Журнал обчисл. прикл. матем.
Вип. 1 (100), 2010, стор. 10-28

J. Numer. Appl. Math.
No. 1 (100), 2010, pp. 10-28

FINITE-PARAMETRIC SOLUTIONS OF A CLASS OF HAMMERSTEIN NONLINEAR INTEGRAL EQUATIONS RELATED TO PHASE PROBLEM

UDC 519.6

O. O. BULATSYK, YU. P. TOPOLYUK AND N. N. VOITOVICH

Анотація. В роботі розглядається клас нелінійних інтегральних рівнянь типу Гаммерштейна, в яких нелінійний множник під інтегралом містить аргумент невідомої комплексної функції. Розв'язки таких рівнянь описуються скінченим числом комплексних параметрів, які є оберненими величинами до нулів поліномів відповідних степенів. Ці параметри визначаються із системи, яка складається із одного інтегрального рівняння і скінченного числа трансцендентних рівнянь. Встановлено існування еквівалентних груп розв'язків нелінійних інтегральних рівнянь, що розглядаються. Одержано необхідні умови для точок галуження і системи рівнянь для їх обчислення. Наведено числові результати для конкретного прикладу.

Abstract. The paper considers a class of nonlinear integral equation of the Hammerstein type involving the argument of unknown complex function in the nonlinear factor of the integrand. The solutions to such equations are described by a limite numbers of complex parameters being the inverse zeros of polynomials of appropriate degrees. The parameters are determined from a system of equations containing one integral equation and a limited number of transcendental ones. Existence of equivalent groups of solutions is established. The necessary condition for the branching points is obtained, and the systems of equations for their calculation are given. Numerical results for a particular examle are presented.

1. Introduction

Nonlinear integral equations of the Hammerstein type have a linear kernel and nonlinear factor depending on the unknown function in the integrand [1, 2]. They commonly occur in different applications. An interesting class of such equations arises in the phase optimization problems nearly connected to the so-called phase problem [3–5]. The argument of given or unknown complex function participates separately from its modulus in such problems [6, 7]. This argument (phase) appears in the nonlinear part of the integrand. Likely, the first equation of such a type

$$f(\xi) = \int_{-1}^{1} \frac{\sin(c(\xi-\xi'))}{\xi-\xi'} F(\xi') \exp(i \arg f(\xi')) d\xi', \tag{1.1}$$

arisen in a concrete application, was obtained in [8]; here c and F are given real positive number and nonnegative function, respectively. Various similar equations were investigated (mainly numerically) in [9, 10].

In [11, 12] analytical solutions to equation (1.1) were obtained. These solutions depend on the finite number of complex parameters being the inverse zeros of polynomials of appropriate degrees (generating polynomials). The parameters are calculated from a

Key words. Nonlinear integral equation of Hammerstein type, finite-parametric solutions, branching of solutions, phase problem.

system of transcendental equations. The results were extended to several concrete equations in [13, 14]. In [15] they were generalized for the equation

$$f(\xi) = \int_a^b K(\xi,\xi')F(\xi')\exp(i\arg f(\xi'))d\xi'$$

with the kernel

$$K(\xi,\xi') = \frac{s(\xi)q(\xi') - s(\xi')q(\xi)}{\tau(\xi) - \tau(\xi')}, \quad (1.2)$$

where $s(\xi)$, $q(\xi)$, $\tau(\xi)$ are real continuous functions such that the function sets $\{\tau^n(\xi)s(\xi)\}$, $\{\tau^n(\xi)q(\xi)\}$ $(n = 0, 1, \dots)$ are linearly independent.

In this paper a more general integral equation is considered. Its solutions are also generated by polynomials of finite degrees. However, their zeros satisfy a system of transcendental equations, which contain an unknown real function. The system is supplemented by an integral equation, linear with respect to this function. The theorems are formulated and proved, which establish the necessary and sufficient conditions for solutions to the mentioned system of equations to generate solutions to the initial nonlinear equation, as well as construct the equivalent groups of solutions and give a linear homogeneous integral equation for their branching points. The systems of equations for numerical determination of the branching points are obtained. The theory is illustrated on a concrete example, for which the numerical results are presented and analyzed.

2. Formulation of the problem. Finite-parametric representation of solutions

Let us consider a nonlinear integral equation of the Hammerstein type

$$\alpha f + \beta Bf = B[F\exp(i\arg f)], \quad (2.1)$$

where $B : L_2(a,b) \to L_2(a,b)$ is a linear integral operator with the kernel $K(\xi,\xi') \in L_2(a,b)\times L_2(a,b)$ of the form (1.2) and funcions $s(\xi)$, $q(\xi)$, $\tau(\xi)$ satisfying the above conditions, $F(\xi) \in L_2(a,b)$ is a given real nonnegative function, α, β are given real nonnegative constants, $\alpha + \beta \neq 0$. Rewrite equation (2.1) in the form

$$\begin{aligned} \alpha f(\xi) + \beta\int_a^b K(\xi,\xi')f(\xi')d\xi' = \\ = \int_a^b K(\xi,\xi')F(\xi')\exp(i\arg f(\xi'))d\xi'. \end{aligned} \quad (2.2)$$

We find the solutions to (2.2) having no zeros in $[a,b]$ in the form

$$f(\xi) = \gamma\hat{f}(\xi)P_N(\tau). \quad (2.3)$$

Here $\hat{f}(\xi)$ is a real positive function having no zeros in $[a,b]$; γ is an arbitrary complex constant with $|\gamma| = 1$;

$$P_N(\tau) = \prod_{k=1}^{N}(1 - \eta_{Nk}\tau)$$

is a polynomial of a finite degree N with complex nonconjugated pairwise zeros η_{Nk}^{-1}:

$$\eta_{Nk} - \bar{\eta}_{Nm} \neq 0, \quad k, m = 1, 2, ..., N. \tag{2.4}$$

Note that the role of argument in the polynomial P_N plays the function $\tau(\xi)$. From (2.3) it follows

$$\exp(i \arg f(\xi)) = \frac{P_N(\tau)}{|P_N(\tau)|}. \tag{2.5}$$

Theorem 2.1 *A function $f(\xi)$ of the form (2.3) having no zeros in $[a, b]$ is a solution to equation (2.2) if and only if the real positive function $|f(\xi)|$ and complex parameters η_{Nk}, $k = 1, 2, ..., N$, with condition (2.4) satisfy the following integral equation*

$$\alpha\,|f(\xi)| + \frac{\beta}{|P_N(\tau)|}\int_a^b K(\xi,\xi')|f(\xi')|\frac{\text{Re}\,\left[\bar{P}_N(\tau)P_N(\tau')\right]}{|P_N(\tau')|}d\xi' =$$

$$= \frac{1}{|P_N(\tau)|}\int_a^b K(\xi,\xi')F(\xi')\frac{\text{Re}\,\left[\bar{P}_N(\tau)P_N(\tau')\right]}{|P_N(\tau')|}d\xi' \tag{2.6}$$

and the system of transcendental equations

$$\Phi_{Nn}(|f(\xi)|, \eta_{N1}, \eta_{N2}, ... \eta_{NN}) = 0, \quad n = 1, 2, ..., N, \tag{2.7a}$$

$$\Psi_{Nn}(|f(\xi)|, \eta_{N1}, \eta_{N2}, ... \eta_{NN}) = 0, \quad n = 1, 2, ..., N, \tag{2.7b}$$

where

$$\Phi_{Nn} = \int_a^b \tau^{n-1} s(\xi)\frac{F(\xi) - \beta|f(\xi)|}{|P_N(\tau)|}d\xi, \tag{2.8a}$$

$$\Psi_{Nn} = \int_a^b \tau^{n-1} q(\xi)\frac{F(\xi) - \beta\,|f(\xi)|}{|P_N(\tau)|}d\xi; \tag{2.8b}$$

in (2.6) $\tau = \tau(\xi)$, $\tau' = \tau(\xi')$.

Proof. Necessity. Let function (2.3) be a solution to equation (2.2). It is seen from (2.2) that one can put $\gamma = 1$ without loss of generality. Substituting (2.3), (2.5) into (2.2) and multiplying its both sides by $\bar{P}_N(\xi)$, we obtain

$$\alpha \hat{f}(\xi)\,|P_N(\tau)|^2 + \beta\bar{P}_N(\tau)\int_a^b K(\xi,\xi')\hat{f}(\xi')P_N(\tau')d\xi' =$$

$$= \bar{P}_N(\tau)\int_a^b K(\xi,\xi')F(\xi')\frac{P_N(\tau')}{|P_N(\tau')|}d\xi'. \tag{2.9}$$

After taking the real part from (2.9), and dividing both its sides by $|P_N(\tau)|$ with using the equality

$$\hat{f}(\xi)\,|P_N(\tau)| = |f(\xi)| \tag{2.10}$$

we arrive at (2.6).

On the other hand, after taking the imaginary part from (2.9), we have

$$\beta\int_a^b K(\xi,\xi')\hat{f}(\xi')R_{N-1}(\tau,\tau')d\xi' = \int_a^b K(\xi,\xi')\frac{F(\xi')}{|P_N(\tau')|}R_{N-1}(\tau,\tau')d\xi', \tag{2.11}$$

where

$$R_{N-1}(\tau,\tau') = \frac{2i\mathrm{Im}\left[P_N(\tau')\bar{P}_N(\tau)\right]}{\tau-\tau'} = \sum_{k,m=1}^{N} a_{km}\tau^{k-1}(\tau')^{m-1} \tag{2.12}$$

is a symmetrical polynomial of two variables with the matrix of coefficients $A=\{a_{km}\}$. Writing (2.11) in the form

$$\int_a^b K(\xi,\xi')\frac{F(\xi')-\beta\,|f(\xi')|}{|P_N(\tau')|}R_{N-1}(\tau,\tau')d\xi' \equiv 0 \tag{2.13}$$

and using (2.12), we have

$$\sum_{k,m=1}^{N} a_{km}\left[q(\xi')\int_a^b \tau^{k-1}s(\xi)\frac{F(\xi')-\beta\,|f(\xi')|}{|P_N(\tau')|}d\xi - \right.$$
$$\left. - s(\xi')\int_a^b \tau^{k-1}q(\xi)\frac{F(\xi')-\beta\,|f(\xi')|}{|P_N(\tau')|}d\xi\right](\tau')^{m-1} \equiv 0. \tag{2.14}$$

Due to the linear independence of the functions $\{\tau^k s(\xi)\}$, $\{\tau^k q(\xi)\}$ at $k=0,1,\ldots,N-1$, identity (2.14) leads to

$$\sum_{k=1}^{N} a_{km}\Phi_{Nn} = 0, \quad n=1,2,...,N, \tag{2.15a}$$
$$\sum_{k=1}^{N} a_{km}\Psi_{Nn} = 0, \quad n=1,2,...,N, \tag{2.15b}$$

where Φ_{Nn}, Ψ_{Nn} are defined in (2.8). If the coefficients a_{km} would be known, then these equalities could be considered as two independent systems of linear algebraic equations with respect to unknowns Φ_{Nn} and Ψ_{Nn}, respectively, with the same matrix A.

The determinant of A is

$$\det A = (-1)^{[N/2]}\prod_{k,m=1}^{N}(\bar{\eta}_{Nm}-\eta_{Nk}), \tag{2.16}$$

where the squire brackets mean the integer part of the value. This fact follows from the condition 4^0 of the Bezudiant from [16] and is consistent with Theorem 7.8 from [17]. Its immediate proof is given in [12].

Due to condition (2.4), $\det A \neq 0$ and systems (2.15) have only zero solutions. This means that the transcendental equation system (2.7) holds.

Sufficiency. Let the integral equation (2.6) together with the system of transcendental equations (2.7) be satisfied at certain integer N and complex η_{Nk}, $k = 1, 2, ..., N$, being under condition (2.4). We show that then function (2.3) is a solution to equation (2.2).

It follows from (2.7) that equalities (2.15) are fulfilled, and, after using (2.8), identity (2.14) and, hence, (2.13) hold. With the aid (2.12), we obtain from (2.13)

$$\mathrm{Im}\left[\bar{P}_N(\tau)\int_a^b K(\xi,\xi')\frac{F(\xi')-\beta\,|f(\xi')|}{|P_N(\tau')|}P_N(\tau')d\xi'\right]=0, \qquad (2.17)$$

or, after adding the real function $\alpha\hat{f}(\xi)\,|P_N(\tau)|^2$ under the imaginary sign,

$$\mathrm{Im}\left[\alpha\hat{f}(\xi)\,|P_N(\tau)|^2+\bar{P}_N(\tau)\int_a^b K(\xi,\xi')\frac{F(\xi')-\beta\,|f(\xi')|}{|P_N(\tau')|}P_N(\tau')d\xi'\right]=0. \qquad (2.18)$$

Taking into account (2.10) and dividing the both sides of (2.18) by the real positive function $|P_N(\tau)|$, we obtain

$$\mathrm{Im}\left[\alpha|f(\xi)|+\frac{\bar{P}_N(\tau)}{|P_N(\tau)|}\int_a^b K(\xi,\xi')\frac{F(\xi')-\beta\,|f(\xi')|}{|P_N(\tau')|}P_N(\tau')d\xi'\right]=0. \qquad (2.19)$$

On the other hand, the integral equation (2.6) can be written in the form

$$\mathrm{Re}\left[\alpha\,|f(\xi)|+\frac{\bar{P}_N(\tau)}{|P_N(\tau)|}\int_a^b K(\xi,\xi')\frac{F(\xi')-\beta\,|f(\xi')|}{|P_N(\tau')|}P_N(\tau')d\xi'\right]=0. \qquad (2.20)$$

Equalities (2.19) and (2.20) together imply that the expression in their square brackets equals zero, what after multiplying by (2.5) shows that function (2.3) solves integral equation (2.2).

Theorem 2.2 *If the function $f(\xi)$ of the form (2.3) with $\gamma = 1$ solves equation (2.2), then the functions*

$$f_n(\xi)=\hat{f}(\xi)P_N(\tau)\frac{1-\bar{\eta}_{Nn}\tau}{1-\eta_{Nn}\tau}, \quad n=1,2,...,N,$$

solve this equation, too.

Proof. For simplicity, we drop the index N in η_{Nn}. Denote

$$Q_N(\tau)=\hat{P}_{N-1}(\tau)(1-\bar{\eta}_n\tau), \qquad (2.21)$$

where

$$\hat{P}_{N-1}(\tau)=P_N(\tau)/(1-\eta_n\tau); \qquad (2.22)$$

in this notation, $f_n(\xi)=\hat{f}(\xi)Q_N(\tau)$. It is seen that

$$|P_N(\tau)|=|Q_N(\tau)|\,. \qquad (2.23)$$

Let the function $f(\xi)$ of the form (2.3) solve equation (2.2). Then, according to Theorem 2.1, the transcendental equation system (2.7) is fulfilled. In the same time, this system is fulfilled after replacing in it the polynomial $P_N(\tau)$ by any other polynomial of the degree N with the same modulus, in particular, by $Q_N(\tau)$, that is

$$\int_a^b \tau^{n-1} s(\xi) \frac{F(\xi) - \beta\,|f(\xi)|}{|Q_N(\tau)|} d\xi = 0, \qquad n = 1, 2, ..., N, \tag{2.24a}$$

$$\int_a^b \tau^{n-1} q(\xi) \frac{F(\xi) - \beta\,|f(\xi)|}{|Q_N(\tau)|} d\xi = 0, \qquad n = 1, 2, ..., N. \tag{2.24b}$$

Rewrite equation (2.2) in the form

$$\alpha f(\xi) = \int_a^b K(\xi, \xi')\,[F(\xi') \exp(i \arg f(\xi')) - \beta f(\xi')]\, d\xi'. \tag{2.25}$$

In order to show that $f_n(\xi)$ satisfies this equation, we substitute $f_n(\xi')$ instead $f(\xi')$ in its right-hand side and make same identical transformations. Multiplying and dividing this expression by $|Q_N(\tau)|^2$ and using (2.21)-(2.23), we obtain

$$\int_a^b K(\xi, \xi')\,(F(\xi') \exp(i \arg f_n(\xi')) - \beta f_n(\xi'))\, d\xi' =$$

$$= \int_a^b K(\xi, \xi') \frac{F(\xi') - \beta\,|f(\xi')|}{|Q_N(\tau')|} Q_N(\tau') d\xi' = \tag{2.26}$$

$$= \frac{(1 - \bar{\eta}_n \tau)\hat{P}_{N-1}(\tau)\bar{Q}_N(\tau) \int_a^b K(\xi, \xi')\,\left(F(\xi') - \beta\,|f(\xi')|\right) Q_N(\tau') \Big/ |Q_N(\tau')|\, d\xi'}{|Q_N(\tau)|^2}.$$

Since equation system (2.24) holds, then equality (2.17) with substituting $P_N(\xi)$ by $Q_N(\xi)$ holds, as well, that is,

$$\mathrm{Im}\left[\bar{Q}_N(\tau) \int_a^b K(\xi,\xi')\, \frac{F(\xi') - \beta\,|f(\xi')|}{|Q_N(\tau')|}\, Q_N(\tau')\, d\xi'\right] = 0. \tag{2.27}$$

Introduce the function

$$\hat{f}_n(\xi) = \frac{\bar{Q}_N(\tau) \int_a^b K(\xi, \xi')\, \frac{F(\xi') - \beta\,|f(\xi')|}{|Q_N(\tau')|}\, Q_N(\tau')\, d\xi'}{\alpha |Q_N(\tau)|^2};$$

due to (2.27), this function is real. With this notation, identity (2.26) can be rewritten in the form

$$\alpha \hat{f}_n(\xi)(1 - \bar{\eta}_n \tau)\hat{P}_{N-1}(\tau) = \int_a^b K(\xi, \xi')\, \frac{F(\xi') - \beta\,|f(\xi')|}{|Q_N(\tau')|}\, Q_N(\tau')\, d\xi'.$$

With the aid of (2.21), we have

$$\alpha\hat{f}_n(\xi)(1-\bar{\eta}_n\tau)\hat{P}_{N-1}(\tau)=\int\limits_a^b K(\xi,\xi')\frac{F(\xi)-\beta\,|f(\xi)|}{|Q_N(\tau')|}(1-\bar{\eta}_n\tau')\hat{P}_{N-1}(\tau')d\xi'. \qquad (2.28)$$

This equality implies that if $\hat{f}_n(\xi)$ has no zeros in $[a,b]$, then the function $\hat{f}_n(\xi)\hat{P}_{N-1}(\xi)(1-\bar{\eta}_n\xi)$ solves equation (2.2).

In order to complete the proof, it is remained to show that $\hat{f}_n(\xi)=\hat{f}(\xi)$. For this end, we substitute the function $f(\xi)=\hat{f}(\xi)P_N(\xi)$ into (2.25) and use (2.22):

$$\alpha\hat{f}(\xi)(1-\eta_n\tau)\hat{P}_{N-1}(\tau)=\int\limits_a^b K(\xi,\xi')\frac{F(\xi')-\beta\,|f(\xi')|}{|Q_N(\tau')|}(1-\eta_n\tau')\hat{P}_{N-1}(\tau')d\xi'. \qquad (2.29)$$

Multiply the both sides of (2.28) by $(1-\eta_n\tau)$ and the both sides of (2.29) by $(1-\bar{\eta}_n\tau)$, and subtract them from one other. Using the identity

$$(1-\eta_n\tau)(1-\bar{\eta}_n\tau')-(1-\bar{\eta}_n\tau)(1-\eta_n\tau')=(\eta_n-\bar{\eta}_n)(\tau'-\tau),$$

we obtain

$$\begin{aligned}&\left[\hat{f}_n(\xi)-\hat{f}(\xi)\right]\hat{P}_{N-1}(\tau)|1-\eta_n\tau|^2=(\eta_n-\bar{\eta}_n)\times\\&\times\left[q(\xi)\int\limits_a^b s(\xi')\frac{F(\xi')-\beta\,|f(\xi')|}{|Q_N(\tau')|}\hat{P}_{N-1}(\tau')d\xi'-\right.\\&\left.-\,s(\xi)\int\limits_a^b q(\xi')\frac{F(\xi')-\beta\,|f(\xi')|}{|Q_N(\tau')|}\hat{P}_{N-1}(\tau')\,d\xi'\right].\end{aligned} \qquad (2.30)$$

Since $\hat{P}_{N-1}(\tau)$ is a polynomial of the degree $N-1$, the integrals in the right-hand side of (2.30) equal zero owing to (2.7). Under the theorem condition, the functions $\hat{P}_{N-1}(\tau)$ and $|1-\eta_n\tau|^2$ in the left-hand side of (2.30) have no zeros in $[a,b]$, hence $\hat{f}_n(\xi)=\hat{f}(\xi)$ and the proof is completed.

Corollary 2.3 *The solutions to integral equation (2.6) and the system of transcendental equations (2.7) make up the equivalent groups inside which the function $\hat{f}(\xi)$ is the same and the polynomials $P_N(\tau)$ differ only by substitution of any number $s<N$ of the parameters η_k by the complex conjugated ones:*

$$P_N^{(s)}(\tau)=\prod_{m=1}^{s}(1-\eta_{n_m}\tau)\prod_{m=s+1}^{N}(1-\bar{\eta}_{n_m}\tau),$$

where $n_{m_1}\neq n_{m_2}$ if $m_1\neq m_2$. Such polynomials generate the solutions to (2.2) with the same $|f(\xi)|$.

In particular, if the function $f(\xi)$ solves equation (2.2), then $\bar{f}(\xi)$ solves this equation, too; of course, this is easily seen immediately from (2.2).

3. Branching of solutions

Consider the case when the kernel $K(\xi, \xi')$ in equation (2.2) depends on a certain real parameter c. At any fixed value of this parameter, the solutions to (2.2) are expressed in the form (2.3). The number of solutions depends on c and may vary as c varies (we abstract from the solution sets caused by arbitrariness of γ). For determining the branching points $c = c_n$ the following theorem serves.

Theorem 3.1 *Let the integral operator B in equation (2.1) depend on a real positive parameter c and a function $f(\xi)$ of the form (2.3) solve this equation. In order to a value $c = c_0$ is a branching point of this solution it is necessary that the homogeneous integral equation*

$$\lambda_n |f|\, |P_N| v_n = \mathrm{Re}\left(\bar{P}_N D\left[F \frac{P_N}{|P_N|} v_n\right]\right) \tag{3.1}$$

has at this point the eigenvalue $\lambda_n = 1$ of the multiplicity not less than 2. Here the operator $D : L_2(a, b) \to L_2(a, b)$ is

$$D = (\alpha I + \beta B)^{-1} B, \tag{3.2}$$

I is the unit operator.

Proof. Following [1], we apply the perturbation method to equation (2.1). Let at $c = c_0$, $B = B_0$ equation (2.1) have the solution f_0 in the form (2.3). The perturbation $c = c_0 + \varepsilon$ leads to the perturbations of the operator B and the solution f, which, in the first approximations, are $B = B_0 + \varepsilon B_1$ and $f = f_0 + \varepsilon f_1$, respectively. Keeping the terms of the first order with respect to ε, we obtain

$$\exp(i \arg f) = \exp(i \arg f_0) + i\varepsilon \exp(i \arg f_0)\, \mathrm{Im}\, \frac{f_1}{f_0}.$$

Substituting the expressions for B, f, and $\exp(i \arg f)$ into (2.1) and equating the terms at ε, we obtain in the first approximation the following nonhomogeneous equation with respect to f_1:

$$\alpha f_1 + \beta B_0 f_1 - iB_0 \left[F \exp(i \arg f_0)\, \mathrm{Im}\, \frac{f_1}{f_0}\right] = B_1 \left[F \exp(i \arg f_0)\right]. \tag{3.3}$$

The branching is possible only in the case when the function $f_1(\xi)$ cannot be determined uniquely from (3.3), that is, if the homogeneous equation

$$\alpha f_1 + \beta B_0 f_1 = iB_0 \left[F \exp(i \arg f_0) \mathrm{Im}\, \frac{f_1}{f_0}\right] \tag{3.4}$$

holds. The values c_0, at which this fact takes place are the possible branching points. Introduce the real functions u, v, as follows

$$u + iv = \frac{f_1}{f_0}.$$

Then equation (3.4) becomes

$$f_0 \cdot (u + iv) = iD_0 \left[F \exp(i \arg f_0) v\right], \tag{3.5}$$

where D_0 is defined by (3.2) at $B = B_0$. After substituting (2.5) and the equality

$$f_0 = |f_0| \frac{P_N}{|P_N|}$$

into (3.5) and multiplying both its sides by $\bar{P}_N$, we obtain

$$|f_0|\,|P_N| \cdot (u + iv) = i\bar{P}_N D_0 \left[F \frac{P_N}{|P_N|} v \right]. \tag{3.6}$$

Equating the imaginary parts of both side of equation (3.6) leads to the real equation

$$|f_0|\,|P_N| v = \mathrm{Re}\left(\bar{P}_N D_0 \left[F \frac{P_N}{|P_N|} v \right] \right) \tag{3.7}$$

with respect to the real function v. When this function is found, u is immediately calculated from (3.6).

Equation (3.7) can be considered as an eigenvalue problem, nonlinear with respect to the spectral parameter c. It can be rewritten in the form (3.1), as a problem of finding the values of c, at which an eigenvalue λ_n of this linear problem equals the unity.

It is seen that the function $v \equiv 1$ solves (3.1) with $\lambda_n = 1$ for any value of the parameter c. Indeed, according to (2.1) and (3.2), $D\,[FP_N/|P_N|] = f$. Substituting this expression into the right-hand side of (3.7) at $v \equiv 1$ we obtain the identity.

In this way we have established that the homogeneous equation (3.1) has eigenvalue $\lambda_0 = 1$ with the eigenfunction $v_0 \equiv 1$ for any c. This fact is expected, it describes the freedom of the constant γ in (2.3). This means that at the branching point the multiplicity of the eigenvalue $\lambda_n = 1$ in (3.1) must be two or more. The proof is completed.

Note that in certain cases the values $c = c_n$, satisfying the above theorem, may be not the branching points, because the condition used for nonuniqueness f_1 is the necessary only.

There is another way to find the branching points of the solutions to equation (2.2). This way is based on immediate construction of the transcendental equations for these points. In the case when the solution of the type (2.3) branches without changing the degree N of the polynomial P_N, the theory of implicit functions of several variables is applied for this purpose. If the polynomial degree is changed in new solutions, then the transcendental equation system is made up from the equations of the type (2.7) written for the different values of N simultaneously. Of course, integral equation (2.6) should be satisfied as well.

First we consider the case when N does not change at the branching point, that is, new solutions are generated by the polynomials of the same degree as the initial one. We denote $\eta_{Nk} = \eta'_{Nk} + i\eta''_{Nk}$ and consider the real values η'_{Nk}, η''_{Nk} as functions of the parameter c.

Write the integral equation (2.6) in the form

$$\begin{gathered}
\Upsilon(|f(\xi)|, \eta_{N1}, \eta_{N2}, \ldots \eta_{NN}) \equiv \alpha\,|f(\xi)| + \frac{\beta}{|P_N(\tau)|} \times \\
\times \int_a^b |f(\xi')| K(\xi, \xi') \frac{\mathrm{Re}\,\left[\bar{P}_N(\tau) P_N(\tau')\right]}{|P_N(\tau')|} d\xi' - \\
- \frac{1}{|P_N(\tau)|} \int_a^b F(\xi') K(\xi, \xi') \frac{\mathrm{Re}\,\left[\bar{P}_N(\tau) P_N(\tau')\right]}{|P_N(\tau')|} d\xi' = 0,
\end{gathered} \tag{3.8}$$

and use the notation

$$\mathcal{F} = \left\{ \Upsilon, \vec{\Phi}, \vec{\Psi} \right\},$$

where $\vec{\Phi}, \vec{\Psi}$ are the vectors in the left-hand sides of system (2.7), Υ is the left-side of equation (3.8). According to the theory of implicit functions of several variables ([18], Theorem 14.2), the points c, at which the solutions to system (2.2) are found, are determined from the condition

$$J(c) \equiv \det(\mathcal{F}') = 0, \tag{3.9}$$

where

$$\mathcal{F}' = \begin{pmatrix} \left\{ \dfrac{\partial \Phi_{Nj}}{\partial \eta'_{Nk}} \right\}_{j,k=1}^{N} & \left\{ \dfrac{\partial \Phi_{Nj}}{\partial \eta''_{Nk}} \right\}_{j,k=1}^{N} & \left\{ \dfrac{\partial \Phi_{Nj}}{\partial |f|} \right\}_{j}^{N} \\ \left\{ \dfrac{\partial \Psi_{Nj}}{\partial \eta'_{Nk}} \right\}_{j,k=1}^{N} & \left\{ \dfrac{\partial \Psi_{Nj}}{\partial \eta''_{Nk}} \right\}_{j,k=1}^{N} & \left\{ \dfrac{\partial \Psi_{Nj}}{\partial |f|} \right\}_{j}^{N} \\ \left\{ \dfrac{\partial \Upsilon}{\partial \eta'_{Nk}} \right\}_{k=1}^{N} & \left\{ \dfrac{\partial \Upsilon}{\partial \eta''_{Nk}} \right\}_{k=1}^{N} & \dfrac{\partial \Upsilon}{\partial |f|} \end{pmatrix} \tag{3.10}$$

is the Jakobi matrix of these functions. The last block-collumn in the matrix $\mathcal{F}'$ implies the discretized first variations of Φ_{Nj}, Ψ_{Nj}, and Υ with respect to $|f|$. Condition (3.9) is the transcendental equation with respect to c. It is the sought equation for the branching points. Of course, in order to calculate the left-hand side of this equation, we must solve system (2.6)-(2.7). After that the elements of determinant (3.10) become the integrals with explicit integrands.

The points at which the derivative $d\eta_{Nk}/dc$ equals infinity can be present among the roots of equation (3.9). These points can be the isolated bifurcation points, to the left or right of which the continuation of the solution is impossible; they are the points of appearance or disappearance of the solutions, respectively.

Consider the case when a solution $f_N(\xi)$ generated by the polynomial $P_N(\tau)$ branches with changing the polynomial degree (in this section we index the solutions to equation (2.2) by the degree of its generating polynomial). First, we assume that N increases by one at the branching point c_j. Since both the initial solution $f_N(\xi)$ and the off-branched one $f_{N+1}(\xi)$ solve (2.2), the necessary condition for the branching is $f_N(\xi) \equiv f_{N+1}(\xi)$ at $c = c_j$, or, after using (2.5),

$$\frac{P_N(\tau)}{|P_N(\tau)|} = \frac{P_{N+1}(\tau)}{|P_{N+1}(\tau)|}.$$

Then the parameters η_{Nk} of the polynomial $P_N(\tau)$ and $\eta_{N+1,k}$ of $P_{N+1}(\tau)$ are connected by the equalities

$$\eta_{Nk} = \eta_{N+1,k}, \qquad k = 1, \dots, N,$$

with $\operatorname{Im} \eta_{N+1,N+1} = 0$. Besides equation (2.6) and system (2.7), the equations

$$\alpha\, |f(\xi)| + \frac{\beta}{|P_{N+1}(\tau)|} \int\limits_a^b K(\xi,\xi')|f(\xi')| \frac{\operatorname{Re}\left[\bar{P}_{N+1}(\tau) P_{N+1}(\tau')\right]}{|P_{N+1}(\tau')|} d\xi' =$$

$$= \frac{1}{|P_{N+1}(\tau)|} \int\limits_a^b K(\xi,\xi') F(\xi') \frac{\operatorname{Re}\left[\bar{P}_{N+1}(\tau) P_{N+1}(\tau')\right]}{|P_{N+1}(\tau')|} d\xi', \tag{3.11}$$

$$\int_a^b \tau^{n-1} s(\xi) \frac{F(\xi)-\beta|f(\xi)|}{|P_{N+1}(\tau)|} d\xi \;= 0, \qquad n = 1, 2, ..., N+1, \tag{3.12a}$$

$$\int_a^b \tau^{n-1} q(\xi) \frac{F(\xi)-\beta|f(\xi)|}{|P_{N+1}(\tau)|} d\xi \;= 0, \qquad n = 1, 2, ..., N+1 \tag{3.12b}$$

should hold.

Since the new parameter $\eta_{N+1,N+1}$ is real, the integral equation (3.11) coincides with (2.6).

Owing to the identity

$$\frac{1}{1-\eta\tau} = 1 + \frac{\eta\tau}{1-\eta\tau},$$

the nth equation of system (3.12) ($n = 1, 2, \ldots, N$) is a linear combination of the nth equation of system (2.7) and $(n+1)$ th equation of system (3.12). Hence, at the branching point, besides the integral equation (2.6) and the transcendental equations (2.7), only two additional equations

$$\int_a^b \tau^N s(\xi) \frac{F(\xi)-\beta|f(\xi)|}{|P_N(\tau)|(1-\eta_{N+1,N+1}\tau)} d\xi = 0, \tag{3.13a}$$

$$\int_a^b \tau^N q(\xi) \frac{F(\xi)-\beta|f(\xi)|}{|P_N(\tau)|(1-\eta_{N+1,N+1}\tau)} d\xi = 0 \tag{3.13b}$$

should hold. On the whole, we have one real integral equation and $2N+2$ transcendental ones for determining the real function $|f(\xi)|$, N complex parameters η_{Nk}, $k = 1, 2, \ldots, N$ and real $\eta_{N+1,N+1}$ and c_j.

At the branching points where the polynomial degree changes by two, the equalities

$$\eta_{Nk} = \eta_{N+2,k}, \qquad k = 1, \ldots, N,$$

are valid. Besides (2.6) and (2.7), the four additional equations

$$\int_a^b \tau^{n-1} s(\xi) \frac{F(\xi)-\beta|f(\xi)|}{|P_N(\tau)|(1-\eta_{N+2,N+1}\tau)(1-\eta_{N+2,N+2}\tau)} d\xi{=}0, \quad n = N+1, N+2; \tag{3.14a}$$

$$\int_a^b \tau^{n-1} q(\xi) \frac{F(\xi)-\beta|f(\xi)|}{|P_N(\tau)|(1-\eta_{N+2,N+1}\tau)(1-\eta_{N+2,N+2}\tau)} d\xi{=}0, \quad n = N+1, N+2, \tag{3.14b}$$

should be fulfilled with $\eta_{N+2,N+1}$, $\eta_{N+2,N+2}$ satisfying the conditions

$$\eta_{N+2,N+1} = \bar{\eta}_{N+2,N+2} \tag{3.15}$$

or

$$\operatorname{Im}\eta_{N+2,N+1} = \operatorname{Im}\eta_{N+2,N+2} = 0.$$

Hence, we have one real integral equation and $2N+4$ transcendental ones for determining the real function $|f(\xi)|$, N complex parameters η_{Nk}, $k = 1, 2, \ldots, N$, real c_j, and one complex $\eta_{N+2,N+1}$ or two real $\eta_{N+2,N+1}$, $\eta_{N+2,N+2}$. In whole we have one real unknown function and $2N + 3$ real numbers. In general case, the existence of solutions to such a system is low-probable. However, they may exist in the case when $a = -b$, one of the functions $s(\xi)$ and $q(\xi)$ is even and the second one is odd, and

$$F(\xi) = F(-\xi). \tag{3.16}$$

Then the solutions with polynomials of the even modulus are possible. Such a property should be inhered to $P_{N+2}(\tau)$, that is

$$|P_{N+2}(\tau)| = |P_{N+2}(-\tau)|. \tag{3.17}$$

These properties decrease the number of unknowns twice: the parameters $\eta_{N+2,k}$ become imaginary or appear by the pears with opposite signs, whereas, according to (3.15), $\eta_{N+2,k}$, $k = N + 1, N + 2$, are always imaginary with opposite signs:

$$\begin{aligned} \operatorname{Re}\eta_{N+2,N+1} &= \operatorname{Re}\eta_{N+2,N+2} = 0, && (3.18a)\\ \eta_{N+2,N+1} &= \bar{\eta}_{N+2,N+2}. && (3.18b) \end{aligned}$$

On the other hand, conditions (3.17) decrease the number of equations twice, too: N equations of system (2.7) and two additional equations (3.14) become identities, because they have odd integrands in the left-hand side.

Finally, at fulfilling (3.16) the solution branching is possible with increasing the polynomial degree by two if together with integral equation (2.6) the following transcendental equation system holds:

$$\int_{-b}^{b} \tau^{2n-1} s(\xi) \frac{F(\xi) - \beta|f(\xi)|}{|P_N(\tau)|} d\xi = 0, \quad n = 1, 2, \ldots [N/2],$$

$$\int_{-b}^{b} \tau^{2n-2} q(\xi) \frac{F(\xi) - \beta|f(\xi)|}{|P_N(\tau)|} d\xi = 0, \quad n = 1, 2, \ldots [(N+1)/2],$$

$$\int_{-b}^{b} \tau^{2[(N+2)/2]-1} s(\xi) \frac{F(\xi) - \beta|f(\xi)|}{|P_N(\tau)|(1 - \eta_{N+2,N+1}\tau)(1 - \eta_{N+2,N+2}\tau)} d\xi = 0,$$

$$\int_{-b}^{b} \tau^{2[(N+1)/2]} q(\xi) \frac{F(\xi) - \beta|f(\xi)|}{|P_N(\tau)|(1 - \eta_{N+2,N+1}\tau)(1 - \eta_{N+2,N+2}\tau)} d\xi = 0,$$

where η_{Nk}, $k = 1, \ldots, N$, either are imaginary or appear by pears with alternative signs, and $\eta_{N+2,k}$, $k = N + 1, N + 2$, are subject to conditions (3.18). As result, we have one real integral equation, $N+2$ transcendental ones for determining the real function $|f(\xi)|$, and $N + 2$ real numbers.

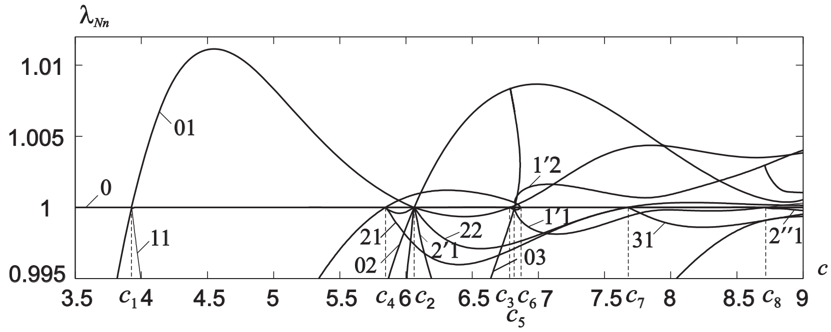

Fig. 1. Eigenvalues of equation (3.1); $F(\xi) = \cos(\xi\pi/2)$, $t = 0.05$.

4. Numerical results

One of particular cases of the general integral equation (2.2) is the equation

$$\begin{aligned}\frac{t}{c}f(\xi) + \int_{-1}^{1} \frac{\sin c(\xi-\xi')}{(\xi-\xi')} f(\xi')d\xi' &= \\ = \int_{-1}^{1} \frac{\sin c(\xi-\xi')}{(\xi-\xi')} F(\xi')\exp(i\arg f(\xi'))d\xi'\end{aligned} \tag{4.1}$$

arisen in the unconditional minimization problem for the functional

$$\sigma_t(u) = \sigma_0(u) + \sigma_1(u),$$

where

$$\sigma_0(u) = \int_{-1}^{1} \left(|f(\xi)| - F(\xi)\right)^2 d\xi, \tag{4.2a}$$

$$\sigma_1(u) = \frac{t}{c}\int_{-1}^{1} |u(x)|^2 dx, \tag{4.2b}$$

$$f(\xi) = \frac{c}{\sqrt{2\pi}} \int_{-1}^{1} u(x)\exp\left(icx\xi\right)dx,$$

$t > 0$ is the regularization parameter [19]. Equation (4.1) is a particular case of (2.2) with

$$s(\xi) = \sin(c\xi), \quad q(\xi) = \cos(c\xi), \quad \tau = \xi, \quad a = -1, \quad b = 1, \quad \alpha = t/c, \quad \beta = 1.$$

Below we give the numerical results for the case

$$F(\xi) = \cos(\xi\pi/2), \quad t = 0.05.$$

Fig. 1 shows the eigenvalues of the homogeneous equation (3.1). The curves are labeled by two numbers Nn, where N is the polynomial degree of the solution from which the branching is finding, and n is the ordinary number of the eigenvalue of equation

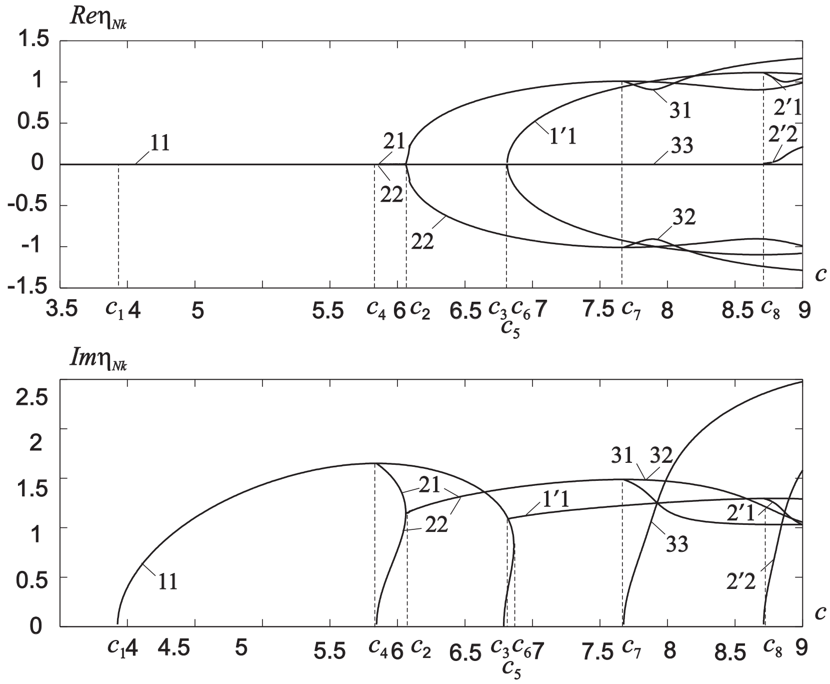

Fig. 2. Real and imaginary parts of parameters η_{Nk}; $F(\xi) = \cos(\xi\pi/2), t = 0.05$

corresponding to this N. As it was noted, the eigenvalue $\lambda_0 = 1$ exists for any c. The points where $\lambda_n = 1$ with $n \neq 0$ are the branching points of solutions to equation (4.1). In the considered range of the values c, different types of branching were observed: without changing N, with N changing by one, and by two. The maximal value of the polynomial degree in this range is $N = 3$. Recall that we consider an example of the even function $F(\xi)$.

The parameters η_{Nk} of the polynomial $P_N(\xi)$ are determined from the integral equation (2.6) and transcendental equation system (2.7). The system was solved by the Newton method. The results are presented in Fig. 2. Only solutions with

$$\operatorname{Im} \eta_{Nk} \geq 0$$

are shown. While analyzing these results we will refer to Fig. 1 for agreeing them with the behavior of the eigenvalues of equation (3.1).

At $N = 0$ the polynomial P_N in (2.3) has the form $P_0 \equiv 1$ and the solution to equation (4.1) becomes real. If such solution $f_0(\xi)$ (having no zeros in $[-1, 1]$) exists, then it can be determined from the linear integral equation obtained from (4.1) at

$$\exp(i \arg f(\xi')) \equiv 1.$$

In our example this solution exists in whole considered range of values c. It has three branching points c_1, c_2, c_3.

At $c = c_1$ curve λ_{01} intersects the line $\lambda = 1$ for the first time. Two solutions $f_1(\xi)$, with $N = 1$ and imaginary conjugated parameters $\pm\eta_{11}$, arise here; at the branching point c_1 these parameters are zero (for simplicity, we denote all solutions from the same equivalent group by one index coinciding with N; if it is needed, this index is primed).

The real function $|f_1(\xi)|$ and parameter c_1 at this point are calculated from real integral equation (2.6) and additional equation

$$\int_{-1}^{1} \cos c\xi (F(\xi) - |f(\xi)|\,) d\xi = 0 \qquad (4.3)$$

following from (3.13).

The first eigenvalue λ_{11} of equation (3.1), corresponding to the solution $f_1(\xi)$ starts at $c = c_1$ from $\lambda = 1$. This means that c_1 can be interpreted as a point at which the solution $f_0(\xi)$ branches from $f_1(\xi)$ as c decreases.

At the point c_2 two eigenvalues λ_{01} and λ_{02} intersect $\lambda = 1$ simultaneously. At this point the solution $f_0(\xi)$ branches with increasing the polynomial degree by two; we denote the new solutions by $f_2(\xi)$. Each of them has two complex parameters η_{21} and η_{22}. Each of these pairs are calculated separately (curves 21, 22 at Fig. 2). At the branching point the parameters η_{21} and η_{22} coincide and are imaginary. The real function $|f_2(\xi)|$ and real parameters η_{21}, c_2 are determined from the integral equation (2.6) and the transcendental equation system (3.14) that becomes

$$\int_{-1}^{1} \cos c\xi \frac{F(\xi) - |f(\xi)|}{(1 - \eta_{21}^2 \xi^2)} d\xi = 0, \qquad (4.4a)$$

$$\int_{-1}^{1} \xi \sin c\xi \frac{F(\xi) - |f(\xi)|}{(1 - \eta_{21}^2 \xi^2)} d\xi = 0. \qquad (4.4b)$$

Note, that this branching is double-sided, the off-branched solution exists for both $c < c_2$ and $c > c_2$. Formally, two new branches of solutions arise at this point, one at $c < c_2$ and other at $c > c_2$. We will return to this point when analyzing other solution with $N = 2$. For $c > c_2$ the parameters η_{21}, η_{22} become complex with the property $\eta_{21} = -\eta_{22}$ (curves 21, 22 in Fig. 2).

At the point c_3 the curve λ_{01} intersects $\lambda = 1$ for the second time. This points is the second root of equation (4.3). However, any new solution does not "branch off" from $f_0(\xi)$ at this point; the solution $f_1(\xi)$ "branches in" here to $f_0(\xi)$ (see curve 11 in Fig. 2). We will return to this point when analyzing solutions with $N = 1$.

The real solution $f_0(\xi)$ has no more branching points. We pass to investigation of the first solution $f_1(\xi)$ branched from it. At $c = c_4$ the curve λ_{11} intersects $\lambda = 1$ for the first time. The solution $f_2(\xi)$ with $N = 2$ branches from the solution $f_1(\xi)$. At the branching point the parameters η_{21} and η_{22} coincide and are imaginary. The real function $|f_2(\xi)|$ and real parameters η_{11}, η_{22}, c_2 are determined from real integral equation (2.6), the transcendental equation system (2.7) and additional equations (3.13). Equation system (2.7) together with (3.13) obtains the form

$$\int_{-1}^{1} \cos c\xi \frac{F(\xi) - |f(\xi)|}{(1 - \eta_{11}^2 \xi^2)^{1/2}} d\xi = 0,$$

$$\int_{-1}^{1} \xi \sin c\xi \frac{F(\xi) - |f(\xi)|}{(1 - \eta_{11}^2 \xi^2)^{1/2}(1 - \eta_{22}\xi)} d\xi = 0,$$

$$\int_{-1}^{1} \xi \cos c\xi \frac{F(\xi) - |f(\xi)|}{(1 - \eta_{11}^2 \xi^2)^{1/2}(1 - \eta_{22}\xi)} d\xi = 0.$$

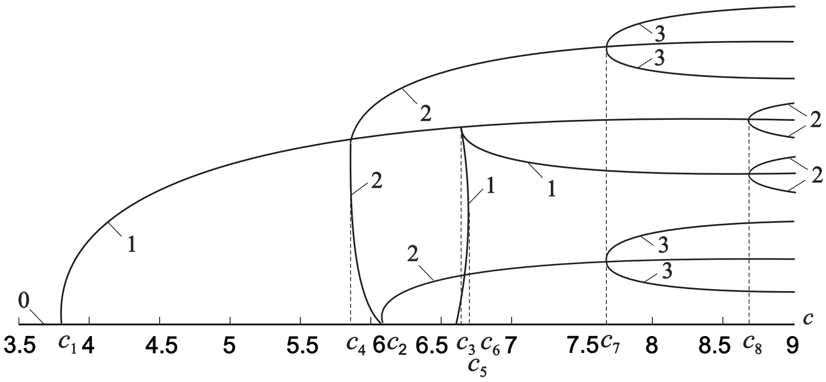

Fig. 3. Graph of solutions to equation (4.1).

Two solutions of this type (with different signs of η_{22}), branched from $f_1(\xi)$ and $\bar{f}_1(\xi)$, make up (together with their complex-conjugated) the equivalent group of four solutions. This group contains two appreciably different functions: $f_2(\xi)$ (with the same signs of η_{21}, η_{22}) and $f_{2'}(\xi)$ (with opposite signs of η_{21}, η_{22}). The corresponding curves of the eigenvalues λ_{21}, $\lambda_{2'1}$, start at $c = c_4$ from $\lambda = 1$, but their behaviors are different. The eigenvalue λ_{21} varies smoothly up to the point c_7, where it intersects the line $\lambda = 1$ secondly, whereas $\lambda_{2'1}$ returns to this line and only touches it at $c = c_2$. At this point the imaginary parameters η_{21}, η_{22} become the complex-conjugated and the solution becomes real, coinciding with $f_0(\xi)$. Therefore, the point c_2 can be treated either as the branching point of the solution $f_0(\xi)$ with increasing the polynomial degree by two, or as the branching point of the solution $f_{2'}(\xi)$ with decreasing the polynomial degree by two.

At the point c_5 the solution $f_1(\xi)$ branches without changing the polynomial degree – two new solutions $f_{1'}(\xi)$ with $N = 1$ branch from it. The off-branched solutions have the complex parameters $\eta_{1'1}$ with the same imaginary part and oppositely-signed real ones (curve $1'1$ in Fig. 2). The function $|f_{1'}(\xi)|$ and real parameters $\eta_{1'1}$, c_5 are determined from integral equation (2.6), the transcendental equation

$$\int_{-1}^{1} \cos c\xi \, \frac{F(\xi) - |f(\xi)|}{(1 - \eta_{11}^2 \xi^2)^{1/2}} \, d\xi = 0$$

obtained from system (2.7), and a transcendental equation following from (3.9). The last one is not written here owing its awkwardness.

The solution $f_1(\xi)$ has no more branching points, and we can investigate the first solution $f_2(\xi)$, branched from it at $c = c_4$. At the point c_7, the curve λ_{21} corresponding to $f_2(\xi)$ intersects the line $\lambda = 1$; the two solutions with $N = 3$ branch from each solution of the considered equivalent group (we denote them by $f_3(\xi)$). Each of the off-branched solutions is generated by a polynomial P_3 with two complex η_{31}, $\eta_{32} = -\bar{\eta}_{31}$ and one imaginary $\pm\eta_{33}$ parameters; at this point $\eta_{31} = \eta_{21}$. The real function $|f_3(\xi)|$ and the parameters η_{21}, η_{33}, c_7 are calculated from the integral equation (2.6) and system (2.7) complemented by equations (3.13) having in this case the form

$$\int_{-1}^{1} \cos c\xi \, \frac{F(\xi) - |f(\xi)|}{|1 - \eta_{21}^2 \xi^2|} d\xi = 0,$$

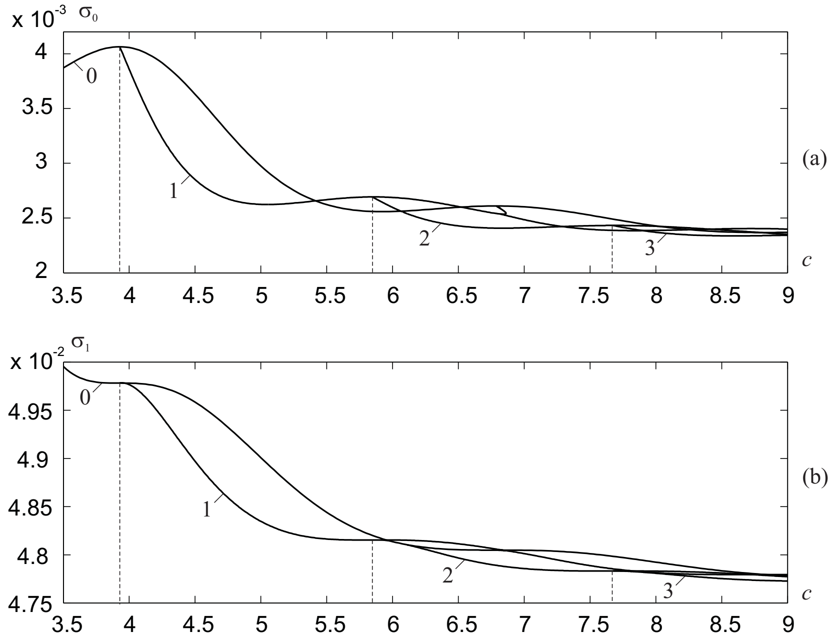

Fig. 4. Comparison of components of σ on different solutions to equation (4.1).

$$\int_{-1}^{1} \xi \sin c\xi \frac{F(\xi) - |f(\xi)|}{|1-\eta_{21}^2\xi^2|} d\xi = 0,$$

$$\int_{-1}^{1} \xi^2 \cos c\xi \frac{F(\xi) - |f(\xi)|}{\left|1-\eta_{21}^2\xi^2\right|(1-\eta_{33}\xi)} d\xi = 0,$$

$$\int_{-1}^{1} \xi^2 \sin c\xi \frac{F(\xi) - |f(\xi)|}{|1-\eta_{21}^2\xi^2|(1-\eta_{33}\xi)} d\xi = 0.$$

The point c_7 is the unique branching point of the solution $f_2(\xi)$ in the considered region.

Consider the solution $f_{1'}(\xi)$ branched from $f_1(\xi)$ at c_5. At $c = c_8$ the curve $\lambda_{1'1}$ corresponding to this solution intersects the line $\lambda = 1$ for the first time. The solution $f_2(\xi)$ with $N = 2$ branches from $f_{1'}(\xi)$. The function $|f_2(\xi)|$, the complex $\eta_{2'1} = \eta_{1'1}$, imaginary $\eta_{2'2}$, and c_8 are calculated from the integral equation (2.6) and system (2.7) complemented by equations (3.13) having in this case the form

$$\int_{-1}^{1} \cos c\xi \frac{F(\xi) - |f(\xi)|}{|1-\eta_{1'1}\xi|} d\xi = 0,$$

$$\int_{-1}^{1} \sin c\xi \frac{F(\xi) - |f(\xi)|}{|1-\eta_{1'1}\xi|} d\xi = 0,$$

$$\int_{-1}^{1} \xi \sin c\xi \frac{F(\xi) - |f(\xi)|}{|1-\eta_{1'1}\xi|\,(1-\eta_{2'2}\xi)} = 0,$$

$$\int_{-1}^{1} \xi \cos c\xi \frac{F(\xi) - |f(\xi)|}{|1-\eta_{1'1}\xi|\,(1-\eta_{2'2}\xi)} d\xi = 0.$$

The branching process for the considered region of c is completed. It is illustrated in Fig. 3 in the graph form. The curve labels coincide with the polynomial degree N. Only the half of the solutions are shown here (to within the complex conjugacy).

Fig. 4 shows the values of the both addends (4.2) in the functional σ. It is seen that the dependencies of both addends are similar. However, the value σ_1 varies in the smaller range than σ_0.

Bibliography

1. Vainberg M. M. Theory of Branching of Solutions of Nonlinear Equations / M. M. Vainberg, V. A. Trenogin.– Moscow: Nauka, 1969. (in Russian).
2. Masujima M. Applied Mathematical Methods in Theoretical Physics / M. Masujima.– 2nd. ed.– Weinhem: WILEY-VCH, 2009.
3. Ferwerda H. A. Problem of the wave front phase reconstruction according to amplitude distribution and coherent functions / H. A. Ferwerda // Inverse Scattering Problems in Optics. Ed. H.P.Baltes.– Berlin: Springer, 1978.
4. Kuznetsova T. I. On the phase problem in optics / T. I. Kuznetsova // Uspekhi Fizicheskikh Nauk.– 1988.– Vol. 154.– P. 677-690. (in Russian).
5. Boikova A. T. Solution of phase retrieval problem by a maximum entropy method / A. T. Boikova // Radiophysics and Quantum Electronics.– 1996.– Vol. 39.– P. 321-327.
6. Pol'skikh S. D. The phase problem: analysis of local extrema and image reconstruction algorithms / S. D. Pol'skikh // Radiotekhnika i Elektronika.– 2008.– Vol. 53.– P. 223-237. (in Russian).
7. Samoilenko M. V. Reconstruction of the amplitude–phase distribution of the field of the received signal in the aperture of a phased antenna array from the measured power / M. V. Samoilenko // Radiotekhnika i Elektronika.– 2009.– Vol. 54.– P. 1058-1063. (in Russian).
8. Voitovich N. N. Antenna synthesis according to prescribed amplitude radiation pattern (V. V. Semenov's method) / N. N. Voitovich // Radiotekhnika i Elektronika.– 1972.– Vol. 17.– P. 2491-2497. (in Russian).
9. Andriychuk M. I. Synthesis of Antennas according to Amplitude Directivity Pattern: Numerical Methods and Algorithms / M. I. Andriychuk, N. N. Voitovich, P. A. Savenko, V. P. Tkachuk.– Kiev: Naukova Dumka, 1993. (in Russian).
10. Savenko P. A. Nonlinear Problems of Radiating Systems Synthesis: Theory and Methods of Solution / P. A. Savenko.– Lviv: IAPMM NAS of Ukraine, 2002. (in Ukrainian).
11. Voitovich N. N. Mean square approximation of compactly supported functions with free phase by functions with bounded spectrum / N. N. Voitovich, O. M. Gis, Y. P. Topolyuk // Dopovidi NAN Ukrainy.– 1999.– Vol. 3.– P. 7-10. (in Ukrainian).
12. Voitovich N. N. Approximation of compactly supported functions with free phase by functions with bounded spectrum / N. N. Voitovich, Y. P. Topolyuk, O. O. Reshnyak // Fields Institute Communications.– 2000.– Vol. 25.– P. 531-541.
13. Bulatsyk O. O. Investigation of branching of solutions to nonlinear equation of the modified phase problem in the case of discrete Fourier transform / O. O. Bulatsyk // Visnuk Lvivskogo univers.– Ser. Prykl. Mat. Inform.– 2003.– Vol. 7.– P. 20-32. (in Ukrainian).
14. Bulatsyk O. O. Properties of nonlinear. Hammerstein integral equations connected with modified phase problem / O. O. Bulatsyk, N. N. Voitovich // Direct and Inverse Problems of Electromagnetic and Acoustic Wave Theory (DIPED-2003).– Proc. of VIII-th Int. Seminar/Workshop, Lviv.– 2003.– P. 135-138.
15. Bulatsyk O. O. Analytic solutions to a class of nonlinear integral equation connected wich modified phase problem / O. O. Bulatsyk, N. N. Voitovich // Information Extraction and Processing – 2003.– Vol. 19.– P. 33-39. (in Ukrainian).

16. Lander F. J. The Bezoitiant and inversion of Hankel and Toeplitz matrices / F. J. Lander // Matematicheskie Issledovaniya.– 1974.– Vol. 2.– P. 69-87. (in Russian).
17. Fiedler M. Special Matrices and their Applications in Numerical Mathematics / M. Fiedler.– Dordrecht / Boston / Lancaster: Martinus Nijhoff Publishers, 1986.
18. Ilyin B. A. Mathematical Analysis / B. A. Ilyin, B. A. Sadovnichiy, V. A. Sendov.– Moscow: Nauka, 2001. (in Russian).
19. Bulatsyk O. O. Generalized nonlinear integral equation of Hammerstein type / O. O. Bulatsyk, Y. P. Topolyuk, N. N. Voitovich // Direct and Inverse Problems of Electromagnetic and Acoustic Wave Theory (DIPED-2009).– Proc. of XIV-th Int. Seminar/Workshop, Lviv.– 2009.– P. 181-185.

Pidstrygach Institute for Applied Problems of Mechanics and Mathematics,
National Academy of Sciences of Ukraine,
3-b Naukova str., Lviv, 79601, Ukraine
E-mail address: voi@iapmm.lviv.ua

Received 11.11.2009