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A REVIEW OF FUNCTIONAL-DISCRETE TECHNIQUE FOR EIGENVALUE PROBLEMS

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V. L. MAKAROV AND N. O. ROSSOKHATA

Abstract. A review of the functional-discrete approach for both linear and nonlinear eigenvalue problems is given. The convergence results are established. The approach is applied to study solutions of the original problems by using the solutions of simpler basic problems. The numerical results are in a good agrement with theoretical ones.

1. Introduction

The known numerical techniques for eigenvalue problems can be divided into two groups: methods based on the direct approximation of the solution of differential equation and methods based on the approximation of the coefficients of differential equation. Both of them have their areas of applicability and hence advantages and drawbacks.

The methods of the first group such as finite difference [1–5], finite elements [6–10] and spectral [11, 12] are extensively treated through both theoretical investigations and well developed soft. The main idea of this approach is replacement of the eigenfunctions by piecewise polynomial functions, which results in the approximation of a differential equation by a system of linear algebraic equations. Under such approach the unbounded differential operator is approximated by the sequence of bounded algebraic operators, that is the eigenvalue problem with infinite spectrum is reduced to the eigenvalue problem with finite spectrum. Hence, because of the different nature of the original differential operator and approximating algebraic operators the numerical solution comes close to the exact one, but may not reflect its physical properties adequately, and so may not be physically realistic. The principal peculiarity of the first approach is the capability to calculate only a restricted number of the first low indexed eigenvalues and corresponding eigenfunctions, that is defined by the size of the grid step. That is why such numerical techniques are effective to find the low-indexed eigenvalues, but not effective to find the high-indexed eigenvalues. The convergence rate also depends on the index of trial eigenvalue and becomes worth as the index increases. Besides, the finite elements and finite difference methods have accuracy saturation [11].

The piece-wise constant and piece-wise polynomial approximation of the coefficients of differential equation, named by Pruess methods, belong to the second group. Firstly this approach was proposed by Kryloff in [13] in 1926 for piece-wise constant approximation and then developed for piece wise polynomials of higher degree [14–21]. The main idea of this approach consists in approximation of the coefficients of differential equation by piece-wise constant, piece-wise linear or piece-wise quadratic functions. The result of such approximation is the capability to write the general solution explicitly on each interval, where the coefficients are polynomials, and to find arbitrary parameters from the matching conditions on the ends of intervals. Under such approximation the unbounded differential operator is replaced by the sequence of the unbounded differential operators, that is, we

† *Key words.* Eigenvalue transmission problem, nonlinear eigenvalue problem, functional-discrete method, Adomian polynomials, basic problem, exponential convergence rate.

get the similar eigenvalue problem with infinite spectrum. Since the nature of the original and approximating operators is the same, the numerical solution tends to and has the same physical properties as the exact one, so the numerical solution is more physically realistic. Contrary to the methods of the first group, there is no restrictions on the number of computable eigenvalues and corresponding eigenfunctions. So the techniques based on such approach are capable to find numerical solution with any desired index and with any desired accuracy. However, in order to find solution of the approximate differential equation in terms of special functions, the degree of the polynomials should be no higher than two [17]. The better accuracy is achieved by the refinement of the interval. That is why this approach is effective for piece-wise smooth coefficients, but not effective for many cases of fast changing coefficients.

A functional-discrete method (FD-method), firstly proposed by V.L. Makarov in [22], is the further development of the differential equation approximation approach and consists of imbedding of the original problem into a parametric set of problems with respect to parameter t in such way that we have a linear eigenvalue problem with constant coefficients for $t = 0$ and the original problem for $t = 1$. The transition from $t = 0$ to $t = 1$ using the Taylor series results in a recurrence algorithm. Thus, for zero iteration we apply Pruess method with piece-wise constant approximation of the differential equation coefficients and then solve obtained in such way basic problem. As a result we receive course approximate solution. For each next iteration step we find a corrections for course solution by solving a boundary value problem for second order differential equation with piecewise constant coefficients and a right hand part constructed by using solutions at previous iterations. So, contrary to Pruess method, in order to improve the accuracy, one does not need to refine the grid, but to repeat the recursion. It provides an exponential convergence rate. The technique is developed in [23, 24, 24–30] for Sturm-Liouville problems with different boundary conditions and potential from $L_\infty(0, 1)$. Works [31–33] are devoted to the investigation of the linear eigenvalue problem with transmission conditions. In [34,35] the approach is extended for the case of potential from $L_1(0, 1)$.

In spite of the numerous publications on the numerical solution of linear eigenvalue problems, to the author's knowledge there are far fewer publications on nonlinear eigenvalue problems (see e.g. [36–42] and the references therein). The reason for this is that the numerical techniques require special attention to approximate their nonlinear parts. One such approach is the Adomian decomposition method proposed in [43] and developed in [44–55]. The technique provides an analytical approximation for quite wide class of nonlinear equations without linearization, perturbation, closure approximation or discretization and doesn't require massive calculations. The exact solution is an infinite series of special polynomials called the Adomian polynomials. The numerical solution, that is the truncated series, has the same physical properties as exact one, so physically is realistic. The numerical experiments have shown exponential convergence of the method. Since both FD-method and the Adomian decomposition technique are based on the same idea of analytical approximation of exact solution as a truncated series and have the same exponential convergence rate, the natural combination of functional-discrete approach for eigenvalue problems with Adomian polynomials for approximation of nonlinear term is explored to solve nonlinear eigenvalue problems in [56–58]. It possesses an important advantage, that the developed algorithm for nonlinear eigenvalue problems is parallelizable and converges with the same (exponential) characteristics as the FD-algorithms for linear problems. The drawback of this approach for nonlinear problems compare to the linear problems is that we can only guarantee a priori its (exponential) convergence starting from some index defined by the convergence condition rather than for all indexes. The idea how to overcome this drawback is proposed and discussed in [59]. Results of numerical experiments supporting this idea are also presented in [59]. Theoretical proofs and further development of this idea is a subject for future research.

Transmission problem for elliptic equations, that is boundary-value problem for two domains in Euclidean n-dimensional space with a portion of their boundaries in common, was posed by M.M. Picone in 1954 [60] for the case of continuous solutions and discontinuous flux at interface points, and then investigated in [61–63]. Then the transmission problem with discontinuous solutions at interface points was studied in [64], existence-uniqueness and smoothness results were generalized for m-order elliptic equations and general boundary conditions. Eigenvalue transmission problem for one-dimensional case was considered in [65–67].

The paper is organized as follows. In Section 2 we describe the FD-method and present the convergence results. Section 3 is devoted to linear eigenvalue transmission problem and numerical technique for the case of potential from $L_1(0,1)$. In Section 4 we present the main results on the nonlinear eigenvalue problems with differential and integral normalizing conditions, including the problems with transmission conditions. In Section 5 we make conclusions about our approach.

2. Functional-discrete approach

The general approach is described and convergence results are proven in [22]. In the Hilbert space H let us consider the following eigenvalue problem

$$(\mathbf{A}+\mathbf{B})u-\lambda u=\theta, \tag{2.1}$$

where $\mathbf{A}=\mathbf{A}^*>\gamma\mathbf{I}$, $\overline{D(\mathbf{A})}=H$, θ is a zero element of the Hilbert space H.

The idea of the approach is to introduce the operator-valued function

$$\mathbf{W}(t)=\overline{\mathbf{B}}+t(\mathbf{B}-\overline{\mathbf{B}}), \qquad t\in[0,1],$$

and imbed equation (2.1) into the parametrical problem set

$$[\mathbf{A}+\mathbf{W}(t)]u(t)-\lambda(t)u(t)=\theta \tag{2.2}$$

with $t>0$.

It is obviously that for $t=1$ the solution of problem (2.2) $(\lambda(1),u(1))$ coincides with the solution of problem (2.1) (λ,u). Hence, we can write the exact solution of problem (2.1) as the Taylor expansion.

$$\lambda_n=\lambda_n(1)=\sum_{j=0}^{\infty}\lambda_n^{(j)}=\overset{m}{\lambda}_n+R_m(\lambda_n), \qquad u_n=u_n(1)=\sum_{j=0}^{\infty}u_n^{(j)}=\overset{m}{u}_n+R_m(u_n), \tag{2.3}$$

where

$$\lambda_n^{(j)}=\frac{1}{j!}\frac{d^j\lambda_n}{dt^j}\bigg|_{t=0} \quad\text{and}\quad u_n^{(j)}=\frac{1}{j!}\frac{d^ju_n}{dt^j}\bigg|_{t=0}.$$

Hence

$$\overset{m}{\lambda}_n=\sum_{j=0}^{m}\lambda_n^{(j)}, \qquad \overset{m}{u}_n=\sum_{j=0}^{m}u_n^{(j)} \tag{2.4}$$

is an approximate solution of range m. To find the terms of series (2.3), from (2.2) we get a recursive sequence of equations

$$(\mathbf{A}+\overline{\mathbf{B}})u_n^{(j+1)}-\lambda_n^{(0)}u_n^{(j+1)}=-(\mathbf{B}-\overline{\mathbf{B}})u_n^{(j)}+\sum_{s=0}^{j}\lambda_n^{(j+1-s)}u_n^{(s)}\equiv F_n^{(j+1)} \tag{2.5}$$

equipped with additional normalizing condition, for example,

$$(u_n^{(j+1)}, u_n^{(0)}) = 0, \qquad j = 0, 1, 2, \dots$$

$\lambda_n^{(j+1)}$ comes from the solvability condition for nonhomogeneous problem (2.5) $(F_n^{(j+1)}, u_n^{(0)}) = 0$, that is :

$$\lambda_n^{(j+1)} = ((\mathbf{B} - \overline{\mathbf{B}})u_n^{(j)}, u_n^{(0)}), \qquad j = 0, 1, 2, \dots \tag{2.6}$$

The initial value $(\lambda_n^{(0)}, u_n^{(0)})$ for the recursive process (2.5) is a solution of the basic problem:

$$(\mathbf{A} + \overline{\mathbf{B}})u_n^{(0)} - \lambda_n^{(0)} u_n^{(0)} = 0, \tag{2.7}$$
$$\|u_n^{(0)}\| = 1.$$

The following convergence result for the case of simple eigenvalues of the basic problem (2.7) is proven in [24].

Theorem 2.1 *([24]). Let* $\mathbf{A} = \mathbf{A}^* \geq \gamma \mathbf{I}$, $\gamma > 0$, $\overline{D(\mathbf{A})} = H$, $\mathbf{B} = \mathbf{B}^* \geq 0$, $\overline{\mathbf{B}} = \overline{\mathbf{B}}^* \geq 0$. *Assume that the eigenvalues of the basic problem (2.7) all are simple, i.e.*

$$0 < \lambda_1^{(0)} < \lambda_2^{(0)} < \dots,$$

and the associated eigenfunctions $\{u_n^{(0)}\}_{n=0}^{\infty}$ *constitute the orthonormal basis in* H. *If the inequality*

$$q_n = 4M_n \|\mathbf{B} - \overline{\mathbf{B}}\| < 1,$$

where

$$M_n = \max\left\{ \frac{1}{\lambda_n^{(0)} - \lambda_{n-1}^{(0)}}, \frac{1}{\lambda_{n+1}^{(0)} - \lambda_n^{(0)}} \right\},$$

is satisfied, then series (2.4) exponentially converges to the corresponding solution (λ_n, u_n) *of the problem (2.1) with the estimates*

$$\|u_n - \overset{m}{u}_n\| \leq \alpha_{m+1} \frac{q_n^{m+1}}{1 - q_n}, \qquad |\lambda_n - \overset{m}{\lambda}_n| \leq \|\mathbf{B} - \overline{\mathbf{B}}\| \alpha_m \frac{q_n^m}{1 - q_n},$$

where $\alpha_m = 2\frac{(2m-1)!!}{(2m+2)!!}$.

The convergence result is generalized for the case of multiply eigenvalues of the basic problem in [23].

3. Linear eigenvalue problems with transmission conditions

Here we discuss the results on numerical treatment of linear eigenvalue transmission problem potential from space L_∞ [31–33] and eigenvalue problems with potential from space L_1 [34, 35].

Let us consider the linear eigenvalue problem

$$\frac{d^2 u_i}{dx^2}(x) + (\lambda - q(x))u_i(x) = 0, \qquad q \geq 0, \quad q \in L_\infty(0, 1), \tag{3.1}$$

$$x \in \Omega_i, \qquad \Omega_1 = (0, x^{(1)}), \quad \Omega_2 = (x^{(1)}, 1)$$

with transmission conditions at matching point $x^{(1)}$

$$\left.\frac{du_i(x)}{dx}\right|_{x=x^{(1)}} = r[u(x^{(1)})], \quad r>0, \qquad i=1,2, \tag{3.2}$$

where $[f(x^{(1)})] = f_2(x^{(1)}) - f_1(x^{(1)})$ is a jump of function at point $x^{(1)}$. We write the boundary conditions in general form

$$\alpha_i u_1(0) + \beta_i u_1'(0) + \gamma_i u_2(1) + \delta_i u_2'(1) = 0, \qquad i=1,2. \tag{3.3}$$

3.1. Numerical algorithm

By introducing the Hilbert spaces $L_2 = L_2(\Omega_1)\times L_2(\Omega_2)$ i $W_2^1 = W_2^1(\Omega_1)\times W_2^1(\Omega_2)$ with inner products and corresponding norms

$$(u,v)_{0,2} = \sum_{i=1}^{2}\int_{\Omega_i} u_i(x)v_i(x)dx, \qquad \|u\|_{0,2} = \left(\sum_{i=1}^{2}\int_{\Omega_i} u_i^2(x)dx\right)^{1/2},$$

$$(u,v)_{2,1} = \sum_{i=1}^{2}\int_{\Omega_i}\left(\frac{du_i(x)}{dx}\frac{dv_i(x)}{dx} + u_i(x)v_i(x)\right)dx + [u(x^{(1)})][v(x^{(1)})],$$

$$\|u\|_{2,1} = \left(\sum_{i=1}^{2}\int_{\Omega_i}\left(\left(\frac{du_i(x)}{dx}\right)^2 + u_i^2(x)\right)dx + [u(x^{(1)})]^2\right)^{1/2},$$

we can write the eigenvalue transmission problem (3.1)-(3.3) as operator equation (2.1). Operators $\mathbf{A}$, $\mathbf{B}$, $\overline{\mathbf{B}}$ are defined as follows

$$<\mathbf{A}u,v> = \sum_{i=1}^{2}\int_{\Omega_i}\frac{du_i(x)}{dx}\frac{dv_i(x)}{dx}dx + r[u(x^{(1)})][v(x^{(1)})] + \frac{du_1(0)}{dx}v_1(0) - \frac{du_2(1)}{dx}v_2(1), \tag{3.4}$$

$$<\mathbf{B}u,v> = \sum_{i=1}^{2}\int_{\Omega_i} q(x)u_i(x)v_i(x)dx, \qquad <\overline{\mathbf{B}}u,v> = \sum_{i=1}^{2}\int_{\Omega_i}\overline{q}(x)u_i(x)v_i(x)dx, \tag{3.5}$$

with domain

$$D = \left\{(u_1,u_2): u_i\in W_2^1(\Omega_i), \left.\frac{du_i}{dx}\right|_{x=x^{(1)}} = r[u(x^{(1)})], i=1,2, r>0\right\}.$$

For the operator $\overline{\mathbf{B}}$ corresponding to any piece-wise constant approximation $\overline{q}(x)$ of function $q(x)$, the basic eigenvalue transmission problem (2.7) can be presented as

$$\frac{d^2u_{ni}^{(0)}(x)}{dx^2} + (\lambda_n^{(0)}(\overline{q}) - \overline{q}(x))u_{ni}^{(0)}(x) = 0, \quad x\in\Omega_i, i=1,2, \tag{3.6}$$

$$\left.\frac{du_{ni}^{(0)}}{dx}\right|_{x=x^{(1)}} = r[u_n^{(0)}(x^{(1)})],$$

$$\alpha_i u_{n1}^{(0)}(0) + \beta_i\frac{u_{n1}^{(0)}(0)}{dx} + \gamma_i u_{n2}^{(0)}(1) + \delta_i\frac{u_{n2}^{(0)}(1)}{dx} = 0,$$

$$\|u_n^{(0)}\| = 1.$$

The corresponding recursive sequence of problems (2.5) can be written as follows

$$\frac{d^2u_{ni}^{(j+1)}(x)}{dx^2} + (\lambda_n^{(0)}(\overline{q}) - \overline{q}(x))u_{ni}^{(j+1)}(x) = \qquad x \in \Omega_i, i = 1,2 \tag{3.7}$$

$$-\sum_{s=0}^{j} \lambda_n^{(j+1-s)}(\overline{q})u_{ni}^{(s)}(x) + (q(x) - \overline{q}(x))u_{ni}^{(j)}(x) = -F_{ni}^{(j+1)}(x),$$

$$\left.\frac{du_{ni}^{(j+1)}}{dx}\right|_{x=x^{(1)}} = r[u_n^{(j+1)}(x^{(1)})],$$

$$\alpha_i u_{n1}^{(j+1)}(0) + \beta_i \frac{u_{n1}^{(j+1)}(0)}{dx} + \gamma_i u_{n2}^{(j+1)}(1) + \delta_i \frac{u_{n2}^{(j+1)}(1)}{dx} = 0.$$

To provide the uniqueness of the solution of problem (3.7), we add the following normalizing condition

$$(u_n^{(j+1)}, u_n^{(0)})_{0,2} = 0, \qquad j = 0, 1, \ldots \tag{3.8}$$

Formula (2.6) for eigenvalue corrections is transformed into

$$\lambda_n^{(j+1)}(\overline{q}) = ((q(x) - \overline{q}(x))u_n^{(j)}, u_n^{(0)})_{0,2}. \tag{3.9}$$

To find numerical solution, we have used the technique of exact difference schemes [68]. In our numerical experiments, we compare the approximate eigenvalues found with FD-method with with approximate eigenvalues of original problem calculated by the method of dichotomy with an error 10^{-23}. They confirm the theoretical findings. The numerical results are presented and discussed in details in [31, 32]. The convergence results for the Dirichlet boundary conditions and periodic boundary conditions are proven and original eigenvalues transmission problems are treated by using the solution of basic problem and FD-method. Then they are extended for the general case (3.3) in [33].

3.2. Properties of the eigensolutions of the basic and the original eigenvalue transmission problems

Here we study some properties of the eigenvalues of operator $\mathbf{A}$ defined by (3.4), that is eigenvalues of the basic problem (3.6) with $\overline{q}(x) \equiv 0$. Then based on them and using Theorem 2.1 on convergence of FD-method, make conclusions about properties of the eigenvalues of the original eigenvalue transmission problem (3.1)-(3.3).

3.2.1. Basic problem

Theorem 3.1 *Assume that matching point* $x^{(1)} = \frac{2n+1}{2(n+k+1)}$, *where* $n, k \in \{0\} \cup N$, *and the coefficients in the boundary conditions (3.3) satisfy the following equalities*

$$\begin{cases} \beta_1\delta_2 - \beta_2\delta_1 = 0, \\ (-1)^{n+k}(\beta_1\gamma_2 - \beta_2\gamma_1 + \alpha_1\delta_2 - \alpha_2\delta_1) + \alpha_1\beta_2 - \alpha_2\beta_1 + \gamma_1\delta_2 - \gamma_2\delta_1 = 0. \end{cases} \tag{3.10}$$

Then there exist exact independent on transmission coefficient r *eigenvalues of operator* $\mathbf{A}$ *defined by (3.4),* $\lambda_{n+k}^{(0)} = \pi^2(n+k+1)^2$.

a). If in addition to (3.10),

$$\begin{cases} (\alpha_1\delta_2 - \alpha_2\delta_1)x^{(1)} - (\beta_1\gamma_2 - \beta_2\gamma_1)(1 - x^{(1)}) = 0, \\ \alpha_1\gamma_2 - \alpha_2\gamma_1 = 0, \end{cases}$$

then there can exist multiple independent on r *eigenvalues of operator* $\mathbf{A}$.

b). If in addition to (3.10), the following conditions

$$\begin{cases} (\alpha_1\delta_2 - \alpha_2\delta_1)x^{(1)} - (\beta_1\gamma_2 - \beta_2\gamma_1)(1 - x^{(1)}) = 0, \\ \alpha_1\gamma_2 - \alpha_2\gamma_1 \neq 0 \end{cases} \tag{3.11}$$

are satisfied, then the difference between independent on r eigenvalues and neighboring ones tends to a constant as the index of trial eigenvalue increases.

c). If in addition to (3.10), inequality

$$(\alpha_1\delta_2 - \alpha_2\delta_1)x^{(1)} - (\beta_1\gamma_2 - \beta_2\gamma_1)(1 - x^{(1)}) \neq 0$$

is satisfied then the difference between any two neighboring eigenvalues increases as the index of trial eigenvalue increases.

Theorem 3.2 *Assume that the matching point $x^{(1)} = \frac{2n+1}{2(n+k)+1}$ with $n, k \in \{0\} \cup N$ and the coefficients in boundary conditions (3.3) satisfy the equalities*

$$\begin{cases} \alpha_1\gamma_2 - \alpha_2\gamma_1 = 0, \\ \alpha_1\beta_2 - \alpha_2\beta_1 + \gamma_1\delta_2 - \gamma_2\delta_1 = 0, \\ r(\beta_1\delta_2 - \beta_2\delta_1) + \beta_1\gamma_2 - \beta_2\gamma_1 = 0. \end{cases} \tag{3.12}$$

Then there exist independent on transmission coefficient r eigenvalues of operator $\mathbf{A}$ *defined by (3.4). They all are simple.*

a). If in addition to (3.12), the inequality

$$\beta_1\delta_2 - \beta_2\delta_1 \neq 0$$

is satisfied, then the difference between any two neighboring eigenvalues increases as the index of trial eigenvalue increases.

b). If in addition to (3.12), the equality

$$\beta_1\delta_2 - \beta_2\delta_1 = 0 \tag{3.13}$$

holds, then the difference between independent on r eigenvalues and neighboring ones tends to a constant as the index of trial eigenvalue increases.

Theorem 3.3 *Assume that the matching point $x^{(1)} = \frac{n}{n+k}$ with $n, k \in N$ and the coefficients in boundary conditions (3.3) satisfy the following equalities*

$$\begin{cases} \alpha_1\gamma_2 - \alpha_2\gamma_1 = 0, \\ (-1)^{n+k}(\alpha_1\delta_2 - \alpha_2\delta_1 - \beta_1\gamma_2 + \beta_2\gamma_1) + \alpha_1\beta_2 - \alpha_2\beta_1 + \gamma_1\delta_2 - \gamma_2\delta_1 = 0. \end{cases} \tag{3.14}$$

Then there exist the independent on transmission coefficient r eigenvalues of operator $\mathbf{A}$ *defined by (3.4).*

a). Under the conditions

$$\begin{cases} \beta_1\delta_2 - \beta_2\delta_1 \neq 0, \\ (\beta_1\gamma_2 - \beta_2\gamma_1)x^{(1)} - (\alpha_1\delta_2 - \alpha_2\delta_1)(1 - x^{(1)}) + r(\beta_1\delta_2 - \beta_2\delta_1) = 0, \end{cases}$$

there can exist multiple independent on r eigenvalues.

b). If besides (3.14) the inequalities

$$\begin{cases} \beta_1\delta_2 - \beta_2\delta_1 \neq 0, \\ (\beta_1\gamma_2 - \beta_2\gamma_1)x^{(1)} - (\alpha_1\delta_2 - \alpha_2\delta_1)(1 - x^{(1)}) + r(\beta_1\delta_2 - \beta_2\delta_1) \neq 0 \end{cases} \tag{3.15}$$

hold, then the difference between independent on r eigenvalues and neighboring ones tends to constant as the index of trial eigenvalue increases.

c). If in addition to (3.14) the condition

$$\begin{cases} \beta_1\delta_2 - \beta_2\delta_1 = 0, \\ (\beta_1\gamma_2 - \beta_2\gamma_1)x^{(1)} - (\alpha_1\delta_2 - \alpha_2\delta_1)(1 - x^{(1)}) \neq 0 \end{cases}$$

is satisfied, then the difference between any two neighboring eigenvalues increases as the index of trial eigenvalue increases.

Example 3.4 *1. For $x^{(1)} = \frac{n}{n+k}$ with $n, k \in N$ and Neimann boundary conditions, the assumptions of Theorem 3.3.b are satisfied.*

2. For Ivonkin-Samarskii boundary conditions and $x^{(1)} = \frac{n}{2m}$, de $m \geq \left[\frac{n+1}{2}\right]$ the assumptions of Theorem 3.3.c are satisfied.

3. For Dirichlet and Neimann boundary conditions and $x^{(1)} = \frac{2n+1}{2(n+k)+1}$, $n, k \in \{0\} \cup N$ the assumptions of Theorem 3.2.b is satisfied.

Theorem 3.5 *Assume that the coefficients in boundary conditions (3.3)satisfy to*

$$\begin{cases} \alpha_1\gamma_2 - \alpha_2\gamma_1 = 0, \\ \beta_1\delta_2 - \beta_2\delta_1 = 0, \\ \alpha_1\delta_2 - \alpha_2\delta_1 \neq 0, \\ \alpha_1\delta_2 - \alpha_2\delta_1 + \beta_1\gamma_2 - \beta_2\gamma_1 = 0, \\ \alpha_1\beta_2 - \alpha_2\beta_1 + \gamma_1\delta_2 - \gamma_2\delta_1 = \pm 2(\alpha_1\delta_2 - \alpha_2\delta_1). \end{cases}$$

Then there exist such eigenvalues of operator $\mathbf{A}$ defined by (3.4) which do not depend on the transmission coefficient r and matching point $x^{(1)}$. The difference between two neighboring eigenvalues increases as the index of trial eigenvalue increases.

The existence of eigenvalues of operator $\mathbf{A}$ with Dirichlet boundary conditions that don't depend on transmission parameter r firstly were obtained experimentally in [69].

Corollary 3.6 *Let assumptions of Theorem 2.1 are satisfied. Then the quality of the convergence of the FD-method (3.6)-(3.9) for eigenvalue transmission problems (3.1)-(3.3) is determined by the difference between neighboring eigenvalues of operator $\mathbf{A}$. It either improves or tends to a constant as the index of trial eigenvalue increases.*

3.2.2. Original problem

Theorems 3.1-3.5 on properties of the eigenvalues of the basic problem and Theorem 2.1 on convergence rate of FD-method lead to the qualitative result about behavior of the eigenvalues of the original problem (3.1)-(3.3).

Theorem 3.7 *Assume that coefficients in boundary conditions (3.3) for the eigenvalue transmission problem (3.1)-(3.3) are such that operators $\mathbf{A}$, $\mathbf{B}$, $\overline{\mathbf{B}}$ defined by (3.4), (3.5) are self-adjoint and operator $\mathbf{A}$ is a positive defined, and $\mathbf{B}$, $\overline{\mathbf{B}}$ are nonnegative defined. Then the difference between two neighboring eigenvalues of the original problem (3.1)-(3.3) is determined by the difference between corresponding neighboring eigenvalues of the basic problem. (3.6).*

Theorem 3.8 *Let assumptions of Theorem 3.7 is fulfilled and let matching point and coefficients in boundary conditions (3.3) obey to one of the following requirements:*

1. $x^{(1)} = \frac{2n+1}{2(n+k+1)}$, $n, k \in \{0\} \cup N$ *and conditions (3.10),(3.11) are satisfied;*
2. $x^{(1)} = \frac{n}{n+k}$, $n, k \in N$ *and conditions (3.14),(3.15)are satisfied;*
3. $x^{(1)} = \frac{2n+1}{2(n+k)+)}$, $n, k \in \{0\} \cup N$ *and conditions (3.12),(3.13) are satisfied.*

Then there exist such pairs of the neighboring eigenvalues of the original problem (3.1)-(3.3), that the difference between them tends to a constant as the index of trial eigenvalue increases.

3.3. Eigenvalue problems with potential in L_1

Previous results have been obtained in assumption that potential $q(x)$ belongs to the space $L_\infty(\Omega)$. However, for many applied problems, for example [70–73], coefficients belong to weaker spaces. Eigenvalue problems with potential from $L_1(\Omega)$ are studied in [34, 35].

Firstly we present the convergence result for Sturm-Liouville problem with Dirichlet boundary conditions.

Let us consider problem (2.1) in functional space $\overset{0}{W}{}_2^1(0,1)$ with operators $\mathbf{A}$, $\mathbf{B}$, $\overline{\mathbf{B}}$ defined as follows

$$< \mathbf{A}u, v >= \int_0^1 \frac{du_i(x)}{dx}\frac{dv_i(x)}{dx}dx, \tag{3.16}$$

$$< \mathbf{B}u, v >= \int_0^1 q(x)u_i(x)v_i(x)dx, \qquad < \overline{\mathbf{B}}u, v >= \int_0^1 \overline{q}(x)u_i(x)v_i(x)dx. \tag{3.17}$$

Theorem 3.9 *a). For* $n \geq \left[\frac{24\sqrt{2}\|q\|_{0,1}}{\pi}\right] + 1$ *numerical solution* $(\overset{m}{\lambda}_n(0), \overset{m}{u}_n(x))$ *received by FD-method (2.4)-(2.7) converges to the corresponding exact solution* $(\lambda_n, u_n(x))$ *of problem (2.1), where operators* $\mathbf{A}$, $\mathbf{B}$, $\overline{\mathbf{B}}$ *are defined by (3.16), (3.17), not worth than a geometric series with denominator* $q_n(0) = 4M_n(0)\|q\|_{0,1}$, *where*

$$M_n(0) = \frac{6\sqrt{2}}{\pi n}.$$

b). For $n \geq \left[\frac{3(c_1^2(\overline{q})+1)\|\overline{q}\|_{0,1}+24c_1(\overline{q})\|q-\overline{q}\|_{0,1}}{\pi}\right] + 1$ *numerical solution* $(\overset{m}{\lambda}_n(\overline{q}), \overset{m}{u}_n(x))$ *received by FD-method (2.4)-(2.7) converges to the corresponding exact solution* $(\lambda_n, u_n(x))$ *of problem (2.1), where operators* $\mathbf{A}$, $\mathbf{B}$, $\overline{\mathbf{B}}$ *are defined by (3.16), (3.17), not worth than a geometric series with denominator* $q_n(\overline{q}) = 4M_n(\overline{q})\|q-\overline{q}\|_{0,1}$, *where*

$$M_n(\overline{q}) = \frac{6c_1(\overline{q})}{\pi n - 3(c_1^2(\overline{q})+1)\|\overline{q}\|_{0,1}}.$$

c). For $1 \leq n \leq \min\left\{\left[\frac{24\sqrt{2}\|q\|_{0,1}}{\pi}\right], \left[\frac{3(c_1^2(\overline{q})+1)\|\overline{q}\|_{0,1}+24c_1(\overline{q})\|q-\overline{q}\|_{0,1}}{\pi}\right]\right\}$ *let choose function* $\overline{q}(x)$ *in such a way that*

$$q_n(\overline{q}) = 4M_n(\overline{q})\|q-\overline{q}\|_{0,1} \leq 1,$$

where

$$M_n(\overline{q}) = c_1^2(\overline{q})\left(\frac{1}{\pi^2\left(1 - \frac{\lambda_n^{(0)}(\overline{q})}{\lambda_{n+1}^{(0)}(\overline{q})}\right)} + \sum_{p=1}^{n-1}\frac{1}{\lambda_n^{(0)}(\overline{q}) - \lambda_p^{(0)}(\overline{q})}\right).$$

Then the numerical solution $(\overset{m}{\lambda}_n(\overline{q}), \overset{m}{u}_n(x))$ *received by FD-method (2.4)-(2.7) converges to the corresponding exact solution* $(\lambda_n, u_n(x))$ *of problem (2.1), where operators* $\mathbf{A}$, $\mathbf{B}$, $\overline{\mathbf{B}}$ *are defined by (3.16), (3.17), not worth than a geometric series with denominator* $q_n(\overline{q})$. *The following estimates*

$$\| \overset{m}{u}_n - u_n \| \le \alpha_{m+1} \frac{q_n^{m+1}}{1-q_n}, \qquad | \overset{m}{\lambda}_n - \lambda_n | \le c_1(\overline{q}) |q - \overline{q}\|_{0,\infty} \frac{q_n^m}{1-q_n},$$

where $\alpha_m = 2\frac{(2m-1)!!}{(2m+2)!!}$ *hold.*

Corollary 3.10 *For the case* $q(x) \in L_1(0,1)$ *the convergence rate of FD-method may became worth as the index of trial eigenvalue increases up to the some index*

$$n_0 = \min\left\{ \left[\frac{8\sqrt{2}\|q\|_{0,1}}{\pi} \right], \left[\frac{(c_1^2(\overline{q})+1)\|\overline{q}\|_{0,1} + 8c_1(\overline{q})\|q-\overline{q}\|_{0,1}}{\pi} \right] \right\}.$$

Starting from index $n_0 + 1$ *it improves as the index of trial eigenvalue increases.*

Now let us consider eigenvalue transmission problem (3.1)-(3.3) with $q(x) \in L_1(\Omega)$ and Dirichlet boundary conditions, that is $\alpha_2 = \beta_1 = \beta_2 = \gamma_1 = \delta_1 = \delta_2 = 0$. The following convergence result is proven.

Theorem 3.11 *a). For index* n, *which satisfy the condition*

$$q_n(0) = \frac{32\|q\|_{0,1}}{\min\{1,r\}\sqrt{\min\{x^{(1)}, 1-x^{(1)}\}}} \left(\frac{2}{r} + \frac{10(1-x^{(1)})}{\pi(2n+1)} \right) < 1, \tag{3.18}$$

the numerical solution $(\overset{m}{\lambda}_n(0), \overset{m}{u}_n(x))$ *(3.6)-(3.9) converges to the corresponding exact solution* $(\lambda_n, u_n(x))$ *of the original problem (3.1)-(3.3) with* $q(x) \in L_1(\Omega)$ *and Dirichlet boundary conditions not worth as a geometric series with denominator* $q_n(0)$ *determined according to (3.18).*

b). If index n *satisfies*

$$c_3(n,r,\overline{q}) = (c_1^2(\overline{q})+1)\|\overline{q}\|_{0,1} \left(\frac{2}{r} + \frac{10(1-x^{(1)})}{\pi(2n+1)} \right) < 1,$$

$$q_n(\overline{q}) = \frac{8c_1(\overline{q})}{1-c_3(n,r,\overline{q})} \left(\frac{2}{r} + \frac{10(1-x^{(1)})}{\pi(2n+1)} \right) \|q-\overline{q}\|_{0,1} < 1, \tag{3.19}$$

simultaneously, then numerical solution $(\overset{m}{\lambda}_n(\overline{q}), \overset{m}{u}_n(x))$ *(3.6)-(3.9) converges to the corresponding exact solution* $(\lambda_n, u_n(x))$ *of the original problem (3.1)-(3.3) with* $q(x) \in L_1(\Omega)$ *and Dirichlet boundary conditions not worth as a geometric series with denominator* $q_n(\overline{q})$ *determined according to (3.19).*

c). For index n *not satisfying conditions a) and b), we choose function* $\overline{q}(x)$ *in such a way that*

$$q_n(\overline{q}) = 4c_1^2(\overline{q}) \left(\frac{1}{\pi^2\left(1 - \frac{\lambda_n^{(0)}(\overline{q})}{\lambda_{n+1}^{(0)}(\overline{q})}\right)} + \sum_{p=1}^{n-1} \frac{1}{\lambda_n^{(0)}(\overline{q}) - \lambda_p^{(0)}(\overline{q})} \right) \|q-\overline{q}\|_{0,1} \le 1. \tag{3.20}$$

Then numerical solution $(\overset{m}{\lambda}_n(\overline{q}), \overset{m}{u}_n(x))$ *(3.6)-(3.9) converges to the corresponding exact solution* $(\lambda_n, u_n(x))$ *of the original problem (3.1)-(3.3) with* $q(x) \in L_1(\Omega)$ *and Dirichlet* *boundary conditions not worth as a geometric series with denominator $q_n(\overline{q})$ determined by (3.20).*

The following estimates

$$\| \overset{m}{u}_n - u_n \| \le \alpha_{m+1} \frac{q_n^{m+1}}{1-q_n}, \qquad | \overset{m}{\lambda}_n - \lambda_n | \le c_1(\overline{q}) |q - \overline{q}\|_{0,\infty} \frac{q_n^m}{1-q_n},$$

with $\alpha_m = 2\frac{(2m-1)!!}{(2m+2)!!}$ hold.

Corollary 3.12 *As it follows from Theorem 3.11, contrary to Sturm-Liouville problem the convergence rate of the FD-method for eigenvalue transmission problem with potential from $L_1(\Omega)$ tends to constant $\alpha(\overline{q}) = \frac{16c_1(\overline{q})\|q-\overline{q}\|_{0,1}}{r-2\|\overline{q}\|_{0,1}(c_1^2(\overline{q})+1)}$, or corresponding, $\alpha(0) = \frac{16c_1(0)\|q\|_{0,1}}{r}$ as the index of trial eigenvalue increases.*

4. Nonlinear eigenvalue problems

Many processes in such areas as quantum mechanics, physics of semiconductors and semiconductor devices, fluid dynamics, sensor technologies are described by nonlinear eigenvalue problems [74–79]. Analytical properties of the nonlinear spectral problems are relatively well studied in [37, 42, 80–83]. In [81, 83] the existence-uniqueness results are established. The asymptotic estimates of the eigenvalues is reported in [42, 82]. Contrary to linear problems, spectrum of nonlinear problems depends on the normalizing condition essentially. That is why we study two kinds of normalizing condition: differential and integral.

4.1. Nonlinear Sturm-Liouville problem

4.1.1. Autonomous potential

Let us consider the following eigenvalue problem

$$u''(x) + \lambda u(x) - N(u(x)) = 0, \ x \in (0,1), \tag{4.1}$$
$$u(0) = 0, \quad u(1) = 0,$$

where $N : \mathbb{R} \to \mathbb{R}$ is an analytical function with respect to $u \in [0, R]$ for each fixed $x \in [0, 1]$ such that

$$N(0) = 0,$$
$$|N_u^{(k)}(u)| = \left| \frac{\mathrm{d^k N(u)}}{\mathrm{du^k}} \right| \le \overline{N}^{(k)}(|u|) \quad u \in \mathbb{R}, \quad \forall k = 0, 1, \dots, \tag{4.2}$$

and $\overline{N}(|u|)$ is a smooth function with non-negative derivatives.

4.1.1.1. Differential normalizing condition

We write differential normalizing condition as

$$u'(0) = M, \tag{4.3}$$

where M is a known constant.

As in linear case we write exact and numerical solution as series (2.3) and (2.4), correspondingly. In order to find the summands of these series, we introduce additionally the new coefficients $A_n^{(j)}(x)$ by

$$N(\sum_{j=0}^{\infty} u_n^{(j)}(x) t^j) = \sum_{j=0}^{\infty} A_n^{(j)}(x) t^j, \qquad A_n^{(j)}(x) = \frac{1}{j!} \left. \frac{\partial^j N(\sum_{j=0}^{\infty} u_n^{(j)}(x) t^j)}{\partial t^j} \right|_{t=0},$$

and then following FD-approach (2.4)-(2.7), obtain the following numerical algorithm.

Coarse approximation $(\lambda_n^{(0)}, u_n^{(0)})$ is a solution of the basic linear eigenvalue problem

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}u_n^{(0)}(x) + \lambda_n^{(0)}u_n^{(0)}(x) = 0, \quad x \in (0,1),$$

$$u_n^{(0)}(0) = 0, \quad u_n^{(0)}(1) = 0, \quad \frac{\mathrm{d}u_n^{(0)}(0)}{\mathrm{d}x} = M,$$

that is,

$$\lambda_n^{(0)} = (n\pi)^2, \quad u_n^{(0)}(x) = M\frac{\sin(n\pi x)}{n\pi}, \quad x \in (0,1), \quad n = 1, 2, \ldots. \tag{4.4}$$

We find corrections $u_n^{(j+1)}(x)$ by solving linear nonhomogeneous boundary value problem

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}u_n^{(j+1)}(x) + \lambda_n^{(0)}u_n^{(j+1)}(x) = -F_n^{(j+1)}(x), \quad x \in (0,1), \tag{4.5}$$

$$u_n^{(j+1)}(0) = 0, \quad u_n^{(j+1)}(1) = 0, \quad \frac{\mathrm{d}u_n^{(j+1)}(0)}{\mathrm{d}x} = 0, \quad j = 0, 1, 2, ...$$

where

$$F_n^{(j+1)}(x) = -\sum_{p=0}^{j}\lambda_n^{(j+1-p)}u_n^{(p)}(x) + A_n^{(j)}(u_n^{(0)}, ..., u_n^{(j)}), \tag{4.6}$$

and

$$A_n^{(j)}(u_n^{(0)}, ..., u_n^{(j)}) = A_n^{(j)}(x) \tag{4.7}$$

$$= \sum_{\alpha_1+\cdots+\alpha_j=j} N_u^{(\alpha_1)}(u_n^{(0)}(x))\frac{[u_n^{(1)}(x)]^{\alpha_1-\alpha_2}}{(\alpha_1-\alpha_2)!}\cdots\frac{[u_n^{(j-1)}(x)]^{\alpha_{j-1}-\alpha_j}}{(\alpha_{j-1}-\alpha_j)!}\cdot\frac{[u_n^{(j)}(x)]^{\alpha_j}}{(\alpha_j)!}, \quad j > 0,$$

$$A_n^{(0)}(u_n^{(0)}) = N(u_n^{(0)})$$

are the Adomian polynomials [84]. We note that

$$N(u_n(x)) = \sum_{j=0}^{\infty}A_n^{(j)}(x). \tag{4.8}$$

The solvability condition for equation (4.5) yields

$$\lambda_n^{(j+1)} = \left\{-\sum_{p=1}^{j}\lambda_n^{(j+1-p)}\int_0^1 u_n^{(p)}(\xi)u_n^{(0)}(\xi)d\xi + \right.$$
$$\left. + \int_0^1 A_n^{(j)}(u_n^{(0)}(\xi), ..., u_n^{(j)}(\xi))u_n^{(0)}(\xi)\mathrm{d}\xi\right\} / \|u_n^{(0)}\|^2, \tag{4.9}$$

where $\|u_n^{(0)}\| = \sqrt{\int_0^1[u_n^{(0)}(x)]^2\mathrm{d}x}$ is a norm in $L_2(0,1)$.

Under such conditions the solution of (4.5) can be written as

$$u_n^{(j+1)}(x) = \int_0^x \frac{\sin[n\pi(x-\xi)]}{n\pi}\left\{-\sum_{p=0}^{j}\lambda_n^{(j+1-p)}u_n^{(p)}(\xi) + A_n^{(j)}(u_n^{(0)}(\xi), ..., u_n^{(j)}(\xi))\right\}\mathrm{d}\xi. \tag{4.10}$$

Thus, numerical algorithm consist in finding (4.4), consecutive computations of (4.9), (4.10) for $j = 0, \ldots, m-1$ and series (2.4).

The following convergence result is established in [57].

Theorem 4.1 *Assume condition (4.2) holds and index n of the trial eigenpair satisfies*

$$\frac{1}{n\pi R} < 1 \tag{4.11}$$

where R is a convergence radius of series for generating function $f(z) = \sum_{j=0}^{\infty} \|u_n^{(j)}\| z^j$. Then the numerical algorithm (4.4),(4.9),(4.10),(2.4) converges exponentially to the exact solution of nonlinear eigenvalue problem (4.1)-(4.3) with error estimates

$$\|u_n - \overset{m}{u}_n\|_\infty \leq \frac{c}{(m+2)^{1+\varepsilon}} \left(\frac{1}{n\pi R}\right)^{m+1}, \qquad |\lambda_n - \overset{m}{\lambda}_n| \leq \frac{c}{R(m+2)^{1+\varepsilon}} \left(\frac{1}{n\pi R}\right)^{m}.$$

Corollary 4.2 *It follows from (4.11) that the convergence improves as the index of trial eigenvalue increases.*

4.1.1.2. Integral normalizing condition

Let us consider problem (4.1) (4.2) with integral normalizing condition

$$\int_0^1 u^2(x)\mathrm{d}x = M \tag{4.12}$$

instead of differential normalizing condition (4.3).

Following the described above FD-approach, we arrive to the algorithm for the numerical solution of problem (4.1), (4.2), (4.12).

Firstly we find coarse approximation

$$\lambda_n^{(0)} = (n\pi)^2, \quad u_n^{(0)}(x) = \sqrt{2M}\sin(n\pi x),\ x \in (0,1), \quad n = 1, 2, \ldots \tag{4.13}$$

which is a solution of the basic linear problem

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} u_n^{(0)}(x) + \lambda_n^{(0)} u_n^{(0)}(x) = 0,\ x \in (0,1),$$

$$u_n^{(0)}(0) = 0,\ u_n^{(0)}(1) = 0,\ \|u_n^{(0)}\| = \sqrt{M}.$$

Then consecutively for $j = 0, 1, ..., m$ we find the corrections

$$\lambda_n^{(j+1)} = \left\{ -\sum_{p=1}^{j} \lambda_n^{(j+1-p)} \int_0^1 u_n^{(p)}(\xi) u_n^{(0)}(\xi)\mathrm{d}\xi + \int_0^1 A_n^{(j)}(u_n^{(0)}(\xi), ..., u_n^{(j)}(\xi)) u_n^{(0)}(\xi)\mathrm{d}\xi \right\} / M, \tag{4.14}$$

where $A_n^{(j)}(u_n^{(0)}(\xi), ..., u_n^{(j)}(\xi))$ is defined according to (4.7), and $u_n^{(j+1)}$ is a corresponding solution of linear nonhomogeneous boundary-value problem

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} u_n^{(j+1)}(x) + \lambda_n^{(0)} u_n^{(j+1)}(x) = -F_n^{(j+1)}(x),\ x \in (0,1), \tag{4.15}$$

$$u_n^{(j+1)}(0) = 0,\ u_n^{(j+1)}(1) = 0,\ \int_0^1 u_n^{(0)}(x) u_n^{(j+1)}(x)\mathrm{d}x = -\frac{1}{2}\int_0^1 \sum_{p=1}^{j} u_n^{(p)}(x) u_n^{(j+1-p)}(x)\mathrm{d}x,$$

with $F_n^{(j+1)}(x)$ determined by (4.6).

The following convergence result is established in [57].

Theorem 4.3 *Assume (4.2) holds and index n of the trial eigenpair satisfies condition*

$$r_n = \frac{1}{n\pi R} < 1 \tag{4.16}$$

where R is a convergence radius of series for generating function $f(z) = \sum_{j=0}^{\infty} \|u_n^{(j)}\| z^j$. Then numerical algorithm (4.13), (4.14), (4.15), (2.4) converges exponentially to the corresponding eigenpair of problem (4.1), (4.2), (4.12) with error estimates

$$\|u_n - \overset{m}{u}_n\|_\infty \leq \frac{c}{(m+2)^{1+\varepsilon}} \left(\frac{1}{n\pi R}\right)^{m+1}, \qquad |\lambda_n - \overset{m}{\lambda}_n| \leq \frac{c}{R(m+2)^{1+\varepsilon}} \left(\frac{1}{n\pi R}\right)^{m}.$$

Corollary 4.4 *It follows from (4.16) that convergence rate of algorithm (4.13), (4.14), (4.15), (2.4) improves as the index of trial eigenvalue increases.*

By using algorithm (4.13), (4.14), (4.15), (2.4), we are able to calculate eigenvalues λ_n and corresponding eigenfunctions $u_n(x)$ for $n = N_0, N_0+1, \ldots$ beginning with some N_0. In [59] the modification of FD-method which allows to find eigenpairs $(\lambda_n, u_n(x))$ for $n = 1, \ldots, N_0 - 1$ is proposed and discussed. The numerical examples confirming the exponential convergence rate of the algorithm are presented.

4.1.2. Potential with both linear and nonlinear terms

Let us consider Sturm-Liouville problem with linear potential $q(x)u(x)$ and nonlinear potential $N(u(x))$ and Dirichlet boundary conditions

$$u''(x) + \lambda u(x) - q(x)u(x) - N(u(x)) = 0, \quad x \in (0,1) \tag{4.17}$$

$$u(0) = u(1) = 0,$$

where $q \in L_2(0,1),\ q(x) \geq 0$, and nonlinear function $N(u)$ satisfy condition (4.2).

4.1.2.1. Differential normalizing condition

Following FD-method we are looking for the numerical solution of problem (4.17), (4.2), (4.3) as truncated series (2.4).

Coarse approximation $(\lambda_n^0(\bar{q}), u_n^0(x))$ is a solution of the basic linear eigenvalue problem

$$\frac{d^2}{dx^2} u_n^{(0)}(x) + (\lambda_n^{(0)}(\bar{q}) - \bar{q}(x)) u_n^{(0)}(x) = 0, \ x \in (0,1) \tag{4.18}$$

$$u_n^{(0)}(0) = u_n^{(0)}(1) = 0, \ \frac{du_n^{(0)}(0)}{dx} = M,$$

where $\bar{q}(x)$ is a piece wise constant approximation of $q(x)$.

Then consecutively for $j = 0, 1, 2 \ldots m$ we find the corrections

$$\lambda_n^{(j+1)}(\bar{q}) = \left\{ -\sum_{p=1}^{j} \lambda_n^{(j+1-p)}(\bar{q}) \int_0^1 u_n^{(p)}(\xi) \cdot u_n^{(0)}(\xi) d\xi + \int_0^1 (q(\xi) - \bar{q}(\xi)) u_n^{(j)}(\xi) \cdot u_n^{(0)}(\xi) d\xi + \right.$$
$$\left. + \int_0^1 A_n^{(j)}(u_n^{(0)}(\xi), \ldots, u_n^{(j)}(\xi)) \cdot u_n^{(0)}(\xi) d\xi \right\} / \left\| u_n^{(0)} \right\|^2, \tag{4.19}$$

where as previous $\|\cdot\|$ denote the norm in $L_2(0,1)$. $A_n^{(j)}(u_n^{(0)}(\xi), \ldots, u_n^{(j)}(\xi))$ is defined according to (4.7).

$u_n^{j+1}(x)$ is a solution of corresponding nonhomogeneous linear boundary value problem

$$\frac{d^2}{dx^2}u_n^{(j+1)}(x) + (\lambda_n^0(\bar{q}) - \bar{q}(x))u_n^{j+1}(x) = F_n^{(j+1)}(x),\ x \in (0,1)$$
$$u_n^{(j+1)}(0) = u_n^{(j+1)}(1) = \frac{du_n^{(j+1)}(0)}{dx} = 0, \tag{4.20}$$

with right hand part

$$F_n^{(j+1)}(x) = -\sum_{p=0}^{j} \lambda_n^{(j+1-p)}(\bar{q})u_n^{(p)}(x) + (q(x) - \bar{q}(x))u_n^{(j)}(x) + A_n^{(j)}(u_n^{(0)}, \dots, u_n^{(j)}), \tag{4.21}$$

The following convergence result is established in [56].

Theorem 4.5 *Assume that condition (4.2) is satisfied and for index n of the trial eigenpair the following condition*

$$r_n = \frac{\|q - \bar{q}\|}{\pi n R} e^{d_n/\pi n} < 1, \tag{4.22}$$

where $d_n = \|\lambda_n^{(0)}(\bar{q}) - \lambda_n^{(0)}(0) - (\bar{q})\|_\infty$, and R is a convergence radius of series for generating function $f(z) = \sum_{j=0}^{\infty} \|u_n^{(j)}\| z^j$ holds. Then numerical algorithm (4.18), (4.19), (4.20), (2.4) converges exponentially to the corresponding exact solution of problem (4.17), (4.2), (4.3) with error estimates

$$\left\| u_n - u_n^{(m)} \right\|_\infty \le \frac{C}{(m+1)^{1+\epsilon}} \left(\frac{\|q - \bar{q}\| \, e^{d_n/\pi n}}{\pi n R} \right)^{n+1},$$
$$\left| \lambda_n - \lambda_n^{(m)} \right| \le \frac{C \, \|q - \bar{q}\|}{R(m+1)^{1+\epsilon}} \left(\frac{\|q - \bar{q}\| \, e^{d_n/\pi n}}{\pi n R} \right)^{n}.$$

Corollary 4.6 *It is proved that*

$$r_n = \frac{e^{d_n/\pi n}}{\pi n} \left(c_1 \|q - \bar{q}\| + c_2 \right).$$

Thus the convergence of FD-method improves as index n of the trial eigenvalue increases, and as the error of approximation of function $q(x)$ by piece wise constant functions decreases, that is for $\|q - \bar{q}\| \to 0$. However, minimal value $r_n^{(0)}$ for fixed n can be reached near boundary where $\|q - \bar{q}\| \to 0$. If $r_n^{(0)} \ge 1$, then the theoretical convergence condition (4.22) is not satisfied even by improvement of approximation $\bar{q}(x)$. Although practically the method can still converge.

4.1.2.2. Integral normalizing condition

Let us consider problem (4.17), (4.2) with integral normalizing condition (4.12). Following FD-approach (2.4)-(2.7), we obtain the numerical algorithm.

Basic linear eigenvalue problem for coarse approximation $(\lambda_n^{(0)}(\bar{q}), u_n^{(0)}(x))$

$$\frac{d^2}{dx^2}u_n^{(0)}(x) + (\lambda_n^{(0)}(\bar{q}) - \bar{q}(x))u_n^{(0)}(x) = 0,\ x \in (0,1) \tag{4.23}$$
$$u_n^{(0)}(0) = u_n^{(0)}(1) = 0,\ \left\| u_n^{(0)} \right\| = \sqrt{M},$$

Correction for eigenvalue $\lambda_n^{(j+1)}(\bar{q})$

$$\lambda_n^{(j+1)}(\bar{q}) = \Big\{ -\sum_{p=1}^{j} \lambda_n^{(j+1-p)}(\bar{q}) \int_0^1 u_n^{(p)}(\xi) u_n^{(0)}(\xi) d\xi + \int_0^1 (q(\xi) - \bar{q}(\xi))\, u_n^{(j)}(\xi) u_n^{(0)}(\xi) d\xi + $$
$$+ \ A_n^{(j)}(u_n^{(0)}(\xi), \ldots, u_n^{(j)}(\xi)) u_n^{(0)}(\xi) d\xi \Big\} / M, \tag{4.24}$$

where $A_n^{(j)}(u_n^{(0)}(\xi), \ldots, u_n^{(j)}(\xi))$ is defined by (4.7).

Corresponding corrections for eigenfunctions $u_n^{(j+1)}(x)$ are the solution of boundary value problem for nonhomogeneous linear differential equation

$$\frac{d^2}{dx^2} u_n^{(j+1)}(x) + (\lambda_n^{(0)}(\bar{q}) - \bar{q}(x)) u_n^{(j+1)}(x) = F_n^{(j+1)}(x), \ x \in (0,1) \tag{4.25}$$
$$u_n^{(j+1)}(0) = u_n^{(j+1)}(1) = 0,$$
$$\int_0^1 u_n^{(0)}(x) u_n^{(j+1)}(x) dx = -\frac{1}{2} \int_0^1 \sum_{p=1}^{j} u_n^{(p)}(x) u_n^{(j+1-p)}(x) dx.$$

Right hand part $F_n^{(j+1)}(x)$ is calculated according to formula (4.21).

Theorem 4.7 *Assume that (4.2) holds and index n of the trial eigenpair satisfies condition*

$$r_n = \frac{\|q - \bar{q}\|}{\pi n R} < 1, \tag{4.26}$$

where R is a convergence radius of series for generating function $f(z) = \sum_{j=0}^{\infty} \|u_n^{(j)}\| z^j$. Then numerical algorithm (4.23)-(4.25), (2.4) converges exponentially to the corresponding exact solution of problem (4.17), (4.2), (4.12) with the error estimates

$$\|u_n - u_n^m\|_\infty \le \frac{C}{(m+1)^{1+\epsilon}} \left[\frac{\|q - \bar{q}\|}{\pi n R} \right]^{m+1}, \ |\lambda_n - \lambda_n^m| \le \frac{C \, \|q - \bar{q}\|}{R(m+1)^{1+\epsilon}} \left[\frac{\|q - \bar{q}\|}{\pi n R} \right]^m.$$

Corollary 4.8 *It is proved that*

$$r_n = \frac{e^{d_n/\pi n}}{\pi n} \left(c_1 \|q - \bar{q}\| + c_2 \right).$$

Thus the convergence of FD-method improves as index n of the trial eigenvalue increases, and as the error of approximation of function $q(x)$ by piece wise constant functions decreases, that is for $\|q - \bar{q}\| \to 0$. However, minimal value $r_n^{(0)}$ for fixed n can be reached near boundary where $\|q - \bar{q}\| \to 0$. If $r_n^{(0)} \ge 1$, then the theoretical convergence condition (4.26) is not satisfied even by improvement of approximation $\bar{q}(x)$. Although practically the method can still converge.

4.2. Nonlinear eigenvalue transmission problem

Let us consider the following nonlinear eigenvalue transmission problem

$$\frac{d^2 u_i(x)}{dx^2} + \lambda u_i(x) - N_i(u_i(x)) = 0, \qquad x \in \Omega_i, \tag{4.27}$$

$$u_1(0) = u_2(1) = 0,$$

$$\frac{du_i(x^{(1)})}{dx} = r[u(x^{(1)})], \quad r > 0, \quad i = 1, 2,$$

where $\Omega_1 = (0, x^{(1)})$, $\Omega_2 = (x^{(1)}, 1)$, $[f(x^{(1)})] = f_2(x^{(1)}) - f_1(x^{(1)})$ is a jump of function at the matching point $x^{(1)}$, $N_i(u_i(x)) : \mathbb{R}^1 \to \mathbb{R}^1$ is an analytical function with respect to u_i such that

$$N_1(0) = 0, \quad |N_{i,u}^{(k)}(u_i)| = \left|\frac{d^k N_i(u_i)}{du_i^k}\right| \le \overline{N}_{i,u}^{(k)}(|u_i|) \quad |u_i| \le C, \quad \forall x \in \Omega_i, \quad C > 0, \tag{4.28}$$

and $\overline{N}_i(u_i)$ is an analytical function with nonnegative derivatives for $u \ge 0$.

4.2.1. Differential normalizing condition

Let us consider problem (4.27) with differential normalizing condition (4.3). According to FD-approach, we write the numerical solution of problem (4.27), (4.2), (4.3) as truncated series (2.4).

Coarse approximation $(\lambda_n^{(0)}, u_{ni}^{(0)}(x), i = 1, 2)$ is a solution of the basic linear eigenvalue transmission problem

$$\frac{d^2 u_{ni}^{(0)}(x)}{dx^2} + \lambda_n^{(0)} u_{ni}^{(0)}(x) = 0, \qquad x \in \Omega_i, \quad i = 1, 2,$$

$$u_{n1}^{(0)}(0) = u_{n2}^{(0)}(1) = 0, \quad \frac{du_{n1}^{(0)}(0)}{dx} = M, \quad \frac{du_{ni}^{(0)}(x^{(1)})}{dx} = r[u_n^{(0)}(x^{(1)})].$$

Using Theorem 3.1 about properties of the eigenvalues of basic transmission problem, we get

$$\lambda_n^{(0)} = \begin{cases} \frac{\pi^2(2k+1)^2}{4(x^{(1)})^2} = \frac{\pi^2(2n+1)^2}{4(1-x^{(1)})^2}, & \frac{x^{(1)}}{1-x^{(1)}} = \frac{2k+1}{2n+1}, \\ \lambda_n^{(0)} : r(\tan(\sqrt{\lambda_n^{(0)}} x^{(1)}) + \tan(\sqrt{\lambda_n^{(0)}}(1 - x^{(1)}))) + \sqrt{\lambda_n^{(0)}} = 0, & \frac{x^{(1)}}{1-x^{(1)}} \ne \frac{2k+1}{2n+1}, \end{cases} \tag{4.29}$$

$k, n \in \{0\} \cup N$.

The corresponding eigenfunctions are following

$$u_{n1}^{(0)}(x) = M\frac{\sin(\sqrt{\lambda_n^{(0)}}x)}{\sqrt{\lambda_n^{(0)}}}, \quad u_{n2}^{(0)}(x) = M\varphi_n \frac{\sin(\sqrt{\lambda_n^{(0)}}(1-x))}{\sqrt{\lambda_n^{(0)}}}, \tag{4.30}$$

where $\varphi_n = \begin{cases} (-1)^{n-k}, & \frac{x^{(1)}}{1-x^{(1)}} = \frac{2k+1}{2n+1}; \\ -\frac{\cos(\sqrt{\lambda_n^{(0)}}x^{(1)})}{\cos(\sqrt{\lambda_n^{(0)}}(1-x^{(1)}))}, & \frac{x^{(1)}}{1-x^{(1)}} \ne \frac{2k+1}{2n+1}. \end{cases}$

Corrections $u_{ni}^{(j+1)}(x)$ $(j = 0, 1, 2, \ldots)$ are the solutions of the transmission problems for linear non-homogenous differential equations:

$$\begin{gathered} \frac{d^2 u_{ni}^{(j+1)}(x)}{dx^2} + \lambda_n^{(0)} u_{ni}^{(j+1)}(x) = -F_{ni}^{(j+1)}(x), \quad x \in \Omega_i, \quad i = 1, 2 \\ u_{n1}^{(j+1)}(0) = u_{n2}^{(j+1)}(1) = 0, \quad \frac{du_{n1}^{(j+1)}(0)}{dx} = 0, \quad \frac{du_{ni}^{(j+1)}(x^{(1)})}{dx} = r[u_n^{(j+1)}(x^{(1)})], \end{gathered} \tag{4.31}$$

where

$$F_{ni}^{(j+1)}(x) = -\sum_{s=0}^{j} \lambda_n^{(j+1-s)} u_{ni}^{(s)} + A_{ni}^{(j)}(u_{ni}^{(0)}, u_{ni}^{(1)}, \ldots, u_{ni}^{(j)})$$

and the Adomian polynomials $A_{ni}^{(j)}(u_{ni}^{(0)},...,u_{ni}^{(j)})$ are defined as

$$A_{ni}^{(j)}(u_{ni}^{(0)},...,u_{ni}^{(j)}) = \sum_{\alpha_1+\cdots+\alpha_j=j} N_{iu_i}^{(\alpha_1)}(u_{ni}^{(0)}(x))\frac{[u_{ni}^{(1)}(x)]^{\alpha_1-\alpha_2}}{(\alpha_1-\alpha_2)!}\cdots\frac{[u_{ni}^{(j-1)}(x)]^{\alpha_{j-1}-\alpha_j}}{(\alpha_{j-1}-\alpha_j)!}\cdot\frac{[u_{ni}^{(j)}(x)]^{\alpha_j}}{(\alpha_j)!}, j>0,$$

$A_{ni}^{(0)}(u_{ni}^{(0)}) = N(u_{ni}^{(0)})$.

The solvability condition for equation (4.31) yields

$$\lambda_n^{(j+1)} = \left\{-\sum_{p=1}^{j}\lambda_n^{(j+1-p)}\sum_{i=1}^{2}\int_{\Omega_i} u_{ni}^{(p)}(\xi)u_{ni}^{(0)}(\xi)d\xi + \right.$$
$$\left. +\sum_{i=1}^{2}\int_{\Omega_i} A_{ni}^{(j)}(u_{ni}^{(0)}(\xi),...,u_{ni}^{(j)}(\xi))u_{ni}^{(0)}(\xi)d\xi\right\}/\|u_n^{(0)}\|^2, \tag{4.32}$$

where $\|u_n\| = \sqrt{\sum_{i=1}^2\int_{\Omega_i} u_{ni}^2(x)dx}$ is the $L_2(\Omega_1\times\Omega_2)$-norm.

Then we can write the solution of non-homogeneous problem (4.31) as

$$u_{n1}^{(j+1)}(x) = \int_0^x \frac{\sin(\sqrt{\lambda_n^{(0)}}(x-\xi))}{\sqrt{\lambda_n^{(0)}}}\times$$
$$\left\{-\sum_{p=0}^{j}\lambda_n^{(j+1-p)}u_{n1}^{(p)}(\xi) + A_{n1}^{(j)}(u_{n1}^{(0)}(\xi),u_{n1}^{(1)}(\xi),\ldots,u_{n1}^{(j)}(\xi))\right\}d\xi,\quad x\in(0,x^{(1)}), \tag{4.33}$$

$$u_{n2}^{(j+1)}(x) = \int_0^{x^{(1)}}\left[\frac{\sin(\sqrt{\lambda_n^{(0)}}(x-\xi))}{\sqrt{\lambda_n^{(0)}}} + \frac{\cos(\sqrt{\lambda_n^{(0)}}(x^{(1)}-x))\cos(\sqrt{\lambda_n^{(0)}}(x^{(1)}-\xi))}{r}\right]\times$$
$$\left\{-\sum_{p=0}^{j}\lambda_n^{(j+1-p)}u_{n1}^{(p)}(\xi)+A_{n1}^{(j)}(u_{n1}^{(0)}(\xi),u_{n1}^{(1)}(\xi),\ldots,u_{n1}^{(j)}(\xi))\right\}d\xi + \int_{x^{(1)}}^x \frac{\sin(\sqrt{\lambda_n^{(0)}}(x-\xi))}{\sqrt{\lambda_n^{(0)}}}\times$$
$$\left\{-\sum_{p=0}^{j}\lambda_n^{(j+1-p)}u_{n2}^{(p)}(\xi) + A_{n2}^{(j)}(u_{n2}^{(0)}(\xi),u_{n2}^{(1)}(\xi),\ldots,u_{n2}^{(j)}(\xi))\right\}d\xi,\quad x\in(x^{(1)},1). \tag{4.34}$$

Thus, numerical algorithm consists in finding coarse approximation (4.29), (4.30) and consecutive computations of (4.32), (4.33), (4.34) for $j=0,1,2,\ldots,m$ and (2.4).

The following convergence result is established in [58].

Theorem 4.9 *Assume that (4.28) holds and the solution of basic eigenvalue problem $\lambda_n^{(0)}$ satisfies*

$$\frac{1}{R}\left(\frac{1}{\sqrt{\lambda_n^{(0)}}}+\frac{1}{r}\right)<1, \tag{4.35}$$

where R is a convergence radius of series for generating function $f(z)=\sum_{j=0}^{\infty}\|u_n^{(j)}\|z^j$. Then the numerical algorithm (4.29),(4.30), (4.32), (4.33), (4.34), (2.4) converges exponentially to the exact solution of problem (4.27), (4.2), (4.3) with error estimates

$$\|u_n - \overset{m}{u}_n\|_\infty \le \frac{c}{(m+1)^{1+\varepsilon}}\left\{\frac{1}{R}\left(\frac{1}{\sqrt{\lambda_n^{(0)}}}+\frac{1}{r}\right)\right\}^{m+1},$$

$$|\lambda_n - \overset{m}{\lambda}_n| \leq \frac{c}{R(m+1)^{1+\varepsilon}} \left\{ \frac{1}{R} \left(\frac{1}{\sqrt{\lambda_n^{(0)}}} + \frac{1}{r} \right) \right\}^m .$$

Corollary 4.10 *Since according to [31]*

$$\sqrt{\lambda_n^{(0)}} = \begin{cases} \frac{\pi(2n+1)}{2(1-x^{(1)})}, & \frac{x^{(1)}}{1-x^{(1)}} = \frac{2k+1}{2n+1}, \\ \frac{\pi(2n+1)}{2(1-x^{(1)})} + \frac{2r}{\pi(2n+1)} + O\left(n^{-2}\right), & \frac{x^{(1)}}{1-x^{(1)}} \neq \frac{2k+1}{2n+1}, \end{cases}$$

then the convergence of the algorithm improves as the index of trial eigenpair increases and tends to the constant defined by the transmission coefficient r.

4.2.2. Integral normalizing condition

Let us consider problem (4.27), (4.28), (4.12). Following FD-approach (2.4)-(2.7) we come to the algorithm for the numerical solution of eigenvalue transmission problem.

Coarse approximation $\lambda_n^{(0)}, u_n^{(0)}(x))$ is a solution of the basic linear eigenvalue transmission problem

$$\frac{d^2 u_{ni}^{(0)}(x)}{dx^2} + \lambda_n^{(0)} u_{ni}^{(0)}(x) = 0, \qquad x \in \Omega_i, \quad i = 1, 2,$$

$$u_{n1}^{(0)}(0) = u_{n2}^{(0)}(1) = 0, \qquad \frac{du_{ni}^{(0)}(x^{(1)})}{dx} = r[u_n^{(0)}(x^{(1)})],$$

$$\|u_n^{(0)}\| = \sqrt{M},$$

that is $\lambda_n^{(0)}$ is determined by (4.29) and corresponding eigenfunctions

$$u_{n1}^{(0)}(x) = \sqrt{2M}\varphi_1 \sin(\sqrt{\lambda_n^{(0)}}x), \qquad u_{n2}^{(0)}(x) = \sqrt{2M}\varphi_2 \sin(\sqrt{\lambda_n^{(0)}}(1-x)), \tag{4.36}$$

where

$$\varphi_1 = \begin{cases} 1, & \frac{x^{(1)}}{1-x^{(1)}} = \frac{2k+1}{2n+1}; \\ \frac{1}{\cos(\sqrt{\lambda_n^{(0)}}x^{(1)})} \left(\frac{x^{(1)}}{\cos^2(\sqrt{\lambda_n^{(0)}}x^{(1)})} + \frac{1-x^{(1)}}{\cos^2(\sqrt{\lambda_n^{(0)}}(1-x^{(1)}))} + \frac{1}{r} \right)^{-1/2}, & \frac{x^{(1)}}{1-x^{(1)}} \neq \frac{2k+1}{2n+1}. \end{cases}$$

$$\varphi_2 = \begin{cases} (-1)^{n-k}, & \frac{x^{(1)}}{1-x^{(1)}} = \frac{2k+1}{2n+1}; \\ -\frac{1}{\cos(\sqrt{\lambda_n^{(0)}}x^{(1)})} \left(\frac{x^{(1)}}{\cos^2(\sqrt{\lambda_n^{(0)}}x^{(1)})} + \frac{1-x^{(1)}}{\cos^2(\sqrt{\lambda_n^{(0)}}(1-x^{(1)}))} + \frac{1}{r} \right)^{-1/2}, & \frac{x^{(1)}}{1-x^{(1)}} \neq \frac{2k+1}{2n+1}. \end{cases}$$

Corrections $u_{ni}^{(j+1)}(x)$ $(j = 0, 1, 2, \ldots)$ we find as a solution of nonhomogeneous linear transmission problem

$$\frac{d^2 u_{ni}^{(j+1)}}{dx^2} + \lambda_n^{(0)} u_{ni}^{(j+1)} = -F_{ni}^{(j+1)}(x), \qquad x \in \Omega_i, \quad i = 1, 2 \tag{4.37}$$

$$u_{n1}^{(j+1)}(0) = u_{n2}^{(j+1)}(1) = 0, \qquad \frac{du_{ni}^{(j+1)}(x^{(1)})}{dx} = r[u_n^{(j+1)}(x^{(1)})],$$

$$\sum_{i=1}^{2} \int_{\Omega_i} u_{ni}^{(0)}(x) u_{ni}^{(j+1)}(x) dx = -\frac{1}{2} \sum_{i=1}^{2} \sum_{p=1}^{j} \int_{\Omega_i} u_{ni}^{(p)}(x) u_{ni}^{(j+1-p)}(x) dx,$$

where $F_{ni}^{(j+1)}(x)$ is defined in the same way as for differential normalizing condition.

From solvability condition for nonhomogeneous equation (4.37) we get corrections for $\lambda_n^{(j+1)}$:

$$\lambda_n^{(j+1)} = \left\{ -\sum_{p=1}^{j} \lambda_n^{(j+1-p)} \sum_{i=1}^{2} \int_{\Omega_i} u_{ni}^{(p)}(\xi) u_{ni}^{(0)}(\xi) d\xi + \right. \\ \left. + \sum_{i=1}^{2} \int_{\Omega_i} A_{ni}^{(j)}(u_{ni}^{(0)}(\xi), ..., u_{ni}^{(j)}(\xi)) u_{ni}^{(0)}(\xi) d\xi \right\} / M. \tag{4.38}$$

Thus, numerical algorithm consists in finding coarse approximation (4.29), (4.36) and consecutive computations of (4.38) and solving (4.37) for $j = 0, 1, 2, \ldots, m$ and (2.4).

The following convergence result is established in [58].

Theorem 4.11 *Assume that (4.28) holds and the solution of basic eigenvalue problem $\lambda_n^{(0)}$ satisfies condition (4.35). Then the numerical algorithm (4.29),(4.36), (4.38), (4.37), (2.4) converges exponentially to the exact solution of problem (4.27), (4.28), (4.12) with error estimates*

$$\|u_n - \overset{m}{u}_n\|_\infty \le \frac{c}{(m+1)^{1+\varepsilon}} \left\{ \frac{1}{R} \left(\frac{1}{\sqrt{\lambda_n^{(0)}}} + \frac{1}{r} \right) \right\}^{m+1},$$

$$|\lambda_n - \overset{m}{\lambda}_n| \le \frac{c}{R(m+1)^{1+\varepsilon}} \left\{ \frac{1}{R} \left(\frac{1}{\sqrt{\lambda_n^{(0)}}} + \frac{1}{r} \right) \right\}^{m}.$$

Corollary 4.12 *The convergence of the algorithm improves as the index of trial eigenpair increases and tends to the constant defined by the transmission coefficient r.*

Corollary 4.13 *As it follows from (4.35) the convergence of the algorithm improves as the transmission coefficient r increases.*

5. Conclusions

Based on FD-approach the recursive algorithms for numerical solution of both linear and nonlinear eigenvalue problems are developed. For nonlinear problems we use the Adomian polynomials for approximation of nonlinear term. Algorithms provide the exponential convergence rate which improves as the index of trial eigenvalue increases. The dependence of convergence on the transmission conditions is studied. Using analytical properties of the solution of basic problems (coarse approximation) and convergence of the method, analytical properties of original problems are studied. Numerical experiments on convergence rate, dependence of eigensolution on transmission parameters, normalizing conditions confirm the theoretical results for both linear and nonlinear problems. They are reported and discussed in details in [31, 32, 34, 56–58].

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Department of Numerical Mathematics, Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereshchenkivs'ka Str., Kyiv-4, 01601, Ukraine
E-mail address: nataross@gmail.com

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