k)^T, \quad k = 0, 1, \dots.$$

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