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AN ITERATIVE METHOD OF GENERALIZED SEPARATION OF VARIABLES FOR SOLVING LINEAR OPERATOR EQUATIONS

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V. M. BILETSKYY

Анотація. Пропонується ітераційний метод узагальненого розділення змінних для розв'язання лінійних операторних рівнянь у гільбертовому просторі, який є тензорним добутком n інших гільбертових просторів. Розв'язок рівняння будується у вигляді ряду, кожен доданок якого є тензорним добутком n елементів відповідних просторів. Такий підхід дозволяє зменшити обчислювальну складність розв'язання задачі. Доводиться збіжність методу та наводяться числові результати.

Abstract. An iterative method of generalized variables separation is proposed for solving linear operator equations in a Hilbert space which is a tensor product of n Hilbert spaces. A solution of an equation is represented as a series where each term is a tensor product of n elements of respective spaces. Such a way of problem solving allows to decrease a computational complexity of a problem. The method convergence is proven and numerical results are given.

1. Introduction

The main idea goes from a separation of variables technique which approximates a function of many variables by a finite sum of products of single variable functions. One of the most universal schemes of many its generalizations is described in [11]. In case of general linear operator equation we will look for a tensor-product representation of a solution. This leads to an approximation of a solution tensor by a sum of rank-one tensors.

A tensor decomposition is an important tool in a scientific computing [6, 8, 12–14, 16]. There are many tensor decomposition methods including Tucker decomposition and CP decomposition. Decompositions of the higher-order tensors have applications in signal processing, numerical linear algebra, numerical analysis, data mining, graph analysis etc. For more applications of the tensor decomposition, see the survey paper [15]. The way of tensor-product representation is widely used to solve a lot of the problems [5, 7, 9, 10].

A different variational iterative methods of generalized separation of variables are used for solving of many-dimensional integral equations in [2, 3, 19]. The algorithm of the method for a Fredholm integral equation of the second kind can be found in [4]. A similar approach is used for an approximation of functions of several variables [17, 18] and for solving of the nonlinear inverse problems [1, 20, 21].

In this paper we will describe the method for the linear operator equation

$$Au = f \tag{1.1}$$

in a Hilbert space H which is a tensor product of n Hilbert spaces $H_j,\ j = 1, \ldots, n$. Then we will prove a convergence of the approximate solution sequence and give some numerical results.

Key words. Tensor-product representation, Generalized separation of variables, Iterative algorithms, Linear operator equation.

2. Method description

Let $H_1, \dots, H_n$ are Hilbert spaces and

$$H = \bigotimes_{j=1}^{n} H_j.$$

Let $A : H \to H$ is a linear continuous operator and there exists inverse continuous operator A^{-1}. In other words $A,\ A^{-1} \in \mathcal{L}(H)$. In this case there exists the only solution of the equation (1.1).

We will be looking for a solution in the form

$$u = \sum_{k=0}^{\infty} u_k \tag{2.1}$$

with

$$u_k = \bigotimes_{j=1}^{n} u_k^{(j)}, \quad u_k^{(j)} \in H_j,$$

where $f_0 = f,\ f_{k+1} = f_k - Au_k$ and

$$u_k = \min_{u^{(1)},\dots,u^{(n)}} \left\| f_k - A\left(\bigotimes_{j=1}^{n} u^{(j)}\right)\right\|. \tag{2.2}$$

We can minimize (2.2) using a simple cyclic iterative procedure. Consider some initial approximation $u^{(1,0)} \otimes \ldots \otimes u^{(n,0)}$. Let us fix all the elements except $u^{(1)}$ and find $u^{(1,1)}$ in accordance with the condition (2.2). Then fix all the elements except $u^{(2)}$ and find $u^{(2,1)}$. And so on. After finding $u^{(n,1)}$ we will switch to $u^{(1)}$ and find $u^{(1,2)}$.

In general we can minimize (2.2) using any other method. The choice can be made according to the properties of a specific problem.

Let us assume that for a problem discretization we will approximate each of n infinite dimensional spaces H_j with m-dimensional spaces V_j. So, H will be approximated with m^n-dimensional space $V = V_1 \otimes \ldots \otimes V_n$. If a finite dimensional version of the problem is solved in a straightforward way we will get m^n by m^n linear equation system. This leads to $O(m^{3n})$ algorithm. And now let us see what is the complexity of the method described above. For each step of the cyclic iterative procedure we must construct m by m linear equation system in a $O(m^{2n})$ time. Thus each cycle requires a $O(n(m^{2n} + m^3))$ operations. This gives the computational advantages as far as $n \geq 2$. Moreover if we fix a number of dimensions of the space V then a single cycle complexity will be $O(n)$.

3. Convergence theorem

Consider the following sequence

$$\{\|f_k\|\}_{k=0}^{\infty}. \tag{3.1}$$

It is obvious that this sequence is limited at the bottom by zero and it is non-increasing due to (2.2)

$$\|f_{k+1}\| = \|f_k - Au_k\| \leq \|f_k - A(0)\| = \|f_k\|.$$

Thus there exists a limit of the sequence (3.1). We will show that this limit is equal to zero and therefore the series (2.1) converges to an exact solution of (1.1). For that purpose we will prove two auxiliary lemmas.

Let us denote the following sets:

- $S_j = \{h \in H_j : \quad \|h\| = 1\}, \quad j = 1, \ldots, n;$
- $B_j = \{h \in H_j : \quad \|h\| \leq 1\}, \quad j = 1, \ldots, n;$
- $S = \{h \in H : \quad h = h_1 \otimes \ldots \otimes h_n, \quad h_j \in S_j, \quad j = 1, \ldots, n\};$
- $B = \{h \in H : \quad h = h_1 \otimes \ldots \otimes h_n, \quad h_j \in B_j, \quad j = 1, \ldots, n\};$
- $X = \left\{\frac{Ah}{\|Ah\|} \in H : \quad h \in S\right\}.$

Lemma 3.1 *X is a total set in H*

$$\overline{span(X)} = H.$$

Proof. Let us consider $h \in H$. It is obvious that S is a total set in H, thus

$$\forall \epsilon > 0 \quad \exists g \in span(S) : \quad \|A^{-1}h - g\| < \epsilon$$

where

$$g = \sum_{k=1}^{N} c_k h_k$$

and $N \in \mathbb{N}$, $c_k \in \mathbb{C}$, $h_k \in S$. Further

$$\|h - Ag\| \leq \|A\| \|A^{-1}h - g\| \leq \|A\|\epsilon.$$

As far as A is a linear operator

$$Ag = \sum_{k=1}^{N} c_k A h_k \in span(X).$$

This proves the lemma. □

Lemma 3.2

$$\forall h \in H \quad \exists x_h \in X : \quad |(h, x_h)| = \sup_{x \in X} |(h, x)| \tag{3.2}$$

Proof. Let us fix $h_0 \in H$. Note that $\forall x \in X : \quad |(h_0, x)| \leq \|h_0\|\|x\| = \|h_0\|$, thus

$$\exists M = \sup_{x \in X} |(h_0, x)|.$$

If $M = 0$ then the proof is trivial. Further we will assume that $M > 0$.

Let us choose a sequence

$$\{x_{k_0}\}_{k_0=1}^{\infty} \subset X : \quad |(h_0, x_{k_0})| \to M, \quad k_0 \to \infty \tag{3.3}$$

where

$$x_{k_0} = \frac{Au_{k_0}}{\|Au_{k_0}\|}, \quad u_{k_0} \in S, \quad k_0 \in \mathbb{N}$$

and

$$u_{k_0} = u_{k_0}^{(1)} \otimes \ldots \otimes u_{k_0}^{(n)}, \quad u_{k_0}^{(j)} \in S_j, \quad k_0 \in \mathbb{N}.$$

Let us denote $c_{k_0} = \|Au_{k_0}\|^{-1}$ and show that a sequence $\{c_{k_0}\}_{k_0=1}^{\infty}$ is bounded.

$$\|Au_{k_0}\| \leq \|A\|\|u_{k_0}\| = \|A\|,$$

$$1 = \|u_{k_0}\| = \|A^{-1}Au_{k_0}\| \le \|A^{-1}\|\|Au_{k_0}\|,$$
$$\|Au_{k_0}\| \ge \|A^{-1}\|^{-1},$$
$$\|A\|^{-1} \le c_{k_0} \le \|A^{-1}\|.$$

Each set B_j is a weak compact set in $H_j,\ j = 1,\dots,n$ and thus for any sequence in B_j there exists its subsequence that is weakly convergent in H_j.

Let us pick out of $\{u^{(1)}_{k_0}\}$ its subsequence $\{u^{(1)}_{k_1}\}$ that is weakly convergent in H_1. According to an index subsequence subsequences $\{u^{(2)}_{k_1}\},\dots,\{u^{(n)}_{k_1}\}$ will be picked out respectively. Then we will pick out of $\{u^{(2)}_{k_1}\}$ its subsequence $\{u^{(2)}_{k_2}\}$ that is weakly convergent in H_2. And then according to an index subsequence we will pick out other subsequences $\{u^{(j)}_{k_2}\}$. And so on. After n such steps we will end up with subsequences $\{u^{(j)}_{k_n}\}$ which are weakly convergent in respective spaces $H_j,\ j = 1,\dots,n$.

Let us pick out of numerical sequence $\{c_{k_0}\}$ its subsequence $\{c_{k_n}\}$ according to respective indexes. This sequence is bounded and thus we can pick out its convergent subsequence $\{c_k\}$. Then let us pick out respective subsequences $\{u^{(j)}_k\},\ j = 1,\dots,n$.

Let us consider a sequence $\{x_k\}_{k=1}^{\infty}$ where

$$x_k = c_k A u_k, \quad u_k = \bigotimes_{j=1}^{n} u^{(j)}_k,$$

$$c_k \to c_* \quad k \to \infty,$$
$$u^{(j)}_k \xrightarrow{w} u^{(j)}_*, \quad k \to \infty, \quad j = 1,\ \dots, n.$$

For the sequence $\{u_k\}_{k=1}^{\infty}$ the following condition holds

$$\forall v \in S, \quad v = \bigotimes_{j=1}^{n} v^{(j)}, \quad v^{(j)} \in S_j :$$

$$(u_k, v) = \prod_{j=1}^{n} (u^{(j)}_k, v^{(j)}) \to \prod_{j=1}^{n} (u^{(j)}_*, v^{(j)}), \quad k \to \infty.$$

As far as S is a total set in H

$$u_k \xrightarrow{w} u_*, \quad k \to \infty$$

where

$$u_* = \bigotimes_{j=1}^{n} u^{(j)}_*.$$

Then

$$\forall h \in H : \quad (Au_k, h) = (u_k, A^*h) \to (u_*, A^*h) = (Au_*, h), \quad k \to \infty$$

and

$$Au_k \xrightarrow{w} Au_*, \quad k \to \infty.$$

Further

$$x_k = c_k A u_k \xrightarrow{w} c_* A u_*, \quad k \to \infty.$$

Taking into consideration the condition (3.3) we will get

$$|(h_0, c_* A u_*)| = M > 0. \tag{3.4}$$

Obviously $c_* A u_* \neq 0$ and as far as $Ker(A) = \{0\}$

$$c_* > 0, \quad u_*^{(j)} \neq 0, \quad j = 1, \ldots, n.$$

Let us denote

$$w_*^{(j)} = \frac{u_*^{(j)}}{\|u_*^{(j)}\|}, \quad j = 1, \ldots, n,$$

$$w_* = \bigotimes_{j=1}^{n} w_*^{(j)},$$

$$d_* = c_* \|A w_*\| \prod_{j=1}^{n} \|u_*^{(j)}\|,$$

$$x_* = \frac{A w_*}{\|A w_*\|} \in X.$$

Then

$$x_k \xrightarrow{w} d_* x_*, \quad k \to \infty. \tag{3.5}$$

According to (3.4)

$$|(h_0, x_*)| = d_*^{-1} |(h_0, d_* x_*)| = d_*^{-1} M \leq M.$$

This shows that $d_* \geq 1$. On the other hand $1 \geq |(x_k, x_*)| \to d_*,\ k \to \infty$. Then $d_* \leq 1$ and thus $d_* = 1$, $|(h_0, x_*)| = M$. This proves the lemma. □

Every element $A(u^{(1)} \otimes \ldots \otimes u^{(n)})$ can be represented in a form of αx and vise versa every element αx can be written as $A(u^{(1)} \otimes \ldots \otimes u^{(n)})$ where

- $u^{(j)} \in H_j, \quad j = 1, \ldots, n;$
- $\alpha \in \mathbb{C};$
- $x \in X.$

Then the definition of the sequence $\{f_k\}_{k=0}^{\infty}$ can be written as

$$f_0 = f, \quad f_{k+1} = f_k - \alpha_k x_k$$

where $\alpha_k \in \mathbb{C}$ and $x_k \in X$.

Considering a functional

$$J_k(\alpha, x) = \|f_k - \alpha x\|^2 = \|f_k\|^2 + |\alpha|^2 - 2Re(f_k, \alpha x) \geq \|f_k\|^2 + |\alpha|^2 - 2|\alpha||(f_k, x)| =$$

$$= \|f_k\|^2 + (|\alpha| - |(f_k, x)|)^2 - |(f_k, x)|^2 \geq \|f_k\|^2 - |(f_k, x_{f_k})|^2$$

it is easy to see the minimum conditions that defines α_k and x_k:

- $\alpha_k = (f_k, x_k);$
- $x_k = x_{f_k}.$

Also $\|f_k\|^2 = \|f_{k-1}\|^2 - |\alpha_{k-1}|^2$. Every element x_k exists due to the lemma 3.2. Now we are ready to formulate the main theorem that will prove the convergence of the series (2.1) to an exact solution of the equation (1.1).

Theorem 3.3 *Let us consider a Hilbert space H and a set $X \in H$ which holds the following conditions:*

- $\forall x \in X : \quad \|x\| = 1;$
- *X is a total set in H;*
- $\forall h \in H \quad \exists x_h \in X : \quad |(h, x_h)| = \sup_{x \in X} |(h, x)|.$

Then

$$\forall h_0 \in H : \quad h_k \to 0, \quad k \to \infty \tag{3.6}$$

where

- $h_{k+1} = h_k - \alpha_k x_k;$
- $\alpha_k = (h_k, x_k);$
- $x_k = x_{h_k}.$

Proof. According to the definition of α_k, x_k and h_k we will get

$$\|h_{k+1}\|^2 = (h_k - \alpha_k x_k, h_k - \alpha_k x_k) = \|h_k\| - |\alpha_k|^2.$$

Then

$$\sum_{k=0}^{\infty} |\alpha_k|^2 \leq \|h_0\|^2 < \infty. \tag{3.7}$$

This results in

$$|\alpha_k| \to 0, \quad k \to \infty. \tag{3.8}$$

Due to the definition

$$\forall x \in X \quad |(h_k, x)| \leq |(h_k, x_k)| = |\alpha_k|.$$

And taking into consideration (3.8) we will get

$$\forall x \in X \quad (h_k, x) \to 0, \quad k \to \infty.$$

X is a total set in H and thus $\{h_k\}$ is weakly convergent to zero in the space H:

$$\{h_k\} \xrightarrow{w} 0, \quad k \to \infty. \tag{3.9}$$

Let us fix $\epsilon > 0$. Due to (3.7)

$$\exists N \in \mathbb{N} : \quad \sum_{k=N}^{\infty} |\alpha_k|^2 < \epsilon.$$

$>$From (3.9) we will get

$$(h_k, h_N) \to 0, \quad k \to \infty.$$

Therefore

$$\exists M > N \quad \forall k > M \quad |(h_k, h_N)| < \epsilon.$$

Let $s \in \mathbb{N}$. Let us denote n_s as an integer number that

$$2^s \leq n_s < 2^{s+1}, \quad |\alpha_{n_s}| = \min_{2^s \leq k < 2^{s+1}} |\alpha_k|.$$

Let us consider a subsequence $\{\|h_{n_s}\|\}_{s=1}^{\infty}$. Then for s such that $2^s > M$ we will get

$$\|h_{n_s}\|^2 = (h_{n_s}, h_N - \sum_{k=N}^{n_s - 1} \alpha_k x_k) \leq |(h_{n_s}, f_N)| + \sum_{k=N}^{n_s - 1} |(h_{n_s}, \alpha_k x_k)| <$$

$$< \epsilon + \sum_{k=N}^{n_s-1} |\alpha_k||(h_{n_s}, x_k)| \le \epsilon + \sum_{k=N}^{n_s-1} |\alpha_k||\alpha_{n_s}| \le \epsilon + \sum_{k=N}^{n_s-1} \frac{|\alpha_k|^2 + |\alpha_{n_s}|^2}{2} <$$

$$< \epsilon + \frac{\epsilon}{2} + 2^s|\alpha_{n_s}|^2 \le \frac{3}{2}\epsilon + \sum_{k=2^s}^{2^{s+1}-1} |\alpha_k|^2 < \frac{5}{2}\epsilon.$$

So,

$$\|h_{n_s}\| \to 0, \quad s \to \infty.$$

As far as $\{h_{n_s}\}$ is a subsequence of monotonous sequence $\{h_k\}$ this results in

$$\|h_k\| \to 0, \quad k \to \infty$$

and proves the theorem. □

4. Numerical results

Let us see how this method works for the following Fredholm integral equation of the second kind in a space $L_2(D)$

$$Au \equiv \int_D 4\sin\left(\sum_{j=1}^{n}(x_j + \widehat{x_j})\right) u(\widehat{x})d\widehat{x} - 2u(x) = 3\cos\left(\sum_{j=1}^{n} x_j\right) \qquad (4.1)$$

where

- $x = (x_1, \ldots, x_n)$;
- $\widehat{x} = (\widehat{x_n}, \ldots, \widehat{x_n})$;
- $D = [a_1, b_1] \times \ldots \times [a_n, b_n]$ – bounded region in $\mathbb{R}^n$;
- $a_j = 0$, $b_j = 1$ for $j = 1, \ldots, n$.

For each $n = 2, 3, \ldots, 9$ the running time of an application that implements an algorithm for the method has been measured. The results are shown in the following table.

n	2	3	4	5	6	7	8	9
Running time, sec.	9.21	14.08	18.97	22.18	29.81	33.03	35.6	44.77

For each n the number of dimensions of the space V that approximates $L_2(D)$ is about 1000. Here we can see that the running time of the application linearly depends on n.

But general question about the rate of the convergence is still under investigation. Also approximation error should be taken into consideration.

Bibliography

1. Balyash Y. A generalized separation of variables in problems of diffraction and antenna synthesis / Y. Balyash, M. Voitovych, S. Yaroshko.– // The 1989 URSI Int. Symp. on Electromagnetic Theory.– 1989.– P. 650-652.
2. Balyash Y. A variational iterative method of solving many-dimensional integral equations / Y. Balyash, M. Voitovych // Integral Equations in Applied Modeling.– 1986.– Vol. 2.– P. 23. (in Russian).
3. Balyash Y. An approximative variational iterative separation of variables in many-dimensional problems / Y. Balyash, M. Voitovych // Waves and Diffraction.– 1985.– Vol. 1.– P. 122. (in Russian).

4. Biletskyy V. A method of generalized separation of variables for solving many-dimensional linear Fredholm integral equation / V. Biletskyy, S. Yaroshko // Direct and Inverse Problems of Electromagnetic and Acoustic Wave Theory.– 2007.– P. 94-97.
5. De Lathauwer L. A multilinear singular value decomposition / L. De Lathauwer, B. De Moor, J. Vandewalle // SIAM J. Matrix Anal. and Appl.– 2000.– Vol. 21.– P. 1253-1278.
6. De Lathauwer L. On the best rank-1 and rank-$(R_1, R_2, \ldots, R_N)$ approximation of higher-order tensors / L. De Lathauwer, B. De Moor, J. Vandewalle // SIAM J. Matrix Anal. and Appl.– 2000.– Vol. 21.– P. 1324-1342.
7. Gavrilyuk I. Hierarchical tensor-product approximation to the inverse and related operators for high-dimensional elliptic problems / I. Gavrilyuk, W. Hackbusch, B. Khoromskij // Computing.– 2005.– Vol. 74.– P. 131-157.
8. Hackbusch W. Hierarchical Kronecker tensor-product approximations / W. Hackbusch, B. Khoromskij, E. Tyrtyshnikov // Journal of Numerical Mathematics.– 2005.– Vol. 13.– P. 119-156.
9. Hackbusch W. Tensor-product approximation to operators and functions in high dimensions / W. Hackbusch, B. Khoromskij // Journal of Complexity.– 2007.– Vol. 23.– P. 697-714.
10. Hackbusch W. Use of tensor formats in elliptic spectral problems / W. Hackbusch, B. Khoromskij, S. Sauter, E. Tyrtyshnikov.– MPI MIS Leipzig.– 2008.– Preprint 78/2008.
11. Kaleniuk P. Generalized method of variables separation / P. Kaleniuk, Y. Baranetskyy, Z. Nytrebych.– Kyiv: Naukova dumka, 1993. (in Russian).
12. Khoromskij B. Low rank Tucker-type tensor approximation to classical potentials / B. Khoromskij, V. Khoromskaia // Central European Journal of Mathematics.– 2007.– Vol. 5.– P. 523-550.
13. Kolda T. Multilinear operators for higher-order decompositions / T. Kolda.– Sandia National Laboratories.– 2006.– Technical Report SAND2006-2081.
14. Kolda T. Scalable tensor decompositions for multi-aspect data mining / T. Kolda, J. Sun // Proceedings of the 8-th IEEE International Conference on Data Mining.– 2008.– P. 363-372.
15. Kolda T. Tensor decompositions and applications / T. Kolda, B. Bader.– Sandia National Laboratories.– 2007.– Technical Report SAND2007-6702.
16. Navasca C. Swamp reducing technique for tensor decompositions / C. Navasca, L. De Lathauwer, S. Kinderman // Proceedings of the 16-th European Signal Processing Conference.– 2008.
17. Pospelov V. About approximation of a function of some arguments by products of functions of a single argument / V. Pospelov // IAM Academy of Science.– 1978.– No. 32.– 20 p.– Preprint. (in Russian).
18. Verlan A. To the question of convergence of variational iterative method of approximation of function of two arguments / A. Verlan, I. Serikova // Accuracy and Reliability of Cybernetic Systems.– 1975.– Vol. 3.– P. 10-12. (in Russian).
19. Voitovych M. A variational iterative method of variables separation for solving many-dimensional integral equations / M. Voitovych, S. Yaroshko // Math. methods and phis.-mec. spaces.– 1997.– Vol. 4.– P. 122-126. (in Ukrainian).
20. Voitovych M. Synthesis of two-dimensional antennas grid with generalized separation of variables / M. Voitovych // Radiotechnics and Electronics.– 1988.– Vol. 12.– P. 2637-2640. (in Russian).
21. Voitovych M. The numerical solution of a problem of two-dimensional antennas grid synthesis / M. Voitovych, S. Yaroshko // Radiotechnics and Electronics.– 1991.– Vol. 1.– P. 192-196. (in Russian).

Ivan Franko National University of Lviv,
1 Universytets'ka Str., Lviv, 79000, Ukraine
E-mail address: vbiletskyy@gmail.com

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