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EXACT ORDER OF ALGORITHMIC COMPLEXITY FOR FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND

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ABSTRACT. The present article is devoted to study of the algorithmic complexity for Fredholm integral equations of the second kind with kernels of Sobolev's type of smoothness. The exact order of algorithmic complexity was obtained for the case of anisotropic smoothness kernels.

1. Problem statement

Let $G = [0, 2\pi]$, $L_2 = L_2(G)$, be a space of squared summable on G functions with a standard norm $\|\cdot\|$ and a standard inner product $(\cdot, \cdot)$.

The basis functions in $L_2(G)$ are defined by formulas

$$e_0(t) = \frac{1}{\sqrt{2\pi}},$$

$$e_n(t) = \frac{1}{\sqrt{\pi}} \cos(nt),$$

$$e_{-n}(t) = \frac{1}{\sqrt{\pi}} \sin(nt)$$

where $t \in [0, 2\pi]$, $n \in \mathbb{N}$. The Fourier coefficients of $f \in L_2(G)$ are defined in a standard way

$$\hat{f}(i) = (f, e_i), \quad i \in \mathbb{Z}.$$

Let $s, r \in \mathbb{R}^+$. By H^r we denote the space of one-variable functions with norm

$$\|f\|_r := (\sum_{i \in \mathbb{Z}} (1 + |i|^2)^r \hat{f}^2(i))^{1/2},$$

and by $\mathcal{H}^{r,s}(G^2)$, $\quad G^2 = [0, 2\pi] \times [0, 2\pi]$, we denote the space of two-variables functions with norm:

$$\|k\|_{r,s} := (\sum_{i,j \in \mathbb{Z}} (1 + |i|^{2r} + |j|^{2s}) \hat{k}^2(i,j))^{1/2},$$

where

$$\hat{k}(i,j) = \int_{G^2} k(t,\tau) e_i(t) e_j(\tau) dt d\tau.$$

It is clear that spaces H^r and $\mathcal{H}^{r,s}(G^2)$ are generalizations for standard Sobolev spaces of one- and two-variables functions correspondingly which have r bounded derivatives under the metrics of L_2 (for H^r) or r bounded derivatives with respect to the first variable and s bounded derivatives with respect to the second variable (for $\mathcal{H}^{r,s}(G^2)$).

Further, by $\mathcal{H}_1^{r,s}$, H_1^r we mean unit balls in the spaces $\mathcal{H}^{r,s}(G^2)$ and H^r correspondingly.

† *Key words.* Integral equation, algorithmic complexity, representation of information, Gelfand number.

Consider the Fredholm integral equation

$$g(t) - T_k g(t) = f(t) \tag{1.1}$$

where

$$T_k g(t) = \int\limits_G k(t,\tau) g(\tau) d\tau. \tag{1.2}$$

Let $\alpha_1 > 0$, $\alpha_2 > 1$, $\alpha = (\alpha_1, \alpha_2)$. Then by

$$\mathcal{H}_\alpha^{r,s} = \{k \in \mathcal{H}^{r,s}(G^2) : \|k\|_{r,s} \le \alpha_1, \quad \|(I - T_k)^{-1}\|_{L_2 \to L_2} \le \alpha_2\}$$

denote a set of observable kernels of (1.1), (1.2) such that integral equation (1.1) is solvable.

We denote by $X_\alpha^{r,s}$ the equation class (1.1) with free terms $f \in H_1^r$ and with kernels $k(t,s) \in \mathcal{H}_\alpha^{r,s}$, i.e.,

$$X_\alpha^{r,s} = \mathcal{H}_\alpha^{r,s} \times H_1^r.$$

The present article is devoted to the study of algorithmic complexity of approximate solving equations (1.1) from the class $X_\alpha^{r,s}$. In IBC-theory ([4]) by the algorithmic complexity we mean the minimal quantity of elementary arithmetic operations required for finding the approximate solution with given accuracy. Finding the algorithmic complexity is rather intricate problem. Often one has to be satisfied only with its lower and upper estimates. So one of the main aims of IBC-theory is obtaining exact order estimates of the algorithmic complexity for wide range of problems. In the article [3] such estimate was obtained for equations from the class $X_\alpha^{r,r}$, i.e., for the case of isotropic smoothness kernels ($r = s$). The aim of the present article is to extend these studies on the equation class $X_\alpha^{r,s}, \quad s > r$.

Following [2, p.203], we give a strict problem statement about algorithmic complexity of integral equation (1.1).

Any set $N = (N_1, N_2)$ of continuous functionals such that

$$N_1 = \{\lambda_i \in (\mathcal{H}_\alpha^{r,s})^*, \ i = 1, \ldots, n_1, \}, \ N_2 = \{\sigma_j \in (H_1^r)^*, \ j = 1, \ldots, n_2\}$$

is called the representation of information about equation (1.1) from the class $X_\alpha^{r,s}$. By $\operatorname{card}(N)$ denote the number of all functionals from the set N. For a fixed set N to each equation (1.1) we assign numerical vector

$$N(k,f) = (\lambda_1(k), \lambda_2(k), \ldots, \lambda_{n_1}(k), \sigma_1(f), \sigma_2(f), \ldots, \sigma_{n_2}(f)),$$

which is considered as information about equation (1.1).

By the algorithm φ of approximate solving equation (1.1) we mean any operator $\varphi : N(k,f) \mapsto \varphi(N(k,f))$ where $\varphi(N(k,f))$ is some element of the space L_2. This element is taken for approximate solution of equation (1.1).

We denote by $\operatorname{card}(\varphi)$ the number of arithmetic operations required for realization of the algorithm φ. Moreover, for a fixed set N we denote by $\Phi(N)$ a set of algorithms φ defined on N. The error of the algorithm $\varphi \in \Phi(N)$ on the class $X_\alpha^{r,s}$ is defined as

$$e(X_\alpha^{r,s}, \varphi) = \sup_{(k,f) \in X_\alpha^{r,s}} \|S(k,f) - \varphi(N(k,f))\|$$

where $S(k,f) = (I - T_k)^{-1} f$ is the exact solution of equation (1.1).

Let

$$e_n(X_\alpha^{r,s}) = \inf_{N : \operatorname{card}(N) \le n} \ \inf_{\substack{\varphi \in \Phi(N) \\ \operatorname{card}(\varphi) \le n}} e(X_\alpha^{r,s}, \varphi).$$

The quantity $e_n(X_\alpha^{r,s})$ is the minimal error which can be achieved performing at most n arithmetic operations on the values of at most n information functionals.

The algorithmic complexity of approximate solution of equations (1.1) from the class $X_\alpha^{r,s}$ is

$$\mathrm{comp}(\varepsilon, X_\alpha^{r,s}) = \inf\{n: \ e_n(X_\alpha^{r,s}) \le \varepsilon\}$$

which is an inverse quantity to $e_n(X_\alpha^{r,s})$ in fact. Our following studies are devoted to determining the exact order estimates for $e_n(X_\alpha^{r,s})$ and $\mathrm{comp}(\varepsilon, X_\alpha^{r,s})$. For a fixed vector $N = (N_1, N_2)$ we introduce the following quantity

$$r(X_\alpha^{r,s}, N) = \inf_{\varphi\in\Phi(N)} e(X_\alpha^{r,s}, \varphi)$$

called the radius of information $N(k, f)$. From this definition it follows that the so-called nth minimal radius of information

$$r_n(X_\alpha^{r,s}) = \inf_{N:\mathrm{card}(N)\le n} r(X_\alpha^{r,s}, N)$$

is a lower bound for the algorithmic complexity, i.e.,

$$e_n(X_\alpha^{r,s}) \ge r_n(X_\alpha^{r,s}). \tag{1.3}$$

2. Auxiliary evidence

Our further arguments will be essentially using the relation between the minimal radius of information $r_n(X_\alpha^{r,s})$ and Gelfand numbers of a certain operator. To describe this relation we need some definitions and facts from the theory of s-numbers of operators in Banach spaces. Section 2 is devoted to these questions. A detailed description of this theory can be found in the monograph [1].

Let $V,\ E$ be Banach spaces. Recall that for an operator $S \in L(V, E)$ the nth approximation number of S are defined by the formula

$$a_n(S: V \to E) = \inf_{\substack{A\in L(V,E)\\ \mathrm{rank}(A)<n}} \|S - A\|_{V\to E}.$$

The following symmetric property of the approximation numbers will be useful for us.

Proposition 2.1 *[1, p.168] For all $S \in L(V, E)$*

$$a_n(S: V \to E) \ge a_n(S^*: E^* \to V^*),$$

where S^* is an adjoint operator to S.

For an operator $S \in L(V, E)$ by the nth Gelfand number of S is called by the quantity

$$c_n(S: V \to E) = \inf_{\substack{a_i\in V^*\\ i=1,2,\dots,n-1}} \ \sup_{\substack{f\in V, \|f\|_V\le 1\\ a_i(f)=0,\ i=1,2,\dots,n-1}} \|Sf\|_E.$$

Proposition 2.2 *[1, p. 169] For all operators $S_1 \in L(V_1, V),\ S_2 \in L(E, E_1)$ and $S \in L(V, E)$ the relation*

$$c_n(S_2 S S_1: V_1 \to E_1) \le \|S_2\|_{E\to E_1} \cdot c_n(S: V \to E) \cdot \|S_1\|_{V_1\to V}$$

holds true.

The properties of Gelfand numbers for diagonal operators are especially important for us.

So, let D be the diagonal operator. Recall that the operator $D \in L(l_{p_1}(\mathbb{Z}^2), l_{p_2}(\mathbb{Z}^2))$ $(1 \le p_1, p_2 \le \infty)$ is called diagonal if there are numbers $\xi_{k,m} \ge 0$ such that

$$Db_{k,m} = \xi_{k,m} b_{k,m}.$$

Here $l_{p_i}(\mathbb{Z}^2)$ is a Banach space of all p_i-summable sequences with elements from $\mathbb{Z}^2$. Let $b_{k,m}$ be the basis in the space $l_{p_1}(\mathbb{Z}^2)$.

Let $\lambda_1 \ge \lambda_2 \dots \lambda_n \ge \dots$ be the numbers $\xi_{k,m}$ arranged in nonincreasing order. Then the next result holds.

Theorem 2.3 *[1, p.150] If* $\lim\limits_{n\to\infty} \lambda_n = 0$ *then*

$$a_n(D:\ l_1 \to l_2) = \sup\left\{ \left(\frac{h-n+1}{\sum\limits_{k=1}^{h} \lambda_k^{-2}} \right)^{1/2} :\ h = n, n+1, \dots \right\}.$$

Corollary 2.4 *If* $\lim\limits_{n\to\infty} \lambda_n = 0$ *then*

$$c_n(D:\ l_2(\mathbb{Z}^2) \to l_\infty(\mathbb{Z}^2)) \ge \frac{\lambda_{2n}}{\sqrt{2}}.$$

Proof. Since D is a diagonal operator then D^* is also a diagonal operator with the same set of λ_k. From Proposition 2.1 taking into account that $l_2(\mathbb{Z}^2)$ is Hilbert space we obtain

$$c_n(D : l_2(\mathbb{Z}^2) \to l_\infty(\mathbb{Z}^2)) \ge a_n(D^* : l_1(\mathbb{Z}^2) \to l_2(\mathbb{Z}^2)).$$

Then from the last inequality and Theorem 2.3 we conclude that

$$c_n(D:\ l_2(\mathbb{Z}^2) \to l_\infty \mathbb{Z}^2) \ge \sup_{h=n,n+1,\dots} \left(\frac{h-n+1}{\sum\limits_{k=1}^{h} \lambda_k^{-2}} \right)^{1/2} \ge \left(\frac{n+1}{\sum\limits_{k=1}^{2n} \lambda_k^{-2}} \right)^{1/2} \ge$$

$$\ge \left(\frac{n+1}{2n\lambda_{2n}^{-2}} \right)^{1/2} \ge \frac{1}{\sqrt{2}} \lambda_{2n}.$$

The corollary is proved.□

Let us describe the result from [3] determining a relation between the minimal radius of information and Gelfand numbers.

Let B_K and B_V be the unit balls of the linear normed spaces K and V correspondingly with centres in zero. Suppose that there are constants $\rho = (\rho_1, \rho_2, \rho_4, \rho_5, \rho_6)$, $\rho_i > 0$, $\quad i = \overline{1,6}$ such that

$$\rho_1 B_K \subset K_0 \subset \rho_2 B_K \tag{2.1}$$

and for all $k \in K_0, \quad K_0 \subset K$

$$\|T_k\|_{E\to E} \le \rho_3, \qquad \|(I - T_k)^{-1}\|_{E\to E} \le \rho_4, \tag{2.2}$$

$$\|T_k\|_{V\to V} \le \rho_5, \qquad \|(I - T_k)^{-1}\|_{V\to V} \le \rho_6. \tag{2.3}$$

Let's consider the operator $\Psi :\ K \to L(V, E)$ mapping to each $k \in K$ the linear operator

$$\Psi_k = T_k J_V,$$

where J_V is the embedding operator from V into E.

Theorem 2.5 *[3] If conditions (2.2)-(2.3) are true then for the class $X_0 = K_0 \times B_V$*

$$r_n(X_0) \asymp \inf_{n_1+n_2 \le n} \{c_{n_1}(\Psi : K \to L(V,E)) + c_{n_2}(J_V : V \to E)\}.$$

Let check that the class $X_\alpha^{r,s}$ fulfills conditions of Theorem 2.5.

In our notations $E = L_2(G)$, $V = H^r$, $K = X^{r,s}$, $K_0 = \mathcal{X}_\alpha^{r,s}$, $B_K = H_\alpha^{r,s}$, $B_V = H_1^r$.

Lemma 2.6 *For the set $X_\alpha^{r,s} = H_\alpha^{r,s} \times H_1^r$ constants $\rho_i > 0$, $i = \overline{1,6}$ exist such that*

$$\rho_1 \mathcal{H}_1^{r,s} \subset \mathcal{H}_\alpha^{r,s} \subset \rho_2 \mathcal{H}_1^{r,s} \tag{2.4}$$

and for any $k \in \mathcal{H}_\alpha^{r,s}$

$$\|T_k\|_{H^r \to H^r} \le \rho_5, \quad \|(I - T_k)^{-1}\|_{H^r \to H^r} \le \rho_6. \tag{2.5}$$

Proof. Let us prove inequalities (2.5). So,

$$\|T_k\|_{H^r \to H^r} \le \|T_k\|_{L_2 \to H^r} = \sup_{\|f\|=1} \|T_k f\|_{H^r} =$$

$$= \sup_{\|f\|=1} \left\| \int_0^{2\pi} k(t,\tau) f(\tau) d\tau \right\|_{H^r} = \sup_{\|f\|=1} \left\| \int_0^{2\pi} k(t,\tau) \sum_{l=-\infty}^{+\infty} \hat{f}(l) e_l(\tau) d\tau \right\|_{H^r} =$$

$$= \sup_{\|f\|=1} \left[\sum_{n=-\infty}^{+\infty} (1+m^2)^r \left(\int_0^{2\pi} e_m(t) \left[\int_0^{2\infty} k(t,\tau) \sum_{l=-\infty}^{+\infty} \hat{f}(l) e_l(\tau) d\tau \right] dt \right)^2 \right]^{1/2} \le$$

$$\le \sup_{\|f\| \le 1} \left[\sum_{l,m=-\infty}^{+\infty} (1+m^2)^r k^2(t,\tau) dt d\tau \right]^{1/2} \|f\| \le \|k\|_{r,s} \le \alpha_1.$$

Check the second inequality of (2.5):

$$\|(I - T_k)^{-1}\|_{H^r \to H^r} = \|I + T_k(I - T_k)^{-1}\|_{H^r \to H^r} \le 1 + \|T_k(I - T_k)^{-1}\|_{H^r \to H^r} \le$$

$$\le 1 + \|T_k(I - T_k)^{-1}\|_{L_2 \to H^r} \le 1 + \|T_k\|_{L_2 \to H^r} \cdot \|(I - T_k)^{-1}\|_{L_2 \to L_2} \le 1 + \alpha_1 \alpha_2.$$

Finally, let us prove embedding (2.4). By the definition we have

$$\mathcal{H}_\alpha^{r,s} \subset \alpha_1 \mathcal{H}_1^{r,s}.$$

Moreover, if $\rho_1 = \min\{\frac{\alpha_2 - 1}{\alpha_2}, \alpha_1\}$, then for every $k \in \rho_1 \mathcal{H}_1^{r,s}$ we have

$$\|(I - T_k)^{-1}\|_{L_2 \to L_2} = \left\| \sum_{l=0}^{\infty} T_k^l \right\|_{L_2 \to L_2} \le \sum_{l=0}^{\infty} \|T_k\|_{L_2 \to L_2} \le$$

$$\le \sum_{l=0}^{\infty} \|k\|_{r,s}^l \le \sum_{l=0}^{\infty} \rho_1^l = \frac{1}{1 - \rho_1} \le \alpha_2.$$

Thus, we obtain assertion of lemma with

$$\rho_1 = \alpha_1, \quad \rho_2 = \min\{\frac{\alpha_2 - 1}{\alpha_2}, \alpha_1\}, \quad \rho_3 = \alpha_1,$$

$$\rho_4 = \alpha_2, \quad \rho_5 = \alpha_1, \quad \rho_6 = 1 + \alpha_1 \alpha_2.$$

The lemma is proved. □

So, from inequality (1.3), Lemma 2.6 and Theorem 2.5 we obtain

$$e_n(X_\alpha^{r,s}) \ge \inf_{n_1+n_2 \ge n} \{c_{n_1}(\Psi : K \to L(V,E)) + c_{n_2}(J_V : V \to E)\} \ge$$

$$\ge c_n\{\Psi : K \to L(V,E)\}. \tag{2.6}$$

3. Main result

Following [3], let us describe the algorithm of approximate solving the equations (1.1) from the class $X_\alpha^{r,s}$. This algorithm consists in computing the solution of equation

$$\hat{g}(t) - \overline{T}_k \hat{g}(t) = Sf(t), \tag{3.1}$$

where

$$\overline{T}_k \hat{g}(t) = \sum_{l,m=1}^{n_1} e_l(t)\hat{k}(l,m) \int_0^{2\pi} e_m(\tau) g(\tau) d\tau,$$

$$Sf(t) = \sum_{m=1}^{n_2} e_m(t) \hat{f}(m),$$

and information about equation is a set of Fourier coefficients of the form

$$N(k,f) = \{\hat{k}(i,j), \quad \hat{f}(j),\ |i \cdot j| \le n_1,\ |j| \le n_2\}.$$

We denote by $\hat{\varphi}$ the algorithm which assigning to every equation (1.1) the element $\hat{\varphi}(N(k,f)) := \hat{g}(t)$ as approximate solution.

Theorem 3.1 *For any* $r,\ s \in \mathbb{R}^+, \quad s > r,$

$$e_n(X_\alpha^{r,s}) \asymp n^{-r} \log^r n.$$

Optimal order of the quantity $e_n(X_\alpha^{r,s})$ *is reached within the framework of the algorithm* $\hat{\varphi}$.

Proof. Upper estimate of the quantity $e(X_\alpha^{r,r})$ was obtained in [3]. Then it is easy to see that with $s > r$ an embedding

$$X_\alpha^{r,r} \supset X_\alpha^{r,s}$$

occurs. This implies relation

$$e_n(X_\alpha^{r,r}) \ge e_n(X_\alpha^{r,s}).$$

Thus, upper estimate of the quantity $e_n(X_\alpha^{r,s})$ is

$$e_n(X_\alpha^{r,r}) \prec n^{-r} \log^r n$$

obtained in [3]. In this paper it was also established that the order $O(n^{-r}\log^r n)$ for the minimal error is achieved within framework of the algorithm $\hat{\varphi}$.

Thus, it remains to prove the lower estimate of the quantity $e_n(X_\alpha^{r,s})$. In view of inequality (2.6) it is enough to estimate Gelfand numbers of the operator Ψ. To this end we consider the operator $W:\ l_2(\mathbb{Z}^2) \to \mathcal{H}_\alpha^{r,s}$ defined by relation

$$Wb_{i,j} = (1 + |i|^{2r} + |j|^{2s})^{-1/2} e_i(t) e_j(\tau),$$

where $\{b_{i,j}\}$ is the basis in $l_2(\mathbb{Z}^2)$. Show that operator W is an isometry, i.e., that inequality

$$\|b_{i,j} - b_{l,m}\|_{l_2} = \|Wb_{i,j} - Wb_{l,m}\|_{r,s}$$

takes place. So, it is easy to verify that

$$\|b_{i,j} - b_{l,m}\|_{l_2} = \sqrt{2}.$$

From the definition of the set $\mathcal{H}_\alpha^{r,s}$ we have

$$\|Wb_{i,j} - Wb_{l,m}\|_{r,s} = \|(1+|i|^{2r}+|j|^{2s})^{-1/2} e_i(t) e_j(\tau) - (1+|l|^{2r}+|m|^{2s})^{-1/2} e_l(t) e_m(\tau)\|_{r,s} =$$

$= (\sum_{k,n}(1+|k|^{2r}+|n|^{2s})\cdot\hat{\varphi}(k,n)^2)^{1/2}$.

Let us compute $\hat{\varphi}(k,n)$:

$$\hat{\varphi}(k,n) = \int_0^{2\pi}\int_0^{2\pi}(1+|i|^{2r}+|j|^{2s})^{-1/2}e_i(t)e_j(\tau)e_k(t)e_n(\tau)dtd\tau -$$

$$-\int_0^{2\pi}\int_0^{2\pi}(1+|l|^{2r}+|m|^{2s})^{-1/2}e_l(t)e_m(\tau))e_k(t)e_n(\tau)dtd\tau = (1+|l|^{2r}+|m|^{2s})^{-1/2}.$$

Then we obtain

$$\|Wb_{i,j}-Wb_{l,m}\|_{r,s} = [(1+|i|^{2r}+|j|^{2s})(1+|i|^{2r}+|j|^{2s})^{-1}+$$

$$+(1+|l|^{2r}+|m|^{2s})(1+|l|^{2r}+|m|^{2s})^{-1}]^{1/2} = \sqrt{2}.$$

Thus, W is an isometry, whence it appears that $\|W\| = 1$.

Further, by $U : L(H^r, L_2) \to l_\infty(\mathbb{Z}^2)$ we define the operator assigning to each operator $A \in L(H^r, L_2)$ an element from $l_\infty(\mathbb{Z}^2)$ of the following form

$$\{(1+|m|^2)^{-r/2}\cdot(Ae_m, e_l)\}_{m,l\in\mathbb{Z}}.$$

The operator U is an injection with $\|U\| \le 1$. Now we compose an operators U, W with operators $\Psi : \mathcal{H}_\alpha^{r,s} \to L(H^r, L_2(G))$ and consider an operator

$$D = U\Psi W,$$

$$D : l_2(\mathbb{Z}^2) \to l_\infty(\mathbb{Z}^2).$$

Obviously, D is a diagonal operator which acts in the following way:

$$Db_{i,j} = \xi_{i,j}b_{i,j}, \quad \xi_{i,j} = (1+|i|^{2r}+|j|^{2s})^{-1/2}(1+|j|^2)^{-r/2}.$$

Let $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n \ge \ldots$ be the elements of the sequence $\{\xi_{i,j}\}$ arranged in nonincreasing order. Namely,

$$\lambda_n = \inf\{\varepsilon :\ \mathrm{card}\{(i,j) :\ \xi_{i,j} > \varepsilon\} < n\} = \max_{\substack{Q\subseteq\mathbb{Z}^2\\ \mathrm{card}(Q)=n}} \min\{\xi_{i,j} :\ (i,j)\in Q\}.$$

Let us consider

$$Q_n = \{(i,j) :\ \xi_{ij} \ge n^{-\frac{2}{r+s}}\},$$

i.e.,

$$Q_n = \{(i,j) :\ (1+|i|^{2r}+|j|^{2s})^{\frac{1}{r+s}}\cdot(1+|j|^2)^{\frac{r}{r+s}} \le n\}.$$

Then by construction

$$\lambda_{\mathrm{card}(Q_n)} \ge n^{-\frac{r+s}{2}}. \tag{3.2}$$

Indeed,

$$\lambda_{\mathrm{card}(Q_n)} = \max_{\substack{Q\subseteq\mathbb{Z}^2\\ \mathrm{card}(Q)=\mathrm{card}(Q_n)}} \min\{(1+|m|^{2r}+|l|^{2s})^{-\frac{1}{2}}\cdot(1+|l|^2)^{-\frac{r}{2}}\} \ge$$

$\ge \{\xi_{ij} : (i,j)\in Q_n\} \ge n^{-\frac{r+s}{2}}$.

Let us find the order of the quantity $\mathrm{card}(Q_n)$. For this end we consider the set

$$B_n = \{(i,j) : \max\{1,|i|^{2r},|j|^{2s}\}^{\frac{1}{r+s}}\cdot\max\{1,|j|^2\}^{\frac{r}{r+s}} \le n\}.$$

An obvious embedding takes place

$$Q_n \subset B_n \subset Q_{\lceil n\rho \rceil}, \quad \rho = \sqrt[r+s]{3 \cdot 2^r}. \tag{3.3}$$

We estimate card(B_n):

$$\text{card}(B_n) = \text{card}\{(i,j): \ \max\{1, |i|^{2r}, |j|^{2s}\}^{\frac{1}{r+s}} \cdot \max\{1, |j|^2\}^{\frac{r}{r+s}} \le n\} =$$

$$= \text{card}\{(i,j): \ |j| \le n^{1/2}, \quad |i| \le n^{\frac{s}{2r}}\} + \sum_{n^{\frac{s}{2r}} < l \le n^{\frac{s+r}{2r}}} \text{card}\{(i,j): \ |i| = l, \quad |j| \le \frac{n^{\frac{s+r}{2r}}}{l}\} \asymp$$

$$\asymp n^{\frac{r+s}{2r}} + \sum_{n^{\frac{s}{2r}} < l \le n^{\frac{s+r}{2r}}} \frac{n^{\frac{s+r}{2r}}}{l} \asymp n^{\frac{r+s}{2r}} \sum_{n^{\frac{s}{2r}} < l \le n^{\frac{r+s}{2r}}} \frac{1}{l} \asymp n^{\frac{r+s}{2r}} \log n.$$

Now from (3.3) we have

$$\text{card}(Q_n) \asymp n^{\frac{r+s}{2r}} \log n. \tag{3.4}$$

Combining (3.2) and (3.4) we obtain

$$\lambda_{[n^{\frac{r+s}{2r}} \log n]} \asymp n^{-\frac{r+s}{2}},$$

which is equivalent to

$$\lambda_n \asymp (n^{-\frac{r+s}{2}})^{\frac{2r}{r+s}} \log^r n = n^{-r} \log^r n.$$

Then by Corollary 2.4

$$c_n(D: \ l_2(\mathbb{Z}^2) \to l_\infty(\mathbb{Z}^2)) \asymp (2n^{-r}) \log^r(2n) \ge 2^{-r} n^{-r} \log^r n \ge c_0 n^{-r} \log^r n. \tag{3.5}$$

Here and further by c_i we mean positive independent of n constants. Taking into account Proposition 2.2 we obtain

$$c_n(D: l_2(\mathbb{Z}^2) \to l_\infty(\mathbb{Z}^2) \le \|U\| \cdot c_n(\Psi : \mathcal{H}_\alpha^{r,s} \to L(H^r, L_2)) \cdot \|W\|. \tag{3.6}$$

Finally, from (2.6), (3.5), (3.6) it follows that

$$e_n(X_\alpha^{r,s}) \succ n^{-r} \log^r n.$$

This completes the proof of Theorem. □

Directly from Theorem 3.1 it is easy to obtain the following estimate for the algorithmic complexity of approximate solving equations (1.1).

Corollary 3.2 *For the arbitrary* $r, s \in \mathbb{R}^+, \quad s > r$, *the estimate*

$$\text{comp}(\varepsilon, X_\alpha^{r,s}) \asymp \varepsilon^{-r} \log \frac{1}{\varepsilon}$$

is true. The optimal order of the quantity comp$(\varepsilon, X_\alpha^{r,s})$ *on the equation class* $X_\alpha^{r,s}$ *is realized within framework of the algorithm* $\hat{\varphi}$.

Remark 3.3 *It should be noted that problem of the algorithmic complexity for the equations (1.1) from the class* $X_\alpha^{r,s}$ *is studied for a long time. In particular, in 1989 at [5, 6] for the case of the isotropic smoothness kernels of integral equations (1.1) the double-ended estimate*

$$c_1 n^{-r} \le e_n(X_\alpha^{r,r}) \le c_2 n^{-r} \log^r n$$

was established. Generalization of the obtained result on the case of an arbitrary $r, s \in \mathbb{R}$ *for operator equations of the second kind was obtained at [7]:*

$$c_3 n^{-r} \le e_n(X_\alpha^{r,s}) \le c_4 n^{-r} \log^{r+\frac{1}{2}} n.$$

Later the lower estimate was improved by restricting equation classes to the scale $X_\alpha^{r,s}$*. So, at [3] it was computed the quantity* $e_n(X_\alpha^{r,r}) = O(n^{-r} \log^r n)$*. Later ([8]) this estimations was extended to the case of anisotropic smoothness* $r \ge s$*. Finally, the result of this article in the aggregate with results from [8] allows to extend the estimation from [3] on the whole scale of these classes* $X_\alpha^{r,s}, \quad r, s \in \mathbb{R}^+$.

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