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ON MATRIX FUNCTION INTERPOLATION

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ABSTRACT. We consider the problem of interpolation of matrix functions in case of ordinary, Jordan, Hadamard and Frobenius multiplication rule. We give interpolation formulas of Lagrange and Hermite type, obtain sufficient conditions for convergence of interpolation processes and estimate interpolation errors in the class of analytic functions.

Interpolating matrix functions is in concept similar to interpolating functions of a scalar variable. For sure, the theory of matrix function interpolation is more complicated than its subcase, the classical theory of interpolation, which is widely used in the mathematics itself as well as in adjacent areas of science. The theory of matrix function interpolation finds its application when dealing with such problems that are usually solved using matrix apparatus.

1. Lagrange interpolation formulas

Let X be a set of square matrices of fixed size. We introduce an operator $F : X \to Y$, where Y is some given set. Elements of Y can be numbers, matrices, functions etc.

Let $\lambda_0, \lambda_1, \ldots, \lambda_r$ be different eigenvalues of a square matrix A of size m. Suppose the multiplicities of the eigenvalues are $\alpha_0, \alpha_1, \ldots, \alpha_r$ ($\alpha_0+\alpha_1+\ldots+\alpha_r = m$) correspondingly. Then for a function $F(z)$, which is analytic in a domain containing the spectrum of the matrix A, the following Lagrange–Silvester interpolation formula [1] is true:

$$F(A) = \sum_{k=0}^{r} \sum_{\nu=0}^{\alpha_k-1} H_{\nu k}(A) F^{(\nu)}(\lambda_k), \tag{1.1}$$

where $H_{\nu k}(z)$ are knows algebraic polynomials of degree m. This formula lets one restore the function $F(z)$ in the point A when the values $F^{(\nu)}(\lambda_k)$ are known, where $\{\lambda_k\}$ is the spectrum of A. The formula (1.1) may be considered as an interpolation formula of Hermite type for the nodes A_k being the scalar matrices $\lambda_k I$, where I stands for an identity matrix.

Further in this section we'll consider formulas of Lagrange type for particular kinds of nodes such as scalar matrices and others. An algebraic interpolation polynomial of degree n will be denoted by $L_n(F; A) \equiv L_n(A)$ (sometimes instead of n we'll use $0n$ or $n0$ as a subscript). The denotation for trigonometric interpolation polynomial is $T_n(F; A) \equiv T_n(A)$.

Let A, $A_k \in X$, $k = 0, 1, \ldots, n$. Suppose that the matrices $(A_k - A_\nu)$, $k \neq \nu$, are invertible. Then

$$L_n(F; A) = \sum_{k=0}^{n} l_{nk}(A) l_{nk}^{-1}(A_k) F(A_k), \tag{1.2}$$

[†]*Key words.* Interpolation, matrix functions, interpolation matrix polynomials, interpolation error estimate.

where $l_{nk}(A) = (A - A_0)(A - A_1)\cdots(A - A_{k-1})(A - A_{k+1})\cdots(A - A_n)$. The formula of trigonometric interpolation for the nodes $A_k \in X$, $k = 0, 1, \ldots, 2n$, under the assumption that the matrices $\sin\frac{1}{2}(A_k - A_\nu)$, $k \neq \nu$, are invertible, looks the following way:

$$T_n(A) = \sum_{k=0}^{2n} \psi_k(A)\psi_k^{-1}(A_k)F(A_k), \tag{1.3}$$

where $\psi_k(A) = \sin\frac{A-A_0}{2}\cdots\sin\frac{A-A_{k-1}}{2}\sin\frac{A-A_{k+1}}{2}\cdots\sin\frac{A-A_{2n}}{2}$.

More general formulas of Lagrange interpolation can be given (in the algebraic case) in such a form as

$$L_n(F;A) = \sum_{k=0}^{n} l_{nk}(A)l_{nk}^{-1}(A_k)F(A_k), \tag{1.4}$$

where $l_{nk}(A) = B_{k0}(A-A_0)B_{k1}\cdots B_{nk-1}(A-A_{k-1})B_{nk}(A-A_{k+1})B_{nk+1}\cdots B_{nn-1}(A-A_n)B_{nn}$, $B_{n\nu}$ are some fixed matrices. For the trigonometric case one obtains

$$T_n(A) = \sum_{k=0}^{2n} \psi_k(A)\psi_k^{-1}(A_k)F(A_k), \tag{1.5}$$

where $\psi_k(A) = B_{k0}\sin\frac{A-A_0}{2}B_{k1}\cdots B_{kk-1}\sin\frac{A-A_{k-1}}{2}B_{kk}\sin\frac{A-A_{k+1}}{2}B_{kk+1}\cdots B_{k2n}\sin\frac{A-A_{2n}}{2}B_{k2n+1}$; here, as in the algebraic case, $B_{k\nu}$ are given matrices and A_k, $k, \nu = 0, 1, \ldots, 2n$, are the nodes of interpolation.

It is easy to check that the interpolation conditions for the formulas (1.2)–(1.5) and for further formulas are met: one should substitute A_k in place of A, each time taking into account the structure of fundamental interpolation polynomials.

Formulas for sets of matrices with other multiplication rules Let's give interpolation formulas, that contain Jordan, Hadamard and Frobenius matrix multiplications, denoted by the symbols $\circ$, $\bullet$ and $\diamond$ correspondingly.

The operation of Jordan multiplication of the square matrices A and B is defined by the equity $A \circ B = \frac{1}{2}(AB + BA)$. This operation is nonassociative in general: there exist such matrices A, B and C that their associator $(A \circ B) \circ C - A \circ (B \circ C)$ is nonzero.

Let $l_{nk}(A) = (A - A_0) \circ (A - A_1) \circ \ldots \circ (A - A_{k-1}) \circ (A - A_{k+1}) \circ \ldots \circ (A - A_n)$ and the order of taking products in this expression be fixed. Then for the formulas

$$L_{0n}(A) = \sum_{k=0}^{n} F(A_k) \circ \left\{ l_{nk}^{-1}(A_k) \circ l_{nk}(A) \right\}, \tag{1.6}$$

$$L_{n0}(A) = \sum_{k=0}^{n} \left\{ F(A_k) \circ l_{nk}^{-1}(A_k) \right\} \circ l_{nk}(A) \tag{1.7}$$

the following equities take place:

$$L_{0n}(A_k) = L_{n0}(A_k) = F(A_k),\ k = 0, 1, \ldots, n.$$

Remark 1.1 *In the formulas* (1.6) *and* (1.7) *one should take matrix products in curly brackets at first. Also in the formula* (1.7) *we suppose that the associator of the matrices* $F(A_k)$, $l_{nk}^{-1}(A_k)$ *and* $l_{nk}(A_k)$ *equals zero.*

In particular, the formula of linear interpolation can be written as

$$L_{01}(A) = F(A_0) + [F(A_1) - F(A_0)] \circ \left\{ (A_1 - A_0)^{-1} \circ (A - A_0) \right\}.$$

This formula is invariant under the polynomials

$$P_{01}(A) = D \circ \left\{(A_1 - A_0)^{-1} \circ A\right\} + C,$$

where D and C are arbitrary matrices. One may write $P_{01}(A)$ using ordinary matrix multiplication:

$$\begin{aligned} P_{01}(A) &= \frac{1}{4} D \left[(A_1 - A_0)^{-1} A + A(A_1 - A_0)^{-1}\right] \\ &+ \frac{1}{4}\left[(A_1 - A_0)^{-1} A + A(A_1 - A_0)^{-1}\right] D + C. \end{aligned}$$

Let's give one more interpolation formula

$$L_n(A) = \sum_{k=0}^{n} F(A_k) \circ \tilde{l}_{nk}(A),$$

where

$$\begin{aligned} \tilde{l}_{nk}(A) = &\left\{(A - A_0) \circ (A_k - A_0)^{-1}\right\} \circ \ldots \circ \left\{(A - A_{k-1}) \circ (A_k - A_{k-1})^{-1}\right\} \\ &\circ \left\{(A - A_{k+1}) \circ (A_k - A_{k+1})^{-1}\right\} \circ \ldots \circ \left\{(A - A_n) \circ (A_k - A_n)^{-1}\right\}, \end{aligned}$$

in which one may replace circles with dots and get matrix interpolation formulas containing both Jordan and ordinary multiplication.

Further we'll consider interpolation formulas for the case of Hadamard matrix multiplication. The Hadamard product of matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of same size is $A \bullet B = [a_{ij} b_{ij}]$. Hadamard matrix multiplication is also called Schur multiplication. It is, obviously, commutative and associative. The role of identity matrix is played by J all the elements of which are unitary: $A \bullet J = J \bullet A = A$. The denotation $A^{\overset{\bullet}{-1}}$ stands for the Hadamard inverse matrix of A: $A \bullet A^{\overset{\bullet}{-1}} = A^{\overset{\bullet}{-1}} \bullet A = I$. Note that in general it is not uniquely defined. If the diagonal elements a_{ii}, $i = 1, 2, \ldots, m$, of the matrix A of size m are nonzero, then $A^{\overset{\bullet}{-1}} = \operatorname{diag}\left[\frac{1}{a_{ii}}\right] = \operatorname{diag}\left\{\frac{1}{a_{11}}, \frac{1}{a_{22}}, \ldots, \frac{1}{a_{mm}}\right\}$. Let also $A^{(-1)} = \left[\frac{1}{a_{ij}}\right]$. The matrix $A^{(-1)}$ exists if all the elements of A are nonzero.

Let $q_k(A) = (A - A_0) \bullet \ldots \bullet (A - A_{k-1}) \bullet (A - A_{k+1}) \bullet \ldots \bullet (A - A_n)$, $k = 0, 1, \ldots, n$, where $A = [a_{ij}]$. Suppose the interpolation nodes $A_k = \left[a_{ij}^k\right]$ and the matrices $F(A_k) = \left[f_{ij}^k\right]$, $k = 0, 1, \ldots, n$, are square and of same size, and for $q_k(A_k)$ there exists Hadamard inverse matrix $q_k^{\overset{\bullet}{-1}}(A_k)$. Then for the formula

$$\begin{aligned} L_{0n}(A) &= \sum_{k=0}^{n} F(A_k) \left\{ q_k^{\overset{\bullet}{-1}}(A_k) \bullet q_k(A) \right\} \\ &= \sum_{k=0}^{n} F(A_k) \operatorname{diag} \left[\frac{(a_{ii} - a_{ii}^0) \cdots (a_{ii} - a_{ii}^{k-1}) \left(a_{ii} - a_{ii}^{k+1}\right) \cdots (a_{ii} - a_{ii}^n)}{(a_{ii}^k - a_{ii}^0) \cdots (a_{ii}^k - a_{ii}^{k-1}) \left(a_{ii}^k - a_{ii}^{k+1}\right) \cdots (a_{ii}^k - a_{ii}^n)} \right], \end{aligned}$$

where the product of $F(A_k)$ and any of the matrices $\left\{ q_k^{\overset{\bullet}{-1}}(A_k) \bullet q_k(A) \right\}$ can be ordinary or Jordan, the following conditions are met: $L_{0n}(A_\nu) = F(A_\nu)$, $\nu = 0, 1, \ldots, n$.

For the formula

$$\begin{aligned} L_{n0}(A) &= \sum_{k=0}^{n} F(A_k) \bullet q_k^{(-1)}(A_k) \bullet q_k(A) \\ &= \sum_{k=0}^{n} \left[\frac{\left(a_{ij} - a_{ij}^0\right) \cdots \left(a_{ij} - a_{ij}^{k-1}\right) \left(a_{ij} - a_{ij}^{k+1}\right) \cdots \left(a_{ij} - a_{ij}^n\right)}{\left(a_{ij}^k - a_{ij}^0\right) \cdots \left(a_{ij}^k - a_{ij}^{k-1}\right) \left(a_{ij}^k - a_{ij}^{k+1}\right) \cdots \left(a_{ij}^k - a_{ij}^n\right)} f_{ij}^k \right], \end{aligned}$$

where only Hadamard matrix multiplication is used, interpolation conditions also hold, because $q_k^{(-1)}(A_k) \bullet q_k(A_\nu) = \delta_{k\nu} J$, where $\delta_{k\nu}$ is Kronecker's delta. Here we assume that the matrices $(A_k - A_\nu)$, $k \neq \nu$, don't contain zero elements.

For the nodes $A_k = \alpha_k J$ ($\alpha_k \neq \alpha_\nu$ if $k \neq \nu$) the following interpolation formula holds:

$$L_n(A) = \sum_{k=0}^{n} \frac{(A-\alpha_0 J)\bullet \ldots \bullet (A-\alpha_{k-1}J)\bullet(A-\alpha_{k+1}J)\bullet\ldots\bullet(A-\alpha_n J)}{(\alpha_k-\alpha_0)\cdots(\alpha_k-\alpha_{k-1})(\alpha_k-\alpha_{k+1})\cdots(\alpha_k-\alpha_n)} \bullet F(\alpha_k J).$$

Let's give an interpolation formula containing Frobenius matrix multiplication. Let the matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ be of same size. Their Frobenius product is $A \diamond B = \sum_{i,j} a_{ij} b_{ij}$. This operation is, obviously, commutative and its result is a scalar. Then for the formula

$$L_n(A) = \sum_{k=0}^{n} \frac{l_{nk}(A)}{l_{nk}(A_k)} F(A_k),$$

where

$$\begin{aligned} l_{nk}(A) = {} & \big[(A-A_0)^T \diamond (A_k - A_0)\big] \cdots \big[(A-A_{k-1})^T \diamond (A_k - A_{k-1})\big] \\ & \times \big[(A-A_{k+1})^T \diamond (A_k - A_{k+1})\big] \cdots \big[(A-A_n)^T \diamond (A_k - A_n)\big] \end{aligned}$$

and the interpolation nodes A_k are different, the following conditions hold:

$$L_n(A_\nu) = F(A_\nu), \;\; \nu = 0, 1, \ldots, n.$$

Interpolation formulas for specific kinds of nodes In the above interpolation formulas the biggest computational difficulties are caused by matrix inversion. Let's consider such interpolation nodes for which inverse matrices entering (1.2), (1.3) and other formulas, can be found easily enough, and give an explicit form of these formulas.

If the nodes A_k are the different scalar matrices $a_k I$, Lagrange matrix interpolation formula (1.2) looks as

$$L_n(F;A) = \sum_{k=0}^{n} \frac{(A-a_0 I)\cdots(A-a_{k-1}I)(A-a_{k+1}I)\cdots(A-a_n I)}{(a_k-a_0)\cdots(a_k-a_{k-1})(a_k-a_{k+1})\cdots(a_k-a_n)} F(a_k). \qquad (1.8)$$

This formula is invariant under the matrix polynomial of kind $P_n(A) = \sum_{k=0}^{n} b_k A^k$, where b_k are arbitrary numbers.

If $A_k = a_k I$, $0 \leq a_k < 2\pi$, $k = 0, 1, \ldots, 2n$ ($a_k \neq a_\nu$ if $k \neq \nu$), then the formula of trigonometric interpolation (1.3) also has the simple form

$$T_n(F;A) = \sum_{k=0}^{2n} \frac{\sin\frac{A-a_0 I}{2}\cdots\sin\frac{A-a_{k-1}I}{2}\sin\frac{A-a_{k+1}I}{2}\cdots\sin\frac{A-a_{2n}I}{2}}{\sin\frac{a_k-a_0}{2}\cdots\sin\frac{a_k-a_{k-1}}{2}\sin\frac{a_k-a_{k+1}}{2}\cdots\sin\frac{a_k-a_{2n}}{2}} F(a_k). \qquad (1.9)$$

This formula is exact for trigonometric matrix polynomials of degree n:

$$F(A) = c_0 I + \sum_{k=1}^{n} (c_k \cos kA + d_k \sin kA),$$

where c_k, d_k are certain numbers. In the formulas (1.8) and (1.9) the function $F(z)$ is considered analytic in neighborhoods of the points a_k.

When the diagonal matrices $A_k = \operatorname{diag}\{a_{0k}, a_{1k}, \ldots, a_{mk}\}$ ($a_{ik} \neq a_{i\nu}$ if $k \neq \nu$, $i = 0, 1, \ldots, m$), $k = 0, 1, \ldots, n$, are taken as nodes of interpolation, the following equity holds:

$$L_n(F; A) = \sum_{k=0}^{n} l_{nk}(A) l_{nk}^{-1}(A_k) F(A_k),$$

where $l_{nk}(A) = (A - A_0) \cdots (A - A_{k-1})(A - A_{k+1}) \cdots (A - A_n)$, $l_{nk}^{-1}(A_k) = \operatorname{diag}\left\{\frac{1}{\omega'_{n0}(a_{0k})}, \frac{1}{\omega'_{n1}(a_{1k})}, \ldots, \frac{1}{\omega'_{nm}(a_{mk})}\right\}$, $\omega_{n\nu}(t) = \prod_{j=0}^{n}(t - a_{\nu j})$, $\nu = 0, 1, \ldots, m$.

In the trigonometric case for the diagonal nodes $A_k = \operatorname{diag}\{a_{0k}, a_{1k}, \ldots, a_{mk}\}$, $0 \leq a_{ik} < 2\pi$, $i = 0, 1, \ldots, m$, $k = 0, 1, \ldots, 2n$, interpolation polynomial (1.3) looks as

$$T_n(A) = \sum_{k=0}^{2n} \psi_k(A) \psi_k^{-1}(A_k) F(A_k),$$

where $\psi_k(A) = \sin\frac{A - A_0}{2} \cdots \sin\frac{A - A_{k-1}}{2} \sin\frac{A - A_{k+1}}{2} \cdots \sin\frac{A - A_{2n}}{2}$,

$\psi_k^{-1}(A_k) = \frac{1}{2}\operatorname{diag}\left\{\frac{1}{\Omega'_{2n,0}(a_{0k})}, \frac{1}{\Omega'_{2n,1}(a_{1k})}, \ldots, \frac{1}{\Omega'_{2n,m}(a_{mk})}\right\}$, $\Omega_{2n,\nu}(t) = \prod_{j=0}^{2n} \sin\frac{1}{2}(t - a_{\nu j})$.

Let $A_k = \eta_k I + H$ ($\eta_k \neq \eta_\nu$ if $k \neq \nu$), $k = 0, 1, \ldots, n$, where H is a certain matrix with zero diagonal elements. Then the Lagrange interpolation polynomial takes the form

$$L_n(A) = \sum_{k=0}^{n} \frac{(A - A_0) \cdots (A - A_{k-1})(A - A_{k+1}) \cdots (A - A_n)}{(\eta_k - \eta_0) \cdots (\eta_k - \eta_{k-1})(\eta_k - \eta_{k+1}) \cdots (\eta_k - \eta_n)} F(A_k). \tag{1.10}$$

Given $\eta_k = \cos\frac{2k-1}{2n}\pi$, $k = 1, 2, \ldots, n$, this formula is transformed to the following form:

$$L_{n-1}(A) = \frac{1}{n} \sum_{k=1}^{n} (-1)^{k-1} \sin\frac{2n-1}{2n}\pi \; l_{nk}(A - H) F(\eta_k I + H), \tag{1.11}$$

where $l_{nk}(A) = (A - \eta_1 I) \cdots (A - \eta_{k-1} I)(A - \eta_{k+1} I) \cdots (A - \eta_n I)$.

In the trigonometric case, for the same nodes $A_k = \eta_k I + H$, $0 \leq \eta_k < 2\pi$, $k = 0, 1, \ldots, 2n$, we have

$$T_n(A) = \sum_{k=0}^{2n} \frac{\sin\frac{A-A_0}{2} \cdots \sin\frac{A-A_{k-1}}{2} \sin\frac{A-A_{k+1}}{2} \cdots \sin\frac{A-A_{2n}}{2}}{\sin\frac{\eta_k-\eta_0}{2} \cdots \sin\frac{\eta_k-\eta_{k-1}}{2} \sin\frac{\eta_k-\eta_{k+1}}{2} \cdots \sin\frac{\eta_k-\eta_{2n}}{2}} F(A_k). \tag{1.12}$$

Now let us consider formulas of linear and quadratic interpolation of some other structure for arbitrary matrix nodes A_0 and A_1. Then under the assumption that $P_1(A) = AB_1 + C_1$, where B_1 and C_1 are certain matrices, this problem is solvable if and only if [2] the function $F(A)$ at nodes A_0 and A_1 is representable as $F(A_i) = A_i V$, $i = 0, 1$, where V is a certain matrix. In this case for the matrix polynomials

$$\begin{aligned} L_{10}(A) &= F(A_0) + (A - A_0)\left\{(A_1 - A_0)^+[F(A_1) - F(A_0)] + N\right\}, \\ L_1(A) &= F(A_0) + (A - A_0)B \\ &\quad + (A - A_0)\left\{(A_1 - A_0)^+[F(A_1) - F(A_0) - (A_1 - A_0)B] + N\right\}, \end{aligned}$$

where the matrix $(A_1 - A_0)^+$ is the Moore–Penrose pseudoinverse of $(A_1 - A_0)$, B is a certain matrix, $N \in \ker(A_1 - A_0)$, interpolation conditions $L_{10}(A_i) = L_1(A_i) = F(A_i)$, $i = 0, 1$, are met.

Let's give one more formula of linear interpolation

$$L_1(A) = F(A_0) + \int_0^1 \delta F[A_0 + \tau(A_1 - A_0); A - A_0] d\tau \tag{1.13}$$

and also a formula of quadratic interpolation

$$L_2(A) = L_1(A) + \int_0^1 \int_0^\tau \delta^2 F[A_0 + \tau(A_1 - A_0) + s(A_2 - A_1); (A - A_1)(A - A_0)] ds d\tau, \tag{1.14}$$

where $\delta F[A;h]$ and $\delta^2 F[A;h_1,h_2]$ are Gateaux differentials of first and second order, at the point A, in the directions h and (h_1,h_2) respectively.

To make sure that interpolation conditions hold for $L_1(A)$ and $L_2(A)$, one may use the relations $\delta F[A_0 + \tau h_1; h_1] = \frac{d}{d\tau} F[A_0 + \tau h_1]$ and $\delta^2 F[A_0 + \tau h_1 + s h_2; h_2, h] = \frac{\partial}{\partial s} \delta F[A_0 + \tau h_1 + s h_2; h]$.

Note that the formula (1.13) is exact for the matrix polynomials $P_1(A) = K_{00} + \sum_{i=1}^{m_1} K_{1i} A K_{2i}$ of first degree, whereas (1.14) is exact for the matrix polynomials $P_2(A) = P_1(A) + \sum_{0 \le i,j \le m_1} B_{ij} A C_{ij} A D_{ij}$ of second degree, where K_{ij}, B_{ij}, C_{ij}, D_{ij} are certain matrices, and A, A_0 and A_1 are matrices of same size. One may verify this statement by computing the integrals in (1.13) and (1.14) for $F(A) = P_1(A)$ and $F(A) = P_2(A)$.

Example 1.2 *Let $F(A) = e^A$. Then the formula of linear interpolation* (1.13) *looks the following way:*

$$L_1(A) = e^{A_0} + \int_0^1 \int_0^1 e^{(1-s)[A_0 + \tau(A_1 - A_0)]} (A - A_0) e^{s[A_0 + \tau(A_1 - A_0)]} d\tau ds.$$

If the matrices A, A_0 and A_1 are interchangeable and the inverse $(A_1 - A_0)^{-1}$ exists, then $L_1(A) = e^{A_0} + (A - A_0)(A_1 - A_0)^{-1} \left[e^{A_1} - e^{A_0} \right]$.

The Gateaux differential $\delta F[A;h]$ of this function at the point A in the direction h can be written as $\delta F[A;h] = e^A \int_0^1 e^{-sA} h e^{sA} ds$.

2. Integral representations of Lagrange interpolation formula and of its error in class of analytic functions

While loop integrals of Cauchy type are widely used [3] when interpolating analytic function of a scalar variable, the theory of interpolation of analytic matrix functions is in considerable degree based on matrix integrals of Cauchy type. The problem of interpolation of functions given at nodes, that are scalar matrices, is especially close to the classical problem of function interpolation. We obtain integral representations of interpolation polynomials and of their remainder terms, using the results [4] for the case of nodes being scalar matrices.

Let nodes be the matrices $A_k = \eta_k I + H$ ($\eta_k \ne \eta_\nu$ if $k \ne \nu$), $k = 0, 1, \ldots, n$, where H is a certain matrix with zero diagonal elements, the function $F(z)$ be analytic in the domain D with boundary Γ, and the spectra of the matrices $(A-H)$ and A_k belong to D. Then (1.2) is representable as

$$L_n(A)=\frac{1}{2\pi i}\int\limits_\Gamma \frac{\omega_n(\xi)I-\omega_n(A-H)}{\omega_n(\xi)}(\xi I-A+H)^{-1}F(\xi I+H)d\xi,$$

where $\omega_n(\xi)=(\xi-\eta_0)(\xi-\eta_1)\cdots(\xi-\eta_n)$, and for the error $r_n(A)=F(A)-L_n(A)$ of interpolation the equity

$$r_n(A)=\frac{1}{2\pi i}\int\limits_\Gamma \frac{\omega_n(A-H)}{\omega_n(\xi)}(\xi I-A+H)^{-1}F(\xi I+H)d\xi$$

holds.

If $0\le\eta_k<2\pi$, $k=0,1,\dots,2n$, the trigonometric interpolation polynomial

$$T_n(F;A)=\sum_{k=0}^{2n}\frac{\sin\frac{A-\eta_0 I}{2}\cdots\sin\frac{A-\eta_{k-1}I}{2}\sin\frac{A-\eta_{k+1}I}{2}\cdots\sin\frac{A-\eta_{2n}I}{2}}{\sin\frac{\eta_k-\eta_0}{2}\cdots\sin\frac{\eta_k-\eta_{k-1}}{2}\sin\frac{\eta_k-\eta_{k+1}}{2}\cdots\sin\frac{\eta_k-\eta_{2n}}{2}}F(\eta_k I+H)$$

can be written as

$$T_n(A)=\frac{1}{4\pi i}\int\limits_\Gamma \frac{\Omega_n(\xi)\cos\frac{\xi I-A+H}{2}-\Omega_n(A-H)}{\Omega_n(\xi)}\sin^{-1}\frac{\xi I-A+H}{2}F(\xi I+H)d\xi$$

and the error takes the form

$$r_n(A)=\frac{1}{4\pi i}\int\limits_\Gamma \frac{\Omega_n(A-H)}{\Omega_n(\xi)}\sin^{-1}\frac{1}{2}(\xi I-A+H)F(\xi I+H)d\xi.$$

Moreover, in the last two formulas $F(\xi)$ is a 2π-periodic function, the domain D of analyticity of F is such as depicted in Fig. 1, and $\Gamma=\Gamma_1\cup\Gamma_2$ (see Fig. 1).

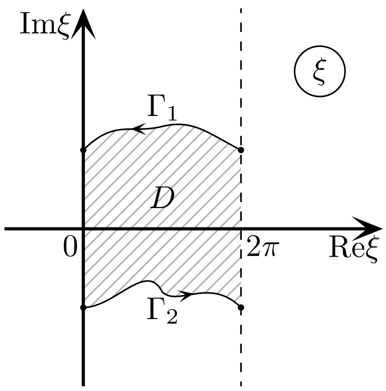

Fig. 1. The domain D.

Let's consider a partial case of the two previous formulas. Let $\eta_k=\frac{2k\pi}{2n+1}$, i.e. the interpolation nodes are $A_k=\frac{2k\pi}{2n+1}I+H$, $k=0,1,\dots,2n$, then $T_n(A)$ can be represented as follows:

$$T_n(A)=\frac{1}{4\pi i}\int\limits_\Gamma \frac{\sin\frac{2n+1}{2}\xi\ \cos\frac{1}{2}(\xi I-A+H)-\sin\frac{2n+1}{2}\xi(A-H)}{\sin\frac{2n+1}{2}\xi}$$
$$\times\sin^{-1}\frac{1}{2}(\xi I-A+H)F(\xi I+H)d\xi, \tag{2.1}$$

$$r_n(A) = \frac{1}{4\pi i} \int\limits_{\Gamma} \frac{\sin \frac{2n+1}{2}(A-H)}{\sin \frac{2n+1}{2}\xi} \sin^{-1} \frac{1}{2}(\xi I - A + H) F(\xi I + H) d\xi. \qquad (2.2)$$

3. Interpolation convergence theorems and error estimates

At first we'll consider convergence of algebraic interpolation on the set S_m of stochastic $m \times m$-matrices. Stochastic matrices turn out an important subclass of nonnegative matrices that finds its application when solving applied problem [5]. The main feature that is further used for investigating the convergence is that absolute values of all eigenvalues of any stochastic matrix don't exceed 1.

Let $A_k = \eta_k I$ be the nodes of interpolation, where η_k are pairwise different numbers, $k = 0, 1, \ldots, p$, $H_n(A)$ be an algebraic matrix polynomial for which the following conditions hold:

$$H_n^{(\nu_k)}(A_k) = F^{(\nu_k)}(A_k), \ \nu_k = 0, 1, \ldots, \alpha_k - 1, \ k = 0, 1, \ldots, p, \qquad (3.1)$$

where α_k is the multiplicity of the node A_k, $\alpha_0 + \alpha_1 + \ldots + \alpha_p = n+1$. Then the following theorem is true [6].

Theorem 3.1 *If the function $F(z)$ is analytic in the circle $|\xi| \le 3$ and $A \in S_m$, then the sequence $\{H_n(A)\}$, $n = 0, 1, 2, \ldots$, where H_n are defined by the equities* (3.1)*, converges to $F(A)$ as $n \to \infty$ for any nodes $A_k = \eta_k I$, $|\eta_k| \le 1$, $k = 0, 1, \ldots, p$.*

The proof of this theorem is based on the estimate of the error of interpolation $r_n(A) = F(A) - L_n(A)$ which is representable, in this case, in the form

$$r_n(A) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{\omega_n(A)}{\omega_n(\xi)} (\xi I - A)^{-1} F(\xi) d\xi,$$

where $\omega_n(A) = \prod\limits_{k=0}^{p} (A - \eta_k I)^{\alpha_k}$, $F(\xi)$ is a function which is analytic in a domain containing the circle $|\xi| \le 3$, Γ is the boundary of the domain of regularity of this function.

Let interpolation nodes be the tridiagonal matrices $A_k \equiv A(a_k, b)$, $k = 0, 1, \ldots, n$, having the elements a_k ($a_k \ne a_\nu$ if $k \ne \nu$) in the main diagonal and the numbers b in the first diagonals above and below the main one. In this case the Lagrange interpolation formula has the form

$$L_n(A) = \sum_{k=0}^{n} \frac{(A - A_0) \cdots (A - A_{k-1})(A - A_{k+1}) \cdots (A - A_n)}{(a_k - a_0) \cdots (a_k - a_{k-1})(a_k - a_{k+1}) \cdots (a_k - a_n)} F(A_k) \qquad (3.2)$$

and for the error $r_n(A)$ of interpolation the integral representation

$$r_n(A) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{\omega_n(A - H)}{\omega_n(\xi)} (\xi I - A - H)^{-1} F(\xi I + H) d\xi \qquad (3.3)$$

holds, where $\omega_n(\xi) = (\xi - a_0)(\xi - a_1) \cdots (\xi - a_n)$, $H = A_k - \alpha_k I$, integration is over the circumference $|\xi| = 1 + 2(b + d) + \varepsilon$, ε is a certain positive number.

Theorem 3.2 *If the function $F(z)$ is analytic in the circle $|z| \le 1 + 2(b+d)$ and $A \in S_m$, then the sequence $L_n(A)$, $n = 0, 1, 2, \ldots$, where L_n are defined by* (3.2)*, converges to $F(A)$ as $n \to \infty$ for any matrix nodes $A_k \equiv A(a_k, b)$, $|a_k| \le d$, $k = 0, 1, \ldots, n$.*

We prove this theorem using the integral representation of the interpolation remainder in the form (3.3) and an estimate of this integral.

4. Estimates of trigonometric interpolation error when nodes are equidistant scalar matrices

Let nodes of interpolation be the matrices

$$A_k = \frac{2k\pi}{2n+1} I, \; k = 0, 1, \ldots, 2n,$$

and $F(x)$ be a 2π-periodic function, which is analytic in the real axis.

In this case the trigonometric interpolation polynomial (1.3) takes the form

$$T_n(A) = \frac{1}{2n+1} \sum_{k=0}^{2n} \sin \frac{2n+1}{2}(A - A_k) \sin^{-1} \frac{1}{2}(A - A_k) F(A_k) \tag{4.1}$$

and for the error $r_n(A)$ of interpolation the formula

$$r_n(A) = \frac{1}{4\pi i} \int\limits_{\Gamma} \frac{\sin \frac{2n+1}{2} A}{\sin \frac{2n+1}{2} \xi} \sin^{-1} \frac{1}{2}(\xi I - A) F(\xi) d\xi, \tag{4.2}$$

where the path of integration $\Gamma = \Gamma_1 \cup \Gamma_2$ consists of two segments $\Gamma_{1,2} = \{\xi : \mathrm{Im}\xi = \pm\varepsilon, \; 0 \le \mathrm{Re}\xi \le 2\pi\}$, is true. Here the numeric parameter ε depends on the function $F(\xi)$.

Theorem 4.1 *If the function $F(x)$ is 2π-periodic and analytic in the real axis and all the eigenvalues of the matrix A belong to the segment $[0, 2\pi]$, then the sequence $\{T_n(A)\}$, $n = 0, 1, 2, \ldots$, where T_n are defined by (4.1), converges to $F(A)$ as $n \to \infty$, for $r_n(A)$ the estimate $\|r_n(A)\| \le M n^{m-1} \exp\{-n\varepsilon\}$ holds (m is the size of A, M is independent of n).*

To prove this theorem we use the inequality

$$\left\| \sin \frac{2n+1}{2} A \right\| \le M_0 n^{m-1},$$

that follows from (1.1), and the inequality

$$\left| \sin \frac{2n+1}{2} (\mathrm{Re}\xi \pm i\varepsilon) \right|^{-1} \le M_1 \exp\left\{ -\frac{2n+1}{2} \varepsilon \right\},$$

where M_0 and M_1 doesn't depend on n.

For the same nodes $A_k = \frac{2k\pi}{2n+1} I$ let's consider the trigonometric polynomial

$$\begin{aligned} T_{2n}(A) = {} & \frac{1}{(2n+1)^2} \sum_{k=0}^{2n} \sin^2 \frac{2n+1}{2}(A - A_k) \\ & \times \left[\sin^{-2} \frac{A - A_k}{2} F(A_k) + 2 \sin^{-1} \frac{A - A_k}{2} \sin^{-1} \frac{A}{2} \sin \frac{A_k}{2} F'(A_k) \right], \end{aligned} \tag{4.3}$$

for which the conditions

$$\begin{aligned} & T_{2n}(A_0) = F(A_0), \\ & T_{2n}(A_k) = F(A_k), \; T'_{2n}(A_k) = F'(A_k), \; k = 0, 1, \ldots, 2n \end{aligned}$$

are met.

The following theorem is proved similarly to theorem 3.

Theorem 4.2 *If the function $F(x)$ is 2π-periodic and analytic in the real axis and all the eigenvalues of the matrix A belong to the segment $[0, 2\pi]$, then the sequence of interpolation polynomials (4.3) converges to $F(A)$ as $n \to \infty$, the following estimate of $r_{2n}(A)$ is true: $\|r_{2n}(A)\| \le Mn^{2m-1} \exp\{-2n\varepsilon\}$.*

In this case the error $r_{2n}(A)$ of the interpolation formula (4.3) is representable in the form

$$r_{2n}(A) = \frac{1}{4\pi i} \int\limits_{\Gamma} \frac{\sin^2 \frac{2n+1}{2} A \sin^{-1} \frac{1}{2} A}{\sin^2 \frac{2n+1}{2} \xi} \sin^{-1} \frac{1}{2}(\xi I - A) \sin \frac{1}{2} \xi d\xi, \tag{4.4}$$

where the integration path is the same as in the formula (4.2). As above, one has to estimate the integral in the equity (4.4). To do it, we use the estimate

$$\left\| \sin \frac{2n+1}{2} A \sin^{-1} \frac{1}{2} A \right\| \le M_2 n^m,$$

that follows from the identity

$$\sin \frac{2n+1}{2} A = \left(I + \sum_{k=1}^{n} \cos kA \right) \sin \frac{1}{2} A.$$

Note that in the given interpolation convergence theorems the required domains of analyticity of interpolated functions are wider than necessary. It is done in order to simplify the proofs.

A possible direction of further research is application of the obtained results to solving numerical problems such as approximate integration of matrix functions.

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