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# AN INTEGRAL EQUATION METHOD FOR A MIXED INITIAL BOUNDARY VALUE PROBLEM FOR UNSTEADY STOKES SYSTEM IN A DOUBLY-CONNECTED DOMAIN

UDC 517.9

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Анотація. Розглядається чисельне розв'язування початково-крайової задачі для рівняння Стокса в плоскій двозв'язній області. Використовуючи перетворення Лагерра, нестаціонарна задача редукується до системи граничних задач для резольвентного рівняння Стокса. За допомогою модифікованих потенціалів отримано систему граничних інтегральних рівнянь з різними сингулярностями в ядрах. Для їх чисельного розв'язування застосовано метод тригонометричних квадратур. Приведено результати чисельних експериментів, які демонструють, що достатньо точне наближення можна отримати за невеликі обчислювальні затрати.

Abstract. We present a novel numerical method for a mixed initial boundary value problem for the unsteady Stokes system in a planar doubly-connected domain. Using a Laguerre transformation the unsteady problem is reduced to a system of boundary value problems for the Stokes resolvent equations. Employing a modified potential approach we obtain a system of boundary integral equations with various singularities and we use a trigonometric quadrature method for their numerical solution. Numerical examples are presented showing that accurate approximations can be obtained with low computational cost.

## 1. Introduction

The Navier–Stokes equations are a mathematical description (non-linear) of the time-evolution of a viscous fluid, and are known to be a very accurate model; however, due to the (mathematical) complexity of these equations, approximations to some special cases are required. The unsteady Stokes system is obtained as a linearized approximation of the full Navier–Stokes system when the Reynolds number is small, see for example [11, 13]. We shall supply this approximation with mixed boundary conditions, i.e. the velocity of the fluid is known on a part of the boundary and the traction on the other part. Mixed boundary value problems occur in many fluid flow problems, see for example [12, 14]. Also, in so-called inverse problems where for example part of the boundary might be overspecified with no data on the remaining part, iterative methods have been proposed which in each iteration step solves direct mixed boundary value problems, see [8, 9]. Thus, it is of importance to have a method for unsteady Stokes system which can give accurate approximations with low computational costs.

Let us formulate the fluid flow situation in a more mathematical way. For simplicity, we only consider doubly-connected domains. Assume that $D_1 \subset \mathbb{R}^2$ and $D_2 \subset \mathbb{R}^2$ are simply connected bounded domains with boundaries $\Gamma_1, \Gamma_2 \in C^3$ such that $\overline{D}_2 \subset D_1$ and let $D := D_1 \backslash \overline{D}_2$ (see Fig. 1). Define $Q := D \times (0, \infty)$ and $\Sigma_\ell = \Gamma_\ell \times (0, \infty)$, $\ell = 1, 2$. To further simplify the presentation, we assume that the fluid is incompressible, that no

*Key words.* Unsteady Stokes system, mixed initial boundary value problem, Laguerre transformation, boundary integral equations, trigonometrical quadratures, Nyström method.

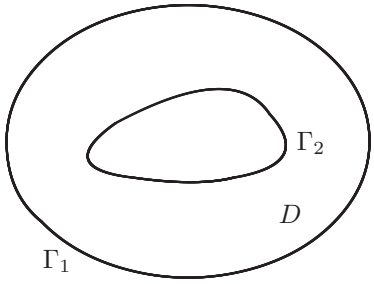


Fig. 1. Doubly-connected domain $D$

sources are present and that the fluid is initially at rest, i.e. we consider the following initial boundary value problem

$$\Delta u - \frac{\partial u}{\partial t} - \nabla p = 0 \quad \text{in} \quad Q, \tag{1.1}$$

$$\operatorname{div} u = 0 \quad \text{in} \quad Q, \tag{1.2}$$

$$u = f_1 \quad \text{on} \quad \Sigma_1, \tag{1.3}$$

$$T(u,p)\nu_2 = f_2 \quad \text{on} \quad \Sigma_2, \tag{1.4}$$

$$u(\cdot,0) = 0 \quad \text{in} \quad D. \tag{1.5}$$

Here $c_r$ is the Reynolds number, $u = (u_1, u_2)$ and $p$ are the unknown functions describing the velocity respectively the (kinematic) pressure, and $f_1$ and $f_2$ are given functions that satisfy the compatibility conditions $f_\ell(x,0) = 0$, for every $x \in D$ and $\ell = 1, 2$. As usual $\Delta$ is the Laplace operator, $T(u,p) = pI - (\nabla + \nabla^\top)u$ is the stress tensor and $\nu_\ell$ is the outward unit normal to the boundary $\Gamma_\ell$, $\ell = 1, 2$.

The classical numerical approximations, i.e. finite differences, boundary element and finite element methods tend to be rather costly for the time-dependent case since the entire solution domain needs to be discretized. There are transformation methods such as the method of Rothe to transform the unsteady system into a Stokes type (steady) system. In this paper, we propose a novel method for bounded smooth planar domains which reduces the unsteady Stokes system to a boundary integral equation over the boundary of the solution domain. This is achieved by employing the so-called Laguerre transformation in the time-variable giving rise to a system involving the Stokes resolvent equations. Note here that some aspects of applying integral equation methods for the stationary case are reflected in [2, 7]. In Section 2 the discretization in time is presented and fundamental solutions are obtained, see Theorem 3. In Section 3 we describe how the equations can be effectively solved using a Nyström type method in combination with the results and fundamental solutions from Section 2. Numerical experiments are given in Section 4 showing that accurate approximations can be obtained with few boundary collocation points.

## 2. Semi-discretization in time and an integral equation method

For the reduction of the unsteady problem (1.1)-(1.5) into a sequence of stationary boundary value problems we employ the Laguerre transformation [1, 4, 6]. We search for the solution of (1.1)-(1.5) in the form of a Fourier-Laguerre series

$$u(x,t) = \kappa^2 \sum_{n=0}^{\infty} u_n(x) L_n(\kappa^2 t) \tag{2.1}$$

and

$$p(x,t) = \kappa^2 \sum_{n=0}^{\infty} p_n(x) L_n(\kappa^2 t), \tag{2.2}$$

where $\kappa > 0$ is a fixed scaling parameter, $L_n$ for $n = 0, 1, \ldots$, are the Laguerre polynomials and $u_n$ and $p_n$ are the Fourier-Laguerre coefficients. It is straightforward to prove that the coefficients $u_n$ and $p_n$ satisfy the following sequence of boundary value problems:

$$\Delta u_n - k^2 u_n - \nabla p_n = k^2 \sum_{m=0}^{n-1} u_m \quad \text{in} \quad D, \tag{2.3}$$

$$\operatorname{div} u_n = 0 \quad \text{in} \quad D, \tag{2.4}$$

$$u_n = f_{1,n} \quad \text{on} \quad \Gamma_1, \tag{2.5}$$

$$T(u_n, p_n)\nu_2 = f_{2,n} \quad \text{on} \quad \Gamma_2, \tag{2.6}$$

where $f_{1,n}$ and $f_{2,n}$ are the Fourier-Laguerre coefficients of $f_1$ and $f_2$, respectively, for $n = 0, 1, \ldots$ . When $n = 0$ the sum should be removed from the right-hand side in (2.3).

**Theorem 2.1** *The sequence of boundary value problems* (2.3)-(2.6) *has at most one solution.*

*Proof.* Firstly we show uniqueness of a classical solution of (2.3)-(2.6) when $n = 0$. From the first Green's formula (see [15]) in the case of homogeneous boundary conditions we have

$$\int_D \left[ 2|\mathcal{D}u_0(x)|^2 + \kappa^2 u_0^2(x) \right] dx = 0,$$

where $\mathcal{D}u := \frac{1}{2}(\nabla u + (\nabla u)^\top)$. Thus $u_0 = 0$ in $D$. Then from the equation (2.3) we obtain that $p_0$ is a constant in $D$, and by using the boundary value condition (2.6) it follows that $p_0 = 0$ on $\Gamma_2$, which then implies that $p_0 = 0$ in $D$. The statement of the theorem follows by mathematical induction. □

Now we wish to reduce the sequence of boundary value problems (2.3)-(2.6) into boundary integral equations. The use of the classical potential approach for these problems leads to integral equations with volume integrals over the domain $D$. Thus, the usual important advantage of the integral equation method, i.e. the reduction of the problem dimension, cannot be obtained for this system with the standard approach. To overcome this difficulty we first define the fundamental solution for the sequence of boundary value problems (2.3)-(2.6) in the following way:

**Definition 2.2** *The sequence of pairs* $(E_n, e)$ *consisting of a* $2 \times 2$ *matrix* $E_n(x,y) = (E_{n,1}(x,y), E_{n,2}(x,y))$ *with columns* $E_{n,1}, E_{n,2}, n = 0, 1, \ldots$, *and a vector* $e(x,y) = (e_1(x,y), e_2(x,y))$ *is called a fundamental solution for the sequence of systems* (2.3)-(2.4) *if:*

$$\begin{cases} \Delta_x E_{n,\ell} - \kappa^2 \sum\limits_{k=0}^{n} E_{k,\ell} - \nabla_x e = \delta(x-y) I_\ell, \\ \operatorname{div}_x E_{n,\ell} = 0, \quad \ell = 1, 2. \end{cases} \tag{2.7}$$

*Here* $I = (I_1, I_2)$ *is the* $2 \times 2$ *identity matrix,* $\delta$ *denotes the Dirac function and the differentiation in* (2.7) *is taken with respect to the* $x$*-variable.*

To shorten the presentation we introduce the following polynomials, which we will use to represent $E_n$:

$$\nu_n(r) = \sum_{m=0}^{[n/2]} a_{n,2m} r^{2m}, \quad \omega_n(r) = \sum_{m=0}^{[(n-1)/2]} a_{n,2m+1} r^{2m+1}$$

for $n = 0, 1, \ldots$, where the coefficients $a_{n,m}$ satisfy the recurrence relations

$$a_{n,0} = 1, \quad n = 0, 1, \ldots,$$

$$a_{n,n} = \frac{\kappa}{2n} a_{n-1,n-1}, \quad n = 1, 2, \ldots,$$

$$a_{n,m} = \frac{1}{2\kappa m} \left\{ 4 \left[ \frac{m+1}{2} \right]^2 a_{n,m+1} + \kappa^2 a_{n-1,m-1} \right\}, \quad m = n-1, \ldots, 1.$$

Here $[r]$ defines the integer part of a real number $r \geq 0$. In addition, we introduce the following sequences of functions

$$\Phi_n(\kappa, r) = K_0(\kappa r)\nu_n(r) + K_1(\kappa r)\omega_n(r),$$

$$\Psi_0(\kappa, r) = \frac{1}{\kappa^2} \left( \ln \frac{1}{r} - K_0(\kappa r) \right),$$

$$\Psi_n(\kappa, r) = -\frac{1}{\kappa^2} \sum_{m=0}^{n} (-1)^m C_n^m \Phi_m(\kappa, r), \quad n = 1, 2, \ldots$$

and

$$Q_n(\kappa, r) = \sum_{m=0}^{n} (-1)^m C_n^m \big[ \Phi_m(\kappa, r) - \Phi_{m-1}(\kappa, r) \big],$$

where $K_0$ and $K_1$ are modified Hankel functions and $C_n^k$ are binomial coefficients.

**Theorem 2.3** *The pair $(E_n, e)$ with*

$$E_n(x, y) = -c_\pi [Q_n(\kappa, |x-y|) I + \mathrm{grad}_x \mathrm{grad}_x^\top \Psi_n(\kappa, |x-y|)] \tag{2.8}$$

*and*

$$e(x, y) = -\frac{c_\pi (x-y)}{|x-y|^2} \tag{2.9}$$

*is a fundamental solution of* (2.3)-(2.4) *in the sense of* Defintion 2. *Here we have set* $c_\pi = (2\pi)^{-1}$.

*Proof.* We follow the ideas of the proof for the analogous case when the sequence is received by Rothe's method [3]. Using the Fourier transform in two variables in (2.7), we receive

$$\begin{cases} -(|\xi|^2 + k^2) \widehat{E}_{n,j} - i\xi \widehat{e}_j = k^2 \sum\limits_{k=0}^{n-1} \widehat{E}_{k,j} + c_\pi I_j, \\ \langle \xi, \widehat{E}_{0,j} \rangle = 0, \end{cases} \tag{2.10}$$

where $\widehat{E}_n$ and $\widehat{e}$ are the Fourier transform of $E_n$ and $e$, respectively. From (2.10) we deduce that

$$\widehat{e}(\xi) = \frac{i c_\pi \xi}{|\xi|^2} \quad \text{and} \quad \widehat{E}_n(\xi) = -c_\pi [I - J(\xi)] \sum_{m=0}^{n} \frac{C_n^m (-1)^m k^{2m}}{(|\xi|^2 + k^2)^{m+1}} \tag{2.11}$$

with the matrix $J(w) = \frac{ww^\top}{|w|^2}$ for $w \in \mathbb{R}^2 \setminus \{0\}$. The inverse Fourier transform applied to (2.11) together with some additional recurrence relations give the representations (2.8) and (2.9) (for the details see [1]). □

Now, we define pairs of single- and double-layer potentials for the sequence (2.3)-(2.4):

$$(V_n\varphi)(x) = \sum_{m=0}^{n} \int_{\Gamma_2} E_{n-m}(x,y)\varphi_m(y)ds(y), \quad x \in D, \tag{2.12}$$

$$(v_n\varphi)(x) = \sum_{m=0}^{n} \int_{\Gamma_2} \langle e(x,y), \varphi_m(y)\rangle ds(y), \quad x \in D, \tag{2.13}$$

$$(W_n\varphi)(x) = \sum_{m=0}^{n} \int_{\Gamma_1} [T_y(E_{n-m}, e)(x,y)\nu_1(y)]^\top \varphi_m(y)ds(y), \quad x \in D \tag{2.14}$$

and

$$(w_n\varphi)(x) = \sum_{m=0}^{n} \int_{\Gamma_1} \langle [T_y(e,0)(x,y)\nu_1(y)]^\top, \varphi_m(y)\rangle ds(y), \quad x \in D, \tag{2.15}$$

respectively. Here $\varphi = (\varphi_0, \ldots, \varphi_n)$ are unknown densities.

**Theorem 2.4** *The following combination of single- and double-layer potentials*

$$u_n(x) = (W_n\varphi^{(1)})(x) + (V_n\varphi^{(2)})(x), \quad x \in D, \tag{2.16}$$

$$p_n(x) = (w_n\varphi^{(1)})(x) + (v_n\varphi^{(2)})(x), \quad x \in D, \tag{2.17}$$

*is the solution of* (2.3)-(2.6) *if the densities* $\varphi^{(1)}$ *and* $\varphi^{(2)}$ *satisfy the sequence of systems of integral equations*

$$\frac{1}{2}\varphi_n^{(\ell)}(x) + \sum_{k=1}^{2} \int_{\Gamma_k} E_0^{\ell k}(x,y)\varphi_n^{(k)}(y)ds(y) = F_n^\ell(x), \quad x \in \Gamma_\ell, \quad \ell = 1,2 \tag{2.18}$$

*with the righ-hand sides*

$$F_n^\ell(x) = f_{\ell,n}(x) - \frac{1}{2}\sum_{m=0}^{n-1} \varphi_m^{(\ell)}(x) + \sum_{m=0}^{n-1}\sum_{k=1}^{2} \int_{\Gamma_k} E_{n-m}^{\ell k}(x,y)\varphi_m^{(k)}(y)ds(y), \; x \in \Gamma_\ell$$

*for* $\ell = 1,2$. *Here, the following matrices were introduced*

$$E_n^{11}(x,y) = [T_y(E_n, e)(x,y)\nu_1(y)]^\top, \quad E_n^{12}(x,y) = E_n(x,y)$$

*and*

$$E_n^{21}(x,y) = T_x[T_y(E_n, e)(x,y)\nu_1(y)]^\top \nu_2(x), \quad E_n^{22}(x,y) = T_x(E_n, e)(x,y)\nu_1(x).$$

*Proof.* From the Definition 2.2 of a fundamental solution we deduce that the representations (2.16) and (2.17) with $\varphi^{(\ell)} \in C(\Gamma_\ell)$, $\ell = 1,2$, satisfy the equations (2.3) and (2.4). It follows from Theorem 2.3 that the potentials have jump relations on the corresponding boundaries analogously to the potentials for the resolvent Stokes equation (see [15]). Then from the boundary value conditions (2.5) and (2.6) we conclude that the densities solve the integral equations (2.18). □

**Theorem 2.5** *For any two sequences* $\{f_{\ell,n}\}_{n=0}^{\infty}$ *with* $f_{\ell,n}$ *in* $L^2(\Gamma_\ell)$, $\ell = 1, 2$, *the sequence of systems of integral equations* (2.18) *possesses a unique solution* $\varphi_n^{(\ell)} \in L^2(\Gamma_\ell)$, $\ell = 1, 2$.

*Proof.* It is straightforward to check that the kernels $E_n^{\ell\ell}(x, y)$, $\ell = 1, 2$, have a logarithmic singularity for $x = y$, and that the kernels $E_n^{k\ell}(x, y)$ are continuous for $k \neq \ell$. Therefore, the corresponding integral operators in (2.18) are compact as mappings from $L^2(\Gamma_\ell)$ to $L^2(\Gamma_k)$. Then the statement of the theorem follows from the uniqueness Theorem 2.1 and the Riesz theory (see for example [10]). □

## 3. A Nyström method

Assume that the boundary $\Gamma_\ell$ have the following parametric representation

$$\Gamma_\ell = \{x_\ell(s) = (x_{\ell,1}(s), x_{\ell,2}(s)) : \quad s \in [0, 2\pi]\}, \quad \ell = 1, 2,$$

where the functions $x_\ell \in C^2$ are $2\pi$-periodic with $|x'_\ell(s)| > 0$ for all $s$, such that the orientation of $\Gamma_\ell$ is counter-clockwise.

Then, after some additional transformations, we can rewrite the system (2.18) in the parametric form

$$\frac{1}{2}\varphi_n^{(\ell)}(s) + \frac{1}{2\pi}\sum_{k=1}^{2}\int_0^{2\pi} E_0^{\ell k}(s, \sigma)\varphi_n^{(k)}(\sigma)d\sigma = F_n^{\ell}(s), \quad s \in [0, 2\pi], \quad \ell = 1, 2 \tag{3.1}$$

with the corresponding parametric right-hand sides

$$F_n^{\ell}(s) = f_{\ell,n}(x_\ell(s)) - \frac{1}{2}\sum_{m=0}^{n-1}\varphi_m^{(\ell)}(s) + \frac{1}{2\pi}\sum_{m=0}^{n-1}\sum_{k=1}^{2}\int_0^{2\pi} E_{n-m}^{\ell k}(s, \sigma)\varphi_m^{(k)}(\sigma)d\sigma$$

for $s \in [0, 2\pi]$, where $\varphi_n^{(\ell)}(s) := \varphi_n(x_\ell(s))$ and the kernels have the form

$$E_n^{\ell k}(s, \sigma) := c_\pi^{-1} E_n^{\ell k}(x_\ell(s), x_k(\sigma))|x'_k(\sigma)|$$

for $n = 0, 1, \ldots$ . As mentioned above, the diagonal kernels in the system (3.1) have a logarithmic singularity, which we represent in the form:

$$E_n^{\ell\ell}(s, \sigma) = \ln\left(4\sin^2\frac{s-\sigma}{2}\right) E_{1n}^{\ell\ell}(s, \sigma) + E_{2n}^{\ell\ell}(s, \sigma),$$

where the smoothness of the new kernels $E_{kn}^{\ell\ell}$ depends on the smoothness of the boundaries $\Gamma_\ell$.

For the numerical solution of the obtained systems of integral equations we use a Nyström method based on trigonometrical quadratures [10]. We choose $M \in \mathbb{N}$ and an equidistant mesh by setting $s_k := \frac{k\pi}{M}$, $k = 0, \ldots, 2M-1$, and use the following quadrature rules

$$\frac{1}{2\pi}\int_0^{2\pi} f(\sigma)d\sigma \approx \frac{1}{2M}\sum_{i=0}^{2M-1} f(s_i) \tag{3.2}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} f(\sigma)\ln\left(4\sin^2\frac{s_j-\sigma}{2}\right)d\sigma \approx \sum_{k=0}^{2M-1} R_{|k-j|}f(s_k), \tag{3.3}$$

with known weights $R_k$ (see [10]). Using these quadratures in (3.1) and after collocating at the points $s_i$ we receive the following sequence of linear equations with the same matrix and recurrent right-hand sides

$$\frac{1}{2}\tilde{\varphi}_n^{(\ell)}(s_i) + \sum_{k=0}^{2M-1}\Big[(R_{|i-k|}E_{10}^{\ell\ell}(s_i,s_k) + \frac{1}{2M}E_{10}^{\ell\ell}(s_i,s_k))\tilde{\varphi}_n^{(\ell)}(s_k) + \\ + \frac{1}{2M}E_0^{\ell,3-\ell}(s_i,s_k)\tilde{\varphi}_n^{(3-\ell)}(s_k)\Big] = \tilde{F}_n^{\ell}(s_i), \quad i=0,\dots,2M-1 \tag{3.4}$$

for $\ell = 1,2$ and $n = 0,1,\dots,N$. Of course, the approximate values $\tilde{F}_n^{\ell}(s_i)$ for the right-hand sides are also obtained with the help of (3.2) and (3.3). The general convergence analysis and error estimate for the Nyström method in [10] in Sobolev space settings leads to the following result.

**Theorem 3.1** *For every $n = 0,1,\dots$ and sufficient large $M \in \mathbb{N}$ there exists a unique solution of the linear system (3.4) and the following error estimates hold*

$$\|\tilde{\varphi}_n^{(\ell)} - \varphi_n^{(\ell)}\|_q \le C_n M^{q-p}\|\varphi_n^{(\ell)}\|_p, \quad \ell = 1,2,$$

*with $0 < C_n$, $1 \le q \le p$ and $\frac{3}{2} < p$.*

Now the solutions of the stationary problems (2.3)-(2.6) for $n = 0,1,\dots,N$, can be approximated in the following way:

$$\tilde{u}_n(x) = \frac{1}{2M}\sum_{k=0}^{2M-1}\sum_{m=0}^{n}\sum_{\ell=1}^{2}E_{n-m}^{1\ell}(x,x_\ell(s_k))\tilde{\varphi}_m^{(\ell)}(s_k)|x_\ell'(s_k)|, \quad x \in D$$

and

$$\tilde{p}_n(x) = \frac{1}{2M}\sum_{k=0}^{2M-1}\sum_{m=0}^{n} \quad (\langle [T_y(e,0)(x,x_1(s_k))\nu(x_1(s_k))]^\top, \tilde{\varphi}_m^{(1)}(s_k)\rangle|x_1'(s_k)| + \\ + \langle e(x,x_2(s_k)), \tilde{\varphi}_m^{(2)}(s_k)\rangle|x_2'(s_k)|), \quad x \in D$$

and, according to (2.1) and (2.2), the numerical solution of the time-dependent problem (1.1)-(1.5) has the form

$$u_M^N(x,t) = \kappa^2\sum_{n=0}^{N}\tilde{u}_n(x)L_n(\kappa^2 t)$$

and

$$p_M^N(x,t) = \kappa^2\sum_{n=0}^{N}\tilde{p}_n(x)L_n(\kappa^2 t).$$

## 4. Numerical results

For the numerical investigations of (1.1)-(1.5) and (2.3) – (2.6) we choose boundaries $\Gamma_1$ and $\Gamma_2$ with parametric representations (see Fig. 2)

$$\Gamma_1 = \{x_1(s) = (\cos(s),\sin(s)), \quad 0 \le s \le 2\pi\},$$

$$\Gamma_2 = \{x_2(s) = (0.4\cos(s) + 0.3\cos(s)^2, 0.2\sin(s)), \quad 0 \le s \le 2\pi\}.$$

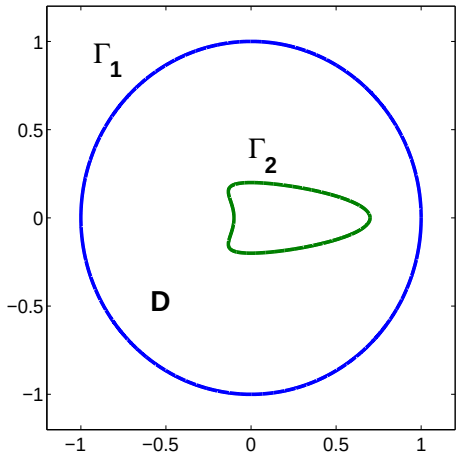


Fig. 2. The domain $D$ with boundaries $\Gamma_1$ and $\Gamma_2$

Tabl. 4.1. Numerical results for the stationary problems (2.3)-(2.6)

| $N$ | $M$ | $\varepsilon_n^1(0.6, 0.2)$ | $\varepsilon_n^2(0.6, 0.2)$ | $\varepsilon_n^p(0.6, 0.2)$ |
|---|---|---|---|---|
| 0 | 16 | 0.000001180867 | 0.000000423457 | 0.000002997811 |
| | 32 | 0.000000000016 | 0.000000000004 | 0.000000000040 |
| 5 | 16 | 0.000001814180 | 0.000001205075 | 0.000005748461 |
| | 32 | 0.000000000018 | 0.000000000028 | 0.000000000087 |
| 10 | 16 | 0.000010341692 | 0.000009842761 | 0.000038041917 |
| | 32 | 0.000000002483 | 0.000000002491 | 0.000000009571 |
| 15 | 16 | 0.000040324993 | 0.000008203909 | 0.000092391487 |
| | 32 | 0.000000000302 | 0.000000002014 | 0.000000002439 |

Tabl. 4.2. Numerical results for $u_M^{N,1}(x, t)$

| $t$ | $M$ | $N = 10$ | $N = 15$ | $N = 20$ |
|---|---|---|---|---|
| 0.0 | 16 | −0.000018743224 | 0.000165411802 | 0.000067816413 |
| | 32 | −0.000018743726 | 0.000165412338 | 0.000067816508 |
| 0.4 | 16 | 0.005678369917 | 0.005593378763 | 0.005598669639 |
| | 32 | 0.005678366292 | 0.005593374726 | 0.005598665603 |
| 0.8 | 16 | 0.017678245104 | 0.017723786862 | 0.017688866743 |
| | 32 | 0.017678243559 | 0.017723785651 | 0.017688865394 |
| 1.2 | 16 | 0.029717050644 | 0.029818467058 | 0.029845344526 |
| | 32 | 0.029717051073 | 0.029818467909 | 0.029845345531 |
| 1.6 | 16 | 0.038784159571 | 0.038800719652 | 0.038842962646 |
| | 32 | 0.038784159947 | 0.038800719901 | 0.038842963051 |
| 2.0 | 16 | 0.043924904863 | 0.043814627782 | 0.043803161052 |
| | 32 | 0.043924903127 | 0.043814625421 | 0.043803158567 |

Tabl. 4.3. Numerical results for $u_M^{N,2}(x,t)$

| $t$ | $M$ | $N=10$ | $N=15$ | $N=20$ |
|---|---|---|---|---|
| 0.0 | 16 | 0.000054526038 | 0.000157807726 | 0.000054253741 |
| | 32 | 0.000054526617 | 0.000157807141 | 0.000054251694 |
| 0.4 | 16 | 0.005297759199 | 0.005243517133 | 0.005249668651 |
| | 32 | 0.005297755441 | 0.005243513789 | 0.005249665335 |
| 0.8 | 16 | 0.016785482373 | 0.016803762350 | 0.016766268306 |
| | 32 | 0.016785471841 | 0.016803751399 | 0.016766256879 |
| 1.2 | 16 | 0.028500415549 | 0.028571805673 | 0.028599560434 |
| | 32 | 0.028500399911 | 0.028571789665 | 0.028599544897 |
| 1.6 | 16 | 0.037490483969 | 0.037520729243 | 0.037566301312 |
| | 32 | 0.037490466302 | 0.037520711851 | 0.037566284473 |
| 2.0 | 16 | 0.042759670018 | 0.042697491723 | 0.042686969330 |
| | 32 | 0.042759653276 | 0.042697475657 | 0.042686952940 |

Tabl. 4.4. Numerical results for $p_M^N(x,t)$

| $t$ | $M$ | $N=10$ | $N=15$ | $N=20$ |
|---|---|---|---|---|
| 0.0 | 16 | −0.005602252652 | −0.002945981631 | −0.001649195637 |
| | 32 | −0.005602254212 | −0.002945990717 | −0.001649210375 |
| 0.4 | 16 | 0.102448026181 | 0.101521606539 | 0.101444012597 |
| | 32 | 0.102448062583 | 0.101521645649 | 0.101444051891 |
| 0.8 | 16 | 0.141287330419 | 0.142271049047 | 0.142740529876 |
| | 32 | 0.141287355059 | 0.142271071006 | 0.142740549924 |
| 1.2 | 16 | 0.139242997263 | 0.140044304467 | 0.139698260091 |
| | 32 | 0.139242996885 | 0.140044301634 | 0.139698258978 |
| 1.6 | 16 | 0.116126075732 | 0.115418162382 | 0.114847970084 |
| | 32 | 0.116126054301 | 0.115418142662 | 0.114847952618 |
| 2.0 | 16 | 0.085094775513 | 0.083530115448 | 0.083659465225 |
| | 32 | 0.085094742647 | 0.083530086983 | 0.083659435730 |

We investigate the stationary and non-stationary problem separately.

**1**. We shall investigate the numerical solution of the stationary problems $(2.3)-(2.6)$ with $c_r = 1$. Choosing boundary functions

$$f_{1,n}(x) = E_{n,1}(x, x^*), \quad x \in \Gamma_1, \quad x^* = (0.9, 0.9)$$

and

$$f_{2,n}(x) = T_x(E_{n,1}(x, x^*), e_1(x, x^*))(x)\nu_2(x),$$

where $E_{n,1}$ is the first column of the fundamental matrix $E_n$ and $e_1$ the first component of fundamental vector $e$, then the solution of $(2.3)-(2.6)$ are $u_n(x) = E_{n,1}(x, x^*)$ and

$p_n(\cdot) = e_n(x, x^*)$ for $x \in D$ and $n = 0, 1, \ldots, N$. Table 4.1 contains the error estimates

$$\varepsilon_n^1(x) := |\widetilde{u}_{n,M}^1(x) - u_n^1(x)|,$$

$$\varepsilon_n^2(x) := |\widetilde{u}_{n,M}^2(x) - u_n^2(x)|$$

and

$$\varepsilon_n^p(x) := |\widetilde{p}_{n,M}(x) - p_n(x)|,$$

where $M$ is the mesh variable and $u_n$ and $p_n$ are the exact solutions of (2.3) – (2.6) with $k^2 = 1$.

**2**. We then consider the time-dependent case and choose the time interval as $[0; 2]$ and observation point $x = (0.6, 0.2)$. The functions $f_1$ and $f_2$ are chosen as:

$$f_1(x,t) = \left(\frac{t^2}{8}e^{-t}, \frac{t^2}{8}e^{-t}\right)^\top, \quad x \in \Gamma_1$$

and

$$f_2(x,t) = 0, \quad x \in \Gamma_2.$$

Note that the function $f_1$ has an analytical expansion in terms of Fourier-Laguerre series, thus the correctness of the numerical presentation of the Laguerre's transformation can be checked.

In the Tables 4.2, 4.3 and 4.4, the numerical solution $(u_M^{N,1}, u_M^{N,2})$, $p_M^N$ of the non-stationary problem (1.1)-(1.5) are presented for different time points at the above observation point $x$.

The obtained results show that we have exponential convergence of the proposed method with respect to the spatial coordinates.

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