Журнал обчисл. прикл. матем.
Вип. 1 (100), 2010, стор. 60-81

J. Numer. Appl. Math.
No. 1 (100), 2010, pp. 60-81

SUPER-EXPONENTIALLY CONVERGENT PARALLEL ALGORITHM FOR EIGENVALUE PROBLEMS FOR THE FOURTH ORDER ODE'S

UDC 519.6

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Анотація. Пропонується новий алгоритм для задачі на власні значення для звичайного диференціального рівняння четвертого порядку. Алгоритм базується на кусково-сталій апроксимації коефіцієнтів диференціального рівняння з наступною рекурентною процедурою, яка базується на ідеї гомотопії. Алгоритм має супер-експоненційну швидкість збіжності, тобто збігається швидше геометричної прогресії, знаменник якої обернено пропорційний порядковому номеру відповідного власного значення. Власні пари можуть бути обчислені паралельно для всіх заданих індексів. Наведені чисельні приклади підтверджують теорію.

Abstract. A new algorithm for eigenvalue problems for the fourth order ordinary differential equations (ODE) is proposed. The algorithm is based on piecewise approximation of the coefficients of differential equation with subsequent recursive procedure adapted from some homotopy considerations. The approach provides an super-exponential convergence rate, i.e the rate of a geometrical progression with a base which is inversely proportional to the index of the eigenvalue under consideration. The eigenpairs can be computed in parallel for all given indexes. Numerical examples are presented to support the theory.

1. Introduction

The eigenvalue problem, i.e. the problem of finding of eigenpairs (eigenvalues (frequencies) and eigenfunctions (vibration shapes)), play an important role in various applications concerned with vibrations and wave processes [1,19]. Such popular methods as the finite-difference method (FD), FEM or variational methods allow one to compute efficiently some lower eigenvalues only. At the same time there are applied problems requiring the computation of a great number (hundreds of thousands) of eigenvalues and eigenfunctions (see e.g. [19, p. 273]).

In the present paper we propose a new parallel super-exponentially convergent algorithm for the numerical solution of the following class of eigenvalue problems:

$$\begin{aligned}
&y^{(4)}(\xi)+g_3(\xi)y^{(3)}(\xi)+g_2(\xi)y''(\xi)+g_1(\xi)y'(\xi)+g_0(\xi)y(\xi)-g(\xi)\lambda y(\xi)=0,\\
&y^{(p)}(0)=y^{(q)}(0)=y^{(r)}(0)=y^{(s)}(0)=0,\\
&0\le p<q\le 3,\qquad 0\le r<s\le 3.
\end{aligned}\tag{1.1}$$

The type of boundary conditions is defined by the four natural numbers $(p,q;r,s)$, $p,q,r,s\in\{0,1,2,3\}$. One of the oldest and probably mostly famous applications of this mathematical model is the description of free and forced vibrations of a Bernoulli-Euler beam [14, 25] (there are also good reasons to call this theory as "The Da Vinci-Euler-Bernoulli Beam Theory" [2]). Euler-Bernoulli beam theory emerged in the middle of the 18th century as a simplification from the linear isotropic theory of elasticity. Due to its simplicity and at the same time to its adequate accuracy (demonstrated by many practical

Key words. Fourth order ordinary differential equation, eigenvalue problem, parallel algorithm, super-exponentially convergent algorithm.

applications, amongst others during assembly of the Eiffel Tower and the Ferris Wheel in the late 19th century) the beam theory became an important tool in the sciences, especially structural and mechanical engineering.

Equation (1.1) as well as the equation in the self-adjoint form

$$\frac{d^2}{d\xi^2}(a(\xi)\frac{d^2}{d\xi^2}y(\xi)) - \frac{d}{d\xi}(b(\xi)\frac{d}{d\xi}y(\xi)) + (c(\xi) - \lambda d(\xi))y(\xi) = 0, \tag{1.2}$$

can be reduced to the form

$$u^{(4)}(x) + k_2(x)u''(x) + k_1(x)u'(x) + k_0(x)u(x) - \lambda u(x) = 0, \tag{1.3}$$

i.e. we can make the coefficient in the front of the third derivative equal to zero and the coefficient at the front of $\lambda u(x)$ equal to one. It is possible since due to the variable transform $\xi = \varphi(x), y(\xi) = v(x)u(x)$ (compare with the Liouville transform [8] for the second order differential equation) we have two free functions. This transform carries (1.1) over to (1.3) with

$$\varphi'(x) = g^{-\frac{1}{4}}(\xi), \quad v(x) = [\varphi']^{\frac{3}{2}}(x)e^{-\frac{1}{4}\int\limits_{\xi_0}^{\xi} g_3(\varphi)d\varphi}, \quad x = \int\limits_{\xi_0}^{\xi} g^{\frac{1}{4}}(\varphi)d\varphi,$$

$$k_2(x) = \Big[\frac{45}{32}g^{-\frac{5}{2}}(\xi)\left[g'\right]^2(\xi) - \frac{5}{4}g^{-\frac{3}{2}}(\xi)g''(\xi) - \frac{3}{8}(\xi)g^{-\frac{1}{2}}(\xi)g_3^2(\xi) -$$
$$- \frac{3}{2}g^{-\frac{1}{2}}(\xi)g_3'(\xi) + g^{-\frac{1}{2}}(\xi)g_2(\xi)\Big]_{\xi=\varphi(x)},$$

$$k_1(x) = \Big[-\frac{225}{64}g^{-\frac{15}{4}}(\xi)\left[g'\right]^3(\xi) + \frac{75}{16}g^{-\frac{11}{4}}(\xi)g'(\xi)g''(\xi) - \frac{5}{4}g^{-\frac{7}{4}}(\xi)g'''(\xi) +$$
$$+ \frac{3}{16}g^{-\frac{7}{4}}(\xi)g'(\xi)g_3^2(\xi) + \frac{3}{4}g^{-\frac{7}{4}}(\xi)g'(\xi)g_3'(\xi) + \frac{1}{8}g^{-\frac{3}{4}}(\xi)g_3^3(\xi) - g^{-\frac{3}{4}}(\xi)g_3''(\xi) -$$
$$- \frac{1}{2}g^{-\frac{7}{4}}(\xi)g'(\xi)g_2(\xi) - \frac{1}{2}g^{-\frac{3}{4}}(\xi)g_3(\xi)g_2(\xi) + g^{-\frac{3}{4}}(\xi)g_1(\xi)\Big]_{\xi=\varphi(x)},$$

$$k_0(x) = \Big[\frac{16929}{4096}g^{-5}(\xi)\left[g'\right]^4(\xi) - \frac{1881}{256}g^{-4}(\xi)\left[g'\right]^2(\xi)g''(\xi) + \frac{33}{16}g^{-3}(\xi)g'(\xi)g'''(\xi) +$$
$$+ \frac{99}{64}g^{-3}(\xi)\left[g''\right]^2(\xi) - \frac{3}{8}g^{-2}(\xi)g^{(4)}(\xi) - \frac{99}{512}g^{-3}(\xi)\left[g'\right]^2(\xi)g_3^2(\xi) -$$
$$- \frac{99}{128}g^{-3}(\xi)\left[g'\right]^2(\xi)g_3'(\xi) - \frac{3}{64}g^{-2}(\xi)g'(\xi)\left[g_3\right]^3(\xi) + \frac{3}{8}g^{-2}(\xi)g'(\xi)g''{}_3(\xi) +$$
$$+ \frac{9}{64}g^{-2}(\xi)g''(\xi)g_3^2(\xi) + \frac{9}{16}g^{-2}(\xi)g''(\xi)g'{}_3(\xi) - \frac{3}{256}g^{-1}(\xi)g_3^4(\xi) +$$
$$+ \frac{3}{32}g^{-1}(\xi)g_3^2(\xi)g'{}_3(\xi) + \frac{3}{16}g^{-1}(\xi)\left[g_3'\right]^2(\xi) - \frac{1}{4}g^{-1}(\xi)g'''{}_3(\xi) +$$
$$+ \frac{33}{64}g^{-3}(\xi)\left[g'\right]^2(\xi)g_2(\xi) - \frac{3}{8}g^{-2}(\xi)g''(\xi)g_2(\xi) + \frac{3}{16}g^{-2}(\xi)g'(\xi)g_3(\xi)g_2(\xi) +$$
$$+ \frac{1}{16}g^{-1}(\xi)g_3^2(\xi)g_2(\xi) - \frac{1}{4}g^{-1}(\xi)g_3'(\xi)g_2(\xi) - \frac{3}{8}g^{-2}(\xi)g'(\xi)g_1(\xi) -$$
$$- \frac{1}{4}g^{-1}(\xi)g_3(\xi)g_1(\xi) + g^{-1}(\xi)g_0(\xi)\Big]_{\xi=\varphi(x)}.$$

Mechanics often use the following method for solving the equations (1.1)-(1.2) with the corresponding boundary conditions: the interval $(0, 1)$ is covered by a grid $\omega = \{t_i :$

$i = 1, ..., N-1,\ 0 = t_0 < t_1 < ... < t_{N-1} < t_N = 1\}$ with a maximal step $h = \max_{i=1,...,N}(t_i - t_{i-1})$, the variable coefficients on each subinterval are replaced by constants (for example, by some fixed values of the corresponding variable coefficients) and the solution of such problem is accepted as an approximate solution of (1.1) or (1.2). The basic idea here is the approximation of the differential equation(i.e. its coefficients). The corresponding methods for the second-order Sturm-Liouville problems are known as the Pruess methods [19] because S.Pruess in 1973 provided a rigorous convergence and error analysis [20]. But in fact variants of this method were in use since the beginning of the past century and for the piecewise coefficient approximation the method was theoretically justified for linear second order ordinary differential equations (as the so called method of "tronsons") by N.N. Bogolyubov and A.N. Krylov in 1927 [17].

The higher order eigenvalue problems were treated e.g. in [1, 6, 7, 27]. Theoretical and numerical methods of solving eigenvalue problems for higher-order equations (especially of those whose coefficients have considerable variation) have not been developed to a desirable extent or are altogether absent. Some constructive approaches to the solution of self-conjugate fourth-order eigenvalue problems with different types of boundary conditions have been suggested in [1].

Let us estimate the accuracy of the method of "tronsons"for the following test eigenvalue problem

$$\begin{aligned} &\frac{d^2}{dt^2}(a(t)\frac{d^2}{dt^2}v(t)) - \lambda v(t) = 0, \\ &v(0) = v(1) = \frac{d^2}{dt^2}v(0) = \frac{d^2}{dt^2}v(1) = 0 \end{aligned} \tag{1.4}$$

with $0 < \kappa \le a(t) \le K < \infty$. Following the described method we consider instead of (1.4) the problem

$$\begin{aligned} &\frac{d^2}{dt^2}(\overline{a}(t)\frac{d^2}{dt^2}v^{(0)}(t)) - \lambda^{(0)}v^{(0)}(t) = 0, \\ &v^{(0)}(0) = v^{(0)}(1) = \frac{d^2}{dt^2}v^{(0)}(0) = \frac{d^2}{dt^2}v^{(0)}(1) = 0 \end{aligned} \tag{1.5}$$

with $\bar{a}(t) = \min_{t\in[t_{i-1},t_i]} a(t)$, $i = 1, ..., N$. At the discontinuity points of the coefficient $\overline{a}(t)$ we require that the following consistency conditions hold:

$$[v(t)]_{t=t_i} = [\frac{d}{dt}v(t)]_{t=t_i} = [\bar{a}(t)\frac{d^2}{dt^2}v(t)]_{t=t_i} = [\frac{d}{dt}\ (\ \bar{a}(t)\frac{d^2}{dt^2}v(t)\ )]_{t=t_i} = 0,$$

where $[w(t)]_{t=t_i} = w(t_i + 0) - w(t_i - 0)$ is the jump of $w(t)$ at the point $t = t_i$. It can be shown (see e.g. [8]) that the spectra of the both problems (1.4)-(1.5) are discrete and the eigenvalues can be ordered so that

$$\lambda_1 \le \lambda_2 \le ... \le \lambda_n \le ... \qquad \lambda_1^{(0)} \le \lambda_2^{(0)} \le ... \le \lambda_n^{(0)} \le ...$$

We are interesting in the error due to replacement of (1.4) by (1.5) (see e.g. [26] for the algorithm to its solution). With this aim let us consider the following auxiliary problem

$$\begin{aligned} &\frac{\partial^2}{\partial t^2}\left(\bar{a}(t,s)\frac{\partial^2}{\partial t^2}v(t,s)\right) - \lambda(s)v(t,s) = 0, \\ &v(0,s) = v(1,s) = \frac{\partial^2}{\partial t^2}v(0,s) = \frac{\partial^2}{\partial t^2}v(1,s) = 0,\ \ s \in [0,1], \end{aligned} \tag{1.6}$$

where $\bar{a}(t,s) = \bar{a}(t) + s(a(t) - \bar{a}(t))$. We have obviously that

$$v(t,1) = v(t),\ \lambda(1) = \lambda,\ v(t,0) = v^{(0)}(t),\ \lambda(0) = \lambda^{(0)}$$

and we normalize the solution by

$$\int_0^1 v^2(t,s)dt = 1, \quad \forall s \in [0,1].$$

Since the parameter s is analytically included in (1.6) we can differentiate these equations with respect to s:

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2}\left(\bar{a}(t,s)\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial s}v(t,s)\right) - \lambda(s)\frac{\partial}{\partial s}v(t,s) = \\
&= -\frac{\partial^2}{\partial t^2}\left((a(t) - \bar{a}(t))\frac{\partial^2}{\partial t^2}v(t,s)\right) + \frac{d}{ds}\lambda(s)v(t,s), \\
&\frac{\partial}{\partial s}v(0,s) = \frac{\partial}{\partial s}v(1,s) = \frac{\partial^2}{\partial t^2}\frac{\partial}{\partial s}v(0,s) = \frac{\partial^2}{\partial t^2}\frac{\partial}{\partial s}v(1,s) = 0,\ s \in [0,1]
\end{aligned}$$

The solvability condition with taking into account the normalization condition and the integration by parts lead to the formula

$$\frac{d\lambda(s)}{ds} = \int_0^1 (a(t) - \bar{a}(t))\Big(\frac{\partial^2}{\partial t^2}v(t,s)\Big)^2 dt.$$

On the other hand, the equalities (1.6) imply

$$\lambda(s) = \int_0^1 a(t,s)\left(\frac{\partial^2}{\partial t^2}v(t,s)\right)^2 ds.$$

These two relations yield

$$0 < \frac{d\lambda(s)}{ds} \le \max_{t\in[0,1]}|a(t) - \bar{a}(t)| \int_0^1 \Big(\frac{\partial^2}{\partial t^2}v(t,s)\Big)^2 dt \le \max_{t\in[0,1]}|a(t) - \bar{a}(t)|\frac{\lambda(s)}{\kappa}$$

which together with

$$\lambda(s) = \lambda^{(0)} + \int_0^s \frac{d}{d\eta}\lambda(\eta)d\eta$$

leads to the following estimate

$$0 \le \lambda_n - \lambda_n^{(0)} \le \max_{t\in[0,1]}(a(t) - \bar{a}(t))\frac{\lambda_n}{\kappa} \tag{1.7}$$

Assuming that $a(t) \in C^{(2)}[0,1]$ due to the estimate [2]

$$\lambda_n = \mathcal{O}(n^4)$$

we obtain from (1.7)

$$0 \le \lambda_n - \lambda_n^{(0)} = \mathcal{O}(h\,n^4), \qquad h = \max_{i=\overline{1,N}}(t_i - t_{i-1})$$

(compare with the estimate $|\lambda_n - \lambda_n^{(0)}| \le C\max\{1, n^2\}h$ for the second order Sturm-Liouville problems [19, p. 119]). This estimate shows that the method under consideration is appropriate for not very large n, in fact, for some lowest eigenvalues only.

In order to find numerically the higher eigenvalues we propose an other approach described below which we will (following [11, 12]) refer to as the FD-method. This approach is based on the perturbation and homotopy ideas for which we preliminary transform equation (1.2) to form (1.3). Let us briefly explain the ideas of perturbation and homotopy for the eigenvalue problem

$$(A + B)u - \lambda u = \theta, \tag{1.8}$$

in a Hilbert space H with the null-element θ under the assumption that the spectrum of the operator is discrete and we are looking for an eigenpair with a fixed index n. Let $\overline{B}$ be an approximating operator for B in the sense that $\|B - \overline{B}\|$ is bounded by a constant K (or, far better, we can control its value and can make it "small enough") and the eigenvalue problem

$$(A + \overline{B})u_n^{(0)} - \lambda_n^{(0)} u_n^{(0)} = \theta \tag{1.9}$$

is "simpler" then problem (1.8). Following to the homotopy idea we consider the following parametric family of problems

$$(A + W(t))u_n(t) - \lambda_n(t)u_n(t) = \theta, \quad t \in [0, 1] \tag{1.10}$$

with $W(t) = \overline{B} + t(B - \overline{B})$ which contains the both problems (1.8) and (1.9), so that we obviously have

$$u_n(0) = u_n^{(0)}, \quad u_n(1) = u_n.$$

This suggests the idea to look for the solution of (1.10) in the form

$$\begin{aligned} \lambda_n(t) &= \sum_{j=0}^{\infty} \lambda_n^{(j)} t^j, \\ u_n(t) &= \sum_{j=0}^{\infty} u_n^{(j)} t^j, \end{aligned} \tag{1.11}$$

where formally

$$\begin{aligned} \lambda_n^{(j)} &= \frac{1}{j!}\frac{d^j \lambda_n(t)}{dt^j}\,|_{t=0}\ , \\ u_n^{(j)} &= \frac{1}{j!}\frac{d^j u_n(t)}{dt^j}\,|_{t=0}\ . \end{aligned}$$

Setting $t = 1$ in (1.11) we obtain

$$\lambda_n = \sum_{j=0}^{\infty} \lambda_n^{(j)}, \quad u_n = \sum_{j=0}^{\infty} u_n^{(j)}$$

provided that series (1.11) converge for all $t \in [0, 1]$. The truncated sums

$$\overset{N}{\lambda}_n = \sum_{j=0}^{N} \lambda_n^{(j)}, \quad \overset{N}{u} = \sum_{j=0}^{N} u_n^{(j)}$$

are approximations (of rank N) to the exact eigenvalue and eigenfunction of problem (1.8).

In order to find the coefficients we substitute (1.11) into (1.10) and by matching the coefficients in front of the same powers of t we arrive at the following recurrence equations

$$(A + \overline{B})u_n^{(j+1)} - \lambda_n^{(0)} u_n^{(j+1)} = F_n^{(j+1)}, \ j = 0, 1, ... \tag{1.12}$$

with $F_n^{(0)} = 0$ and

$$F_n^{(j+1)} = F_n^{(j+1)}(\lambda_n^{(0)}, ..., \lambda_n^{(j+1)}; u_n^{(0)}, ..., u_n^{(j)}) =$$
$$= -[B - \overline{B}]u_n^{(j)} + \sum_{p=0}^{j} \lambda_n^{(j+1-p)} u_n^{(p)} =$$
$$= \lambda_n^{(j+1)} u_n^{(0)} - [B - \overline{B}]u_n^{(j)} + \sum_{p=1}^{j} \lambda_n^{(j+1-p)} u_n^{(p)},\ j = 0, 1, ... \tag{1.13}$$

For the pair $\lambda_n^{(0)},\ u_n^{(0)}$ we get the so called base problem

$$(A + \overline{B})u_n^{(0)} - \lambda_n^{(0)} u_n^{(0)} = \theta,$$

which is assumed to be "simpler" as the original one and produces the initial data for problem (1.12)-(1.13). For the sake of simplicity we assume that all the eigenvalues $\lambda_n^{(0)}$ of the base problem are simple and correspond to orthonormal eigenvectors $u_n^{(0)}$ (i.e. $(u_n^{(0)}, u_n^{(0)}) = 1, (u_n^{(0)}, u_m^{(0)}) = 0, m \neq n$).

Each of problems (1.12) is solvable provided that

$$\left(F_n^{(j+1)}, u_n^{(0)}\right) = 0, \tag{1.14}$$

and for the sake of uniqueness we choose the solution subject to the orthogonality condition

$$\left(u_n^{(j+1)}, u_n^{(0)}\right) = 0. \tag{1.15}$$

From (1.14) with having regard to (1.15) we obtain

$$\lambda_n^{(j+1)} = \left(\left[B - \overline{B}\right] u_n^{(j)}, u_n^{(0)}\right),\ j = 0, 1, ...$$

Due to (1.15) for the solution of (1.12) we have

$$u_n^{(j+1)} = \hat{u}_n^{(j+1)} = -\sum_{\substack{p=1 \\ p\neq n}}^{\infty} \frac{\left(F_n^{(j+1)}, u_n^{(0)}\right)}{\lambda_n^{(0)} - \lambda_p^{(0)}} u_p^{(0)}.$$

In the present paper we consider a variant of the algorithm when $Au = u^{(4)}(x)$ subject to the corresponding boundary conditions and $\overline{B} \equiv 0$, i.e. the coefficients in the front of the lowest derivatives in (1.3) are approximated by the constant zero. We shall show that we obtain all eigenpairs $\lambda_n, u_n(x)$ with an exponential accuracy for all $n \geq n_0$ beginning with some n_0. Approximating the lowest coefficients by the piece-wise constant functions in the base problem with the following iterative procedure based on a homotopy idea one can obtain the eigenpairs with the arbitrary given indexes. This algorithm will be described and justified in a fortcoming paper. Our approach possesses the following advantages some of which are similar to that of Pruess method for the second-order ODEs or of methods from [1]:

1. It produces an infinite spectrum, unlike matrix methods such as the finite difference (FDM), finite elements (FEM) or variational methods (VM).

2. Better convergence results and error estimates (namely the super-exponential ones) as compared with the FDM, FEM or variational methods.

3. It has advantages over shooting methods based on IVP-solvers, since the last ones have the stiffness problem when $k_0 - \lambda$ is large and negative.

4. One can arrive an arbitrary accuracy with almost optimal computational costs for large indexes independent of the mesh coarseness.

Further we should make difference between various types of boundary conditions because they affect among others such important property as the multiplicity of eigenvalues.

1.1. Boundary conditions of type $(0,2;0,2)$

We begin with the analysis of problem (1.3) subject to the boundary conditions of type $(0,2;0,2)$.

In accordance with our method we look for the solution of the form (see e.g. [11, 13])

$$\lambda_n = \sum_{j=0}^{\infty} \lambda_n^{(j)}, \quad u_n(x) = \sum_{j=0}^{\infty} u_n^{(j)}(x)$$

where $\lambda_n^{(0)}$, $u_n^{(0)}(x)$ is the solution of the base problem and $\lambda_n^{(j)}$, $u_n^{(j)}(x), j = 1, 2, ...$ are then defined from some first order recursion relations. The approximation of rank m is defined as the truncated sum

$$\overset{m}{\lambda}_n = \sum_{j=0}^{m} \lambda_n^{(j)}, \quad \overset{m}{u}_n(x) = \sum_{j=0}^{m} u_n^{(j)}(x).$$

For the problem under consideration we have the base problem

$$\begin{aligned} &\frac{d^4}{dx^4} u^{(0)}(x) - \lambda^{(0)} u^{(0)}(x) = 0, \\ &u^{(0)}(0) = \frac{d^2}{dx^2} u^{(0)}(0) = u^{(0)}(1) = \frac{d^2}{dx^2} u^{(0)}(1) = 0. \end{aligned}$$

with the solution

$$u_n^{(0)}(x) = \sqrt{2}\sin(n\pi x), \qquad \lambda_n^{(0)} = (n\pi)^4, \qquad n = 1, 2 \dots .$$

The functions $u_n^{(j+1)}(x), j = 0, 1, ...$ are defined as solutions of the following recurrence sequence of problems

$$\begin{aligned} &\frac{d^4}{dx^4} u_n^{(j+1)}(x) - \lambda_n^{(0)} u_n^{(j+1)}(x) = -k(x) \frac{d^2}{dx^2} u_n^{(j)}(x) - p(x) \frac{d}{dx} u_n^{(j)}(x) - \\ &- q(x) u_n^{(j)}(x) + \sum_{s=0}^{j} \lambda^{(j+1-s)} u_n^{(s)}(x), \quad x \in (0,1) \qquad (1.16) \\ &u_n^{(j+1)}(0) = \frac{d^2}{dx^2} u_n^{(j+1)}(0) = u_n^{(j+1)}(1) = \frac{d^2}{dx^2} u_n^{(j+1)}(1) = 0, \quad j = 0, 1, \dots, m \end{aligned}$$

and are subject to the following normalizing condition

$$\left(u_n^{(j)}, u_n^{(0)}\right) = \left(u_n^{(j)}, u_n^{(0)}\right)_{L_2(0,1)} = \int_0^1 u_n^{(j)}(x) u_n^{(0)}(x) dx = 0. \qquad (1.17)$$

The solvability condition for problem (1.16) together with (1.17) yields

$$\lambda_n^{(j+1)} = \int_0^1 \left(k(x) \frac{d^2}{dx^2} u_n^{(j)}(x) + p(x) \frac{d}{dx} u_n^{(j)}(x) + q(x) u_n^{(j)}(x) \right) u_n^{(0)}(x) dx. \qquad (1.18)$$

Let us represent the solution of problem (1.16) in the form

$$\begin{aligned} u_n^{(j+1)}(x) &= -\frac{1}{\pi^4}\sum_{l=1,l\neq n}^{\infty}\frac{u_l^{(0)}(x)}{n^4-l^4}\int_0^1\Big(-k(x)\frac{d^2}{d\xi^2}u_n^{(j)}(\xi)-\\ &-p(x)\frac{d}{d\xi}u_n^{(j)}(\xi)+q(\xi)u_n^{(j)}(\xi)+\sum_{s=0}^{j}\lambda_n^{(j+1-s)}u_n^{(s)}(\xi)\Big)u_n^{(0)}(\xi)d\xi=\\ &=-\frac{2}{\pi^4}\sum_{l=1,l\neq n}^{\infty}\frac{\sin(l\pi x)}{n^4-l^4}\int_0^1\Big((l\pi)^2k(\xi)\sin(l\pi\xi)+l\pi\big(-2k'(\xi)+\\ &+p(\xi)\big)\cos(l\pi\xi)+\big(-k''(\xi)+p'(\xi)-q(\xi)\big)\sin(l\pi\xi))u_n^{(j)}(\xi)+\\ &+\sum_{s=0}^{j}\lambda_n^{(j+1-s)}u_n^{(s)}(\xi)\sin(l\pi\xi)\Big)d\xi. \end{aligned} \tag{1.19}$$

Due to (1.18)-(1.19) we obtain the estimate

$$\left|\lambda_n^{(j+1-s)}\right|\le\big((n\pi)^2\left\|k\right\|_\infty+n\pi\left\|-2k'+p\right\|_\infty+\left\|-k''+p'-q\right\|_\infty\big)\left\|u_n^{(j)}\right\| \tag{1.20}$$

and

$$\begin{aligned} \left\|u_n^{(j+1)}\right\| \le \Big\{&\frac{4}{5\pi^2}\frac{1}{(n^4-l^4)^2}\left\|k\right\|_\infty+\\ &+\frac{1}{\pi^4}\frac{1}{(2n+1)((n+1)^2+n^2)}\left\|-k''+p'-q\right\|_\infty+\\ &+\frac{1}{\pi^3}\Big(\sum_{l=1,\,l\neq n}^{\infty}\frac{l^2}{(n^4-l^4)^2}\Big)^{\frac12}\left\|-2k'+p\right\|_\infty\Big\}\left\|u_n^{(j)}\right\|+\\ &+\frac{1}{\pi^4}\frac{1}{(2n+1)((n+1)^2+n^2)}\sum_{s=0}^{j}\left|\lambda_n^{(j+1-s)}\right|\left\|u_n^{(s)}\right\|, \end{aligned} \tag{1.21}$$

where $\|u\|=(u,u)$ and $\|u\|_\infty$ is the usual supremum norm.

It is easy to prove the following inequality

$$\Big(\sum_{l=1,l\neq n}^{\infty}\frac{l^2}{(n^4-l^4)^2}\Big)^{\frac12}\le\frac{1}{2\sqrt{2}n^2}.$$

Substituting (1.21) into the right-hand side of (1.20) we obtain

$$\left\|u_n^{(j+1)}\right\|\le a_n\left\|u_n^{(j)}\right\|+b_n\sum_{s=0}^{j}\left\|u_n^{(j-s)}\right\|\left\|u_n^{(s)}\right\|,\quad j=0,1,\dots,\quad \left\|u_n^{(0)}\right\|=1,$$

where

$$\begin{aligned} a_n &= \frac{4}{5\pi^2}\frac{1}{(n^4-l^4)^2}\left\|k\right\|_\infty+\frac{1}{\pi^4}\frac{1}{(2n+1)((n+1)^2+n^2)}\left\|-k''+p''-q\right\|_\infty+\\ &+\frac{1}{2\sqrt{2}\pi^3}\frac{1}{n^2}\left\|-2k'+p\right\|_\infty,\\ b_n &= \frac{1}{\pi^4}\frac{\big((n\pi)^2\left\|k\right\|_\infty+n\pi\left\|-2k'+p\right\|_\infty+\left\|-k''+p'-q\right\|_\infty\big)}{(2n+1)((n+1)^2+n^2)} \end{aligned}$$

or more coarser

$$\begin{aligned} &\left\| u_n^{(j+1)} \right\| \le \max(a_n, b_n) \Big(\left\| u_n^{(j)} \right\| + \sum_{s=0}^{j} \left\| u_n^{(j-s)} \right\| \left\| u_n^{(s)} \right\| \Big), \quad j = 0, 1, \dots, \\ &\left\| u_n^{(0)} \right\| = 1, \\ &\max(a_n, b_n) = \frac{4}{5\pi^2} \frac{1}{(2n+1)} \|k\|_\infty + O(\frac{1}{n^2}) \end{aligned} \tag{1.22}$$

Applying the method of generating functions we obtain for the solution of (1.22) the following estimate (see e.g. [12, p. 52])

$$\left\| u_n^{(j)} \right\| \le \Big((3+\sqrt{8}) \max(a_n, b_n) \Big)^j \, 2^{-2} \frac{(2j-3)!!}{(2j)!!} \tag{1.23}$$

Substituting (1.23) into (1.21) we further obtain

$$\begin{aligned} &\left| \lambda_n^{(j+1)} \right| \le \big((n\pi)^2 \|k\|_\infty + n\pi \|-2k' + p\|_\infty + \\ &+ \|-k'' + p'' - q\|_\infty \big) \Big((3+\sqrt{8}) \max(a_n, b_n) \Big)^j \, 2^{-2} \frac{(2j-3)!!}{(2j)!!} \end{aligned} \tag{1.24}$$

The inequalities (1.23)-(1.24) imply the next assertion.

Theorem 1.1 *Let $n \in \mathbb{N}$ is such that*

$$r_n = (3+\sqrt{8}) \max(a_n, b_n) < 1,$$

then FD-method is super-exponentially convergent with the estimates

$$\begin{aligned} &\left| \lambda_n - \overset{m}{\lambda}_n \right| \le (r_n)^m \cdot \frac{3+\sqrt{8}}{1-r_n} \cdot \frac{(2m-1)!!}{(2m+2)!!} \\ &\left\| u_n - \overset{m}{u}_n \right\| \le \frac{(r_n)^{m+1}}{1-r_n} \cdot \frac{(2m-1)!!}{(2m+2)!!} \end{aligned}$$

The next example confirms this result.

Example 1.2 *Let us consider problem (1.3) with $k_2(x) = x$, $k_1(x) \equiv 0$, $k_0(x) \equiv 0$. The smallest eigenvalue of this problem computed with the computer algebra system Maple is*

$$\lambda_1^{ex} = 102.3353144965013$$

and our method above provides the results

$$\begin{aligned} &\lambda_1^{(0)} = \pi^4, \quad \left| \lambda_1^{ex} - \lambda_1^{(0)} \right| = 4.926223462, \\ &\lambda_1^{(1)} = \frac{\pi^2}{2}, \quad \left| \lambda_1^{ex} - \overset{1}{\lambda}_1 \right| = \left| \lambda_1^{ex} - \lambda_1^{(0)} - \lambda_1^{(1)} \right| = 0.008578738, \\ &\lambda_1^{(2)} = -\frac{1}{96} \big(1 + \frac{15}{\pi^2} - \frac{48}{\pi^3} - \frac{96}{(e^\pi - 1)\pi^3} \big), \\ &\left| \lambda_1^{ex} - \overset{2}{\lambda}_1 \right| = \left| \lambda_1^{ex} - \lambda_1^{(0)} - \lambda_1^{(1)} - \lambda_1^{(2)} \right| = 0.000086933 \end{aligned}$$

1.2. Boundary conditions of type $(0,1;0,1)$

For equation (1.3) subject to the boundary condition of type $(0,1;0,1)$ the base problem is

$$\frac{d^4u^{(0)}(x)}{dx^4} - \left(\lambda^{(0)}\right)^4 u^{(0)}(x) = 0, \quad u^{(0)}(0) = \frac{du^{(0)}(0)}{dx} = u^{(0)}(1) = \frac{du^{(0)}(1)}{dx} = 0. \quad (1.25)$$

Its eigenvalues $\left(\lambda_n^{(0)}\right)^4, n = 1,2,\ldots \quad \lambda_1^{(0)} < \lambda_2^{(0)} < \ldots < \lambda_n^{(0)} < \ldots$ are simple and are defined as solutions of the equation

$$\cos\sqrt{\lambda_n^{(0)}} - \frac{1}{\cosh\sqrt{\lambda_n^{(0)}}} = 0.$$

The exact eigenfunctions are

$$\begin{aligned} u_n^{(0)}(x) =& N_n^{(0)} \Big\{ [\cosh\lambda_n^{(0)}x - \cos\lambda_n^{(0)}(x)]\big|_{x=1} - [\sinh\lambda_n^{(0)}x - \sin\lambda_n^{(0)}(x)] - \\ &- [\sinh\lambda_n - \sin\lambda_n][\cosh\lambda_n^{(0)}x - \cos\lambda_n^{(0)}x] \Big\} \\ & n = 1,2,\ldots \end{aligned}$$

Here $N_n^{(0)}$ is the normalizing factor defined by the normalizing condition $\|u_n^{(0)}\|_{L_2(0,1)} = 1$. The system of eigenfunctions $\{u_n^{(0)}(x)\}_{n=1,2,\ldots}$ is a complete orthonormal system in $L_2(0,1)$ (see e.g. [16]). In order to compute the approximation of rank m by our method we should solve the following recurrence sequence of problems

$$\begin{aligned} &\frac{d^4}{dx^4} u_n^{(j+1)}(x) - (\lambda_n^{(0)})^4 u_n^{(j+1)}(x) = -k(x)\frac{d^2}{dx^2} u^{(j)}(x) - p(x)\frac{d}{dx} u^{(j)}(x) - \\ &- q(x)u_n^{(j)}(x) + \sum_{s=0}^{j} \lambda_n^{(j+1-s)} u_n^{(s)}(x) \equiv F^{(j+1)}(x), \qquad x \in (0,1) \qquad (1.26) \\ &u_n^{(j+1)}(0) = \frac{du_n^{(j+1)}(0)}{dx} = u_n^{(j+1)}(1) = \frac{du_n^{(j+1)}(1)}{dx} = 0. \end{aligned}$$

The solvability condition $\left(F^{(j+1)}, u_n^{(0)}\right) = 0$ for problem (1.26) yields

$$\begin{aligned} \lambda_n^{(j+1)} &= \sum_{s=1}^{j} \lambda_n^{(j+1-s)} \int_0^1 u_n^{(s)}(x) u_n^{(0)}(x) dx - \\ &- \int_0^1 \big[k(x)\frac{d^2u_n^{(j)}(x)}{dx^2} + p(x)\frac{du_n^{(j)}(x)}{dx} - q(x)u_n^{(j)}(x)\big] u_n^{(0)}(x) dx. \end{aligned} \quad (1.27)$$

The unique solution subject to the additional condition $\left(u_n^{(j+1)}, u_n^{(0)}\right) = 0, \quad j = 1,2...$ is represented by

$$u_n^{(j+1)}(x) = \sum_{p=1,p\neq n}^{\infty} \frac{\int_0^1 F_n^{(j+1)}(\xi) u_p^{(0)}(\xi) d\xi}{\left(\lambda_p^{(0)}\right)^4 - \left(\lambda_n^{(0)}\right)^4} u_p^{(0)}(x). \quad (1.28)$$

Now, the equality (1.27) can be reduced to

$$\lambda_n^{(j+1)} = -\int_0^1 \big[k(x)\frac{d^2u_n^{(j)}(x)}{dx^2} + p(x)\frac{du_n^{(j)}(x)}{dx} - q(x)u_n^{(j)}(x)\big] u_n^{(0)}(x) dx. \quad (1.29)$$

Let us estimate $\lambda_n^{(j+1)}$ and the norm of $u_n^{(j+1)}(x)$ using (1.28)-(1.29). From (1.28) it follows

$$\left\|u_n^{(j+1)}(x)\right\| \le \sum_{p=1,p\neq n}^{\infty} \frac{1}{\left|\left(\lambda_p^{(0)}\right)^4 - \left(\lambda_n^{(0)}\right)^4\right|} \Big\{ ||k||_\infty \left\| \frac{d^2 u_n^{(0)}}{dx^2} \right\| + \\ + ||-2k'+p||_\infty \left\| \frac{du_p^{(0)}}{dx} \right\| + \|-k''+p'-q\|_\infty \Big\} \left\|u_n^{(j)}\right\| + \sum_{s=0}^{j} \left|\lambda_n^{(j+1-s)}\right| \left\|u_n^{(s)}\right\|. \tag{1.30}$$

To reduce inequality (1.30) to more convenient form, we will use the following inequalities

$$\left\|u_n^{(0)}\right\| \le \frac{1}{\sqrt{2}} \left\| \frac{du_p^{(0)}}{dx} \right\| \le \frac{1}{2} \left\| \frac{d^2 u_p^{(0)}}{dx^2} \right\|, \quad p = 1, 2... \tag{1.31}$$

as well as the equality

$$\left\| \frac{d^2 u_p^{(0)}}{dx^2} \right\| = \left[\lambda_p^{(0)}\right]^2, \quad p = 1, 2... \tag{1.32}$$

Note that the inequalities (1.31) are a consequence of the identity

$$\frac{d^k u_p^{(0)}}{dx^k} = \int_0^x \frac{d^{k+1} u_p^{(0)}(\xi)}{d\xi^{k+1}} d\xi, \quad k = 0, 1$$

and equality (1.32) follows from the equations (1.25) with $u^{(0)}(x) = u_p^{(0)}(x), \lambda = \lambda_p$ after scalar multiplication by $u_p^{(0)}(x)$.

Taking into account (1.31), (1.32) we obtain from (1.30)

$$\left\|u_n^{(j+1)}\right\| \le \sum_{p=1,p\neq n}^{n} \frac{1}{\left|\left(\lambda_p^{(0)}\right)^4 - \left(\lambda_n^{(0)}\right)^4\right|} \left\{ \left\|u_n^{(j)}\right\| [\lambda_p^{(0)}]^2 + \sum_{s=0}^{j} \left|\lambda_n^{(j+1-s)}\right| \left\|u_n^{(s)}\right\| \right\} \tag{1.33}$$

Using again (1.31)-(1.32), from (1.29) we have analogously

$$\left|\lambda_n^{(j+1)}\right| \le \alpha [\lambda_n^{(0)}]^2 \left\|u_n^{(j)}\right\|. \tag{1.34}$$

Substitution of (1.34) into (1.33) and introduction of notations

$$M_n = \alpha \sum_{p=1,p\neq n}^{\infty} \left[\frac{\lambda_p^{(0)}}{\left|\left(\lambda_p^{(0)}\right)^4 - \left(\lambda_n^{(0)}\right)^4\right|} \right]^2,$$

$$L_n = \alpha \sum_{p=1,p\neq n}^{\infty} \frac{1}{\left|\left(\lambda_p^{(0)}\right)^4 - \left(\lambda_n^{(0)}\right)^4\right|},$$

leads to the following system of recurrence inequalities

$$\left\|u_n^{(j+1)}\right\| \le M_n \left\|u_n^{(j)}\right\| + L_n \sum_{s=0}^{j} \left\|u_n^{(j-s)}\right\| \left\|u_n^{(s)}\right\| \le \\ \le \max\{M_n, L_n\} \Big\{ \left\|u_n^{(j)}\right\| + \sum_{s=0}^{j} \left\|u_n^{(j-s)}\right\| \left\|u_n^{(s)}\right\| \Big\}$$

Switching over to the corresponding majorant system and using the method of generating functions we obtain the estimate

$$\left\|u_n^{(j+1)}\right\| \le \frac{1}{2}\left\{\max[M_n, L_n](3+\sqrt{8})\right\}^{j+1}\frac{(2j-1)!!}{(2j+2)!!} \tag{1.35}$$

which together with (1.34) implies the estimate

$$\left|\lambda_n^{(j+1)}\right| \le \frac{\alpha}{2}[\lambda_n^{(0)}]^2 q_n^j \frac{(2j-3)!!}{(2j)!!}, \tag{1.36}$$

with

$$q_n = \max\{M_n, L_n\}(3+\sqrt{8}).$$

The estimates (1.35)-(1.36) yield the next assertion.

Theorem 1.3 *Let $n \in \mathbb{N}$ be such that $q_n < 1$, then the FD-method converges super-exponentially as $m \to \infty$ and for the approximations of rank m the following error estimates hold*

$$\left|\lambda_n - \overset{m}{\lambda}_n\right| \le \frac{\alpha}{2}[\lambda_n^{(0)}]^2 \frac{(q_n)^m}{1-q_n}\frac{(2m-3)!!}{(2m)!!}$$

$$\left\|u_n - \overset{m}{u}_n\right\| \le \frac{1}{2}\frac{(q_n)^{m+1}}{1-q_n}\frac{(2m-1)!!}{(2m+2)!!}$$

The following example is the experimental confirmation of this result.

Example 1.4 *Let us consider equation (1.2) with $a(\xi) = 1+\xi$, $b(\xi) = c(\xi) \equiv 0$, $d(\xi) \equiv 1$. After change of variables*

$$\xi = (1+\frac{3}{4}x)^{4/3} - 1, \quad v(\xi) = (1+\frac{3}{4}x)^{-1/6}u(x) \tag{1.37}$$

equation (1.2) passes into the equation

$$u^{(4)}(x) + \frac{13}{18(x+4/3)^2}u''(x) + \frac{13}{9(x+4/3)^3}u'(x) + (-\lambda + \frac{17}{16(x+4/3)^4}u(x)) = 0. \tag{1.38}$$

The boundary conditions of the type $(0,1;0,1)$ after substitution (1.37) switch to the boundary conditions of the same type for equation (1.38) on the interval $(0,a)$ with $a = \frac{4}{3}(2^{3/4}-1)$:

$$u(0) = u'(0) = u(a) = u'(a) = 0. \tag{1.39}$$

The smallest exact eigenvalue of problem (1.38)-(1.39) is

$$\lambda_1^{ex} = 729.5132640790354497.$$

Our method of rank 1 provides the following results:

$$\lambda_1^{(0)} = 729.0804175123859275, \quad \left|\lambda_1^{ex} - \lambda_1^{(0)}\right| = 0.432846566,$$

$$\lambda_1^{(1)} = 0.4329291815470396, \quad \left|\lambda_1^{ex} - \overset{1}{\lambda}_1\right| = \left|\lambda_1^{ex} - \lambda_1^{(0)} - \lambda_1^{(1)}\right| = 0.000082614,$$

Now, let us consider the often used approach where the coefficients of the differential equation are changed by the piece-wise constant functions, for example, let us consider instead of

$$\frac{d^2}{d\xi^2}((1+\xi)\frac{d^2}{d\xi^2}y(\xi)) - \tilde{\lambda}y(\xi) = 0,$$
$$y(0) = y'(0) = y(1) = y'(1) = 0$$

the problem

$$\frac{d^4}{d\xi^4}\tilde{y}(\xi) - \frac{2}{3}\tilde{\lambda}\tilde{y}(\xi) = 0, \qquad \tilde{y}(0) = \tilde{y}'(0) = \tilde{y}(1) = \tilde{y}'(1) = 0.$$

Then we obtain the following approximation for the smallest eigenvalue

$$\tilde{\lambda_1} = 750.8458526104090604, \quad \lambda_1^{ex} - \tilde{\lambda_1} = 21.33258853$$

which is much coarser as the approximation $\lambda_1^{(0)}$ obtained by the FD-method of the rank zero.

Closing this section we note that all eigenvalues of the base problem consisting of the differential equation

$$\frac{d^4u_n^{(0)}(x)}{dx^4} - \lambda_n^{(0)}u_n^{(0)}(x) = 0, \quad x \in (0,1) \tag{1.40}$$

subject to one of the boundary conditions of type $(p,q;r,s)$, $0 \le p \le q \le 3$, $0 \le r \le s \le 3$, are simple excluding the boundary conditions of type $(2,3;2,3)$, where $\lambda^{(0)} = 0$ is a double eigenvalue corresponding to the orthonormal eigenfunctions

$$u_{01}^{(0)}(x) = 1 \quad \text{and} \quad u_{02}^{(0)}(x) = 2\sqrt{3}\left(\frac{1}{2} - x\right)$$

(see e.g. [16]).

Therefore, we consider further the boundary conditions of type $(2,3;2,3)$ and the ones corresponding to a non self-adjoint problem, for example, $(0,1;1,2)$.

2. Multiple eigenvalues and non self-adjoint problems

Let us begin with the base problem (1.40) subject to the boundary conditions of type (2,3;2,3) for which the eigenvalue $\lambda_0^{(0)} = 0$ corresponds to the following two orthonormal eigenfunctions: $u_{01}^{(0)}(x) = 1$ and $u_{02}^{(0)}(x) = 2\sqrt{3}\left(x - \frac{1}{2}\right)$. All other eigenvalues are simple and the base problem is self-adjoint, therefore all considerations above concerning the convergence and estimates of the FD-method for these eigenvalues remain true.

For the sake of simplicity let us consider differential equation (1.3) with $k_2(x) = k_1(x) \equiv 0$, $k_0(x) = k_0(1-x)$ subject to the boundary conditions of type $(2,3;2,3)$. We come from the solution of the base problem in the form

$$u_0^{(0)}(x) = C_{0,1}^{(0)} + C_{0,2}^{(0)}(x - 1/2),$$

and choose the constants $C_{0,1}^{(0)}, C_{0,2}^{(0)}$ from the solvability conditions

$$\int_0^1 F_0^{(1)}(x)dx = 0, \qquad \int_0^1 F_0^{(1)}(x)(x-1/2)dx = 0. \tag{2.1}$$

for the problem (compare with (1.16))

$$\begin{gathered}\frac{d^4u_0^{(1)}(x)}{dx^4} = \left[\lambda_0^{(1)} - k_0(x)\right] u_0^{(0)}(x) \equiv F_0^{(1)}(x), \quad x \in (0,1)\\ \frac{d^2u_0^{(1)}(0)}{dx^2} = \frac{d^3u_0^{(1)}(0)}{dx^3} = \frac{d^2u_0^{(1)}(1)}{dx^2} = \frac{d^3u_0^{(1)}(1)}{dx^3} = 0.\end{gathered} \tag{2.2}$$

The conditions (2.1) yield the system of linear equations

$$\left[\lambda_0^{(1)} - \int_0^1 k_0(x)dx\right] C_{0,1}^{(0)} = 0, \qquad \left[\frac{\lambda_0^{(1)}}{12} - \int_0^1 k_0(x)\,(x-1/2)^2\,dx\right] C_{0,2}^{(0)} = 0$$

with respect to $C_{0,1}^{(0)}, C_{0,2}^{(0)}$ which possesses non-trivial solutions provided that $\lambda_0^{(1)} = \lambda_{0,\pm}^{(1)}$ with

$$\begin{aligned}\lambda_{0,+}^{(1)} &= \int_0^1 k_0(x)dx,\\ \lambda_{0,-}^{(1)} &= 12\int_0^1 k_0(x)\,(x-1/2)^2\,dx.\end{aligned}$$

Without los of generality we consider the case $\lambda_0^{(1)} = \lambda_{0,+}^{(1)}$ only (the case $\lambda_0^{(1)} = \lambda_{0,+}^{(1)}$ can be considered analogously). The recurrent sequence of problems of our method is of the form

$$\begin{gathered}\frac{d^4u_{0,+}^{(j+1)}(x)}{dx^4} = \sum_{p=0}^{j} \lambda_{0,+}^{(j+1-p)} u_{0,+}^{(p)}(x) - k_0(x)u_{0,+}^{(j)}(x) \equiv F_{0,+}^{(j+1)}(x), \quad x \in (0,1)\\ \frac{d^k u^{(j+1)}(0)}{dx^k} = \frac{d^k u^{(j+1)}(1)}{dx^k} = 0, \quad k = 2,3,\end{gathered} \tag{2.3}$$

with the base problem(2.2).

Each of problems (2.3) is solvable provided that

$$\int_0^1 F_{0,+}^{(j+1)}(x)dx = 0, \qquad \int_0^1 F_{0,+}^{(j+1)}(x)(x-1/2)dx = 0.$$

Subordinating the solutions to the additional orthogonality condition

$$\int_0^1 u_{0,+}^{(j)}(x)\,dx = 0,$$

we obtain

$$\begin{aligned}&C_{0,1,+}^{(j)} = 0,\\ &C_{0,2,+}^{(j)} = \alpha\left\{-\frac{1}{12}\sum_{s=1}^{j-1} C_{0,2,+}^{(s)}\,\lambda_{0,+}^{(j+1-s)} + \int_0^1 k_0(x)(x-1/2)\hat{u}_{0,+}^{(j)}(x)dx\right\},\\ &\hat{u}_{0,+}^{(j)}(x) = \sum_{p=1}^{\infty} \frac{1}{\lambda_p}\int_0^1 F_{0,+}^{(j)}(\xi)u_p^{(0)}(\xi)d\xi\, u_p^{(0)}(x),\\ &\lambda_{0,+}^{(j+1)} = \int_0^1 k_0(x)\hat{u}_{0,+}^{(j)}(x)dx,\\ &u_{0,+}^{(j)}(x) = \hat{u}_{0,+}^{(j)}(x) + C_{0,2,+}^{(j)}(x-1/2),\end{aligned} \tag{2.4}$$

where

$$\alpha = \left[\frac{\lambda_{0,+}^{(1)}}{12} - \int_0^1 k_0(x)(x-1/2)^2 dx \right]^{-1}$$

Taking into account that

$$\int_0^1 F_{0,+}^{(j)}(\xi) u_p^{(0)}(\xi)\, d\xi = \int_0^1 \left[\sum_{p=0}^{j-1} \lambda_{0,+}^{(j-p)}\ \hat{u}_{0,+}^{(p)}(\xi) - k_0(\xi)\hat{u}_{0,+}^{(j-1)}(\xi) \right] u_p^{(0)}(\xi)\, d\xi -$$
$$- C_{0,2,+}^{(j-1)} \int_0^1 k_0(\xi)(\xi - 1/2) u_p^{(0)}(\xi)\, d\xi,$$

we obtain from (2.4)

$$\begin{aligned}
&\left\| \hat{u}_{0,+}^{(j)} \right\| \le \frac{1}{\lambda_1} \left[\sum_{p=0}^{j-1} \left| \lambda_{0,+}^{(j-p)} \right| \left\| \hat{u}_{0,+}^{(p)} \right\| + \| k_0(x)(x-1/2) \| \left| C_{0,2,+}^{(j-1)} \right| \right], \\
&\left| C_{0,2,+}^{(j)} \right| \le |\alpha| \left[\frac{1}{12} \sum_{s=1}^{j-1} \left| C_{0,2,+}^{(s)} \right| \left| \lambda_{0,+}^{(j+1-s)} \right| + \| k_0(x)(x-1/2) \| \left\| \hat{u}_{0,+}^{(j)} \right\| \right], \\
&\left| \lambda_{0,+}^{(j+1)} \right| \le \| k_0 \| \left\| \hat{u}_{0,+}^{(j)} \right\|
\end{aligned} \tag{2.5}$$

The substitution of the last inequality into the right-hand side of two other ones yields

$$\begin{aligned}
&\left\| \hat{u}_{0,+}^{(j)} \right\| \le a \left[\sum_{p=0}^{j-1} \left\| \hat{u}_{0,+}^{(j-p-1)} \right\| \left\| \hat{u}_{0,+}^{(p)} \right\| + \left| C_{0,2,+}^{(j-1)} \right| \right] \\
&\left| C_{0,2,+}^{(j)} \right| \le b \left[\sum_{s=0}^{j-1} \left| C_{0,2,+}^{(s)} \right| \left\| \hat{u}_{0}^{(j-s)} \right\| + \left\| \hat{u}_{0,+}^{(j)} \right\| \right],
\end{aligned}$$

where

$$\begin{aligned}
a &= \max \left\{ \frac{\|k_0\|}{\lambda_1}, \qquad \frac{\|k_0(x)(x-1/2)\|}{\lambda_1}, \right\} \\
b &= \max \left\{ |\alpha| \frac{\|k_0\|}{12}, \qquad |\alpha| \, \|k_0(x)(x-1/2)\| \right\}.
\end{aligned}$$

We introduce the following new variables

$$a^{-j} \left\| \hat{u}_{0,+}^{(j)} \right\| = U_j, \qquad a^{-j} \left| C_{0,2,+}^{(j)} \right| = C_j,$$

and consider the following majorant system of equations

$$\begin{aligned}
&\tilde{U}_j = \sum_{p=0}^{j-1} \tilde{U}_{j-p-1} \tilde{U}_p + \tilde{C}_{j-1}, \\
&\tilde{C}_j = b \left[\sum_{s=0}^{j-1} \tilde{C}_s \tilde{U}_{j-s} + \tilde{U}_j \right], \qquad j = 1, 2 ..., \qquad \tilde{U}_0 = 1, \qquad \tilde{C}_0 = b.
\end{aligned} \tag{2.6}$$

Let us apply the $\mathcal{Z}$-transform (see e.g. [3, 5]) to (2.6). Using the linearity and the relationships $\mathcal{Z}\left\{\tilde{U}_n\right\} = \sum\limits_{n=0}^{\infty} \tilde{U}_n z^{-n} \equiv f(z)$, $\mathcal{Z}\left\{\tilde{C}_n\right\} = \sum\limits_{n=0}^{\infty} \tilde{C}_n z^{-n} \equiv g(z)$ as well as

$\mathcal{Z}\left\{\sum\limits_{p=0}^{n}\tilde{U}_{n-p}\tilde{U}_p\right\} = f(z)\cdot g(z)$, $\mathcal{Z}\left\{\tilde{U}_{n+k}\right\} = z^k\left(f(z) - \sum\limits_{j=0}^{k-1} f_j z^{-j}\right)$ we have from (2.6)

$$\begin{aligned} z\,(f(z)-1) &= [f(z)]^2 + g(z), \\ zg(z) &= b\,z\,g(z)\,[f(z)-1] + b\,z\,[f(z)-1]\,. \end{aligned} \tag{2.7}$$

From system of equations (2.7) we obtain the following expression for z as a function of f:

$$z = \frac{f^2\,[1-b(f-1)] + b(f-1)}{(f-1)\,[1-b(f-1)]}. \tag{2.8}$$

One can easily see that $z \to \infty$ as $f \to 1$ or $f \to \left(1+\frac{1}{b}\right)$ and $z(f) > 0\;\forall\; z \in \left(1, 1+\frac{1}{b}\right)$, which implies the existence of a value $f_{\min} \in (1, 1+1/b)$ for which the function $z = z(f)$ arrives its minimum $z_{\min} = z\,(f_{\min})$ on the interval $f \in \left(1, 1+\frac{1}{b}\right)$. The value $R = \frac{1}{z_{\min}}$ is the convergence radius for the power series $f\,(w) = \sum\limits_{n=0}^{\infty}\tilde{U}_n w^n, \quad w = \frac{1}{z}$. This means that there exist positive C, ε such that

$$R^j\,\tilde{U}_j \le \frac{C}{(1+j)^{1+\varepsilon}}$$

Taking into account the notations introduced above as well as the relations (2.4)-(2.5) we have

$$\begin{aligned} &U_j \le \tilde{U}_j \le \frac{C}{(1+j)^{1+\varepsilon}} R^{-j}, \\ &\left\|\hat{u}_{0,+}^{(j)}\right\| \le \frac{C}{(1+j)^{1+\varepsilon}}\left(\frac{a}{R}\right)^j, \\ &\left|\lambda_{0,+}^{(j+1)}\right| \le \|k_0\|\,\frac{C}{(1+j)^{1+\varepsilon}}\left(\frac{a}{R}\right)^j, \\ &\left|C_{0,2,+}^{(j)}\right| \le \frac{C}{(1+j)^{1+\varepsilon}}\left(\frac{a}{R}\right)^j, \\ &\left\|u_{0,+}^{(j)}\right\| \le \left\|u_{0,+}^{(j)}\right\| + \frac{1}{\sqrt{12}}\left|C_{0,2,+}^{(j)}\right| \le \frac{C\left(1+\frac{1}{\sqrt{12}}\right)}{(1+j)^{1+\varepsilon}}\left(\frac{a}{R}\right)^j. \end{aligned} \tag{2.9}$$

At the penultimate inequality we have used that fact that the convergence radii for $f(w), g(w), \quad w = \frac{1}{z}$ are the same.

These inequalities yield the following result.

Theorem 2.1 *Under the assumption*

$$q = \frac{a}{R} < 1 \qquad \left(q = \frac{a}{R} = 1\right)$$

the FD-method for equation (1.3) with the boundary conditions of type $(2,3;2,3)$ converges super-exponentially (converges) and for the approximation of rank m the following error estimates hold true

$$\begin{aligned} \|u_0 - \overset{m}{u}_0\| &\le \frac{C\left(1+\frac{1}{\sqrt{2}}\right)}{(m+2)^{1+\varepsilon}}\frac{(a/R)^{m+1}}{1-a/R} \qquad (= o(1)) \\ |\lambda_0 - \overset{m}{\lambda}_0| &\le \frac{\|K_0\|C}{(m+2)^{1+\varepsilon}}\frac{(a/R)^m}{1-a/R} \qquad (= o(1)) \end{aligned}$$

The following example confirms the assertion of the theorem.

Example 2.2 *Let us consider the eigenvalue problem*

$$\begin{aligned} &u^{(4)}(x) + \left[\left(x - \frac{1}{2}\right)^2 - \lambda\right] u(x), \quad x \in (0,1) \\ &u^{(k)}(0) = u^{(k)}(1) = 0, \qquad k = 2,3. \end{aligned} \tag{2.10}$$

The smallest eigenvalues of problem (2.10) computed with the computer algebra tool Maple is

$$\lambda_{0,-}^{ex} = 0.0833223112249938..., \ \lambda_{0,+}^{ex} = 0.14999891773580...$$

The results obtained by the FD-method of rank $m = \overline{0,3}$ are

$$\overset{0}{\lambda}_0 = 0, \ \overset{1}{\lambda}_{0,-} = \frac{1}{12} = 0.08(3), \ \overset{1}{\lambda}_{0,+} = \frac{3}{20} = 0.15,$$
$$\overset{2}{\lambda}_{0,-} = \frac{7559}{90720} = 0.0833223104056437..., \ \overset{2}{\lambda}_{0,+} = \frac{138599}{924000} = 0.14999891774891...,$$
$$\overset{3}{\lambda}_{0,-} = \frac{163437676007}{1961511552000} = 0.0833223112249098...,$$
$$\overset{3}{\lambda}_{0,+} = \frac{34306024477}{228708480000} = 0.14999891773580...$$

and for the absolut error we have the estimates

$$\left|\lambda_{0,-}^{ex} - \overset{0}{\lambda}_0\right| \le 0.84 \cdot 10^{-1}, \ |\lambda_{0,+}^{ex} - \overset{0}{\lambda}_0| = 0.15,$$
$$\left|\lambda_{0,-}^{ex} - \overset{1}{\lambda}_0\right| \le 0.12 \cdot 10^{-4}, \ |\lambda_{0,+}^{ex} - \overset{1}{\lambda}_0| \le 0.11 \cdot 10^{-5},$$
$$\left|\lambda_{0,-}^{ex} - \overset{2}{\lambda}_0\right| \le 0.83 \cdot 10^{-9}, \ |\lambda_{0,+}^{ex} - \overset{2}{\lambda}_0| \le 0.14 \cdot 10^{-10},$$
$$\left|\lambda_{0,-}^{ex} - \overset{3}{\lambda}_0\right| \le 0.84 \cdot 10^{-13}, \ |\lambda_{0,+}^{ex} - \overset{3}{\lambda}_0| \le 0.39 \cdot 10^{-15}.$$

To conclude this section let us consider the non self-adjoint case of the differential equation (1.3) with the boundary conditions of type $(0,1;1,2)$.

The base problem

$$\begin{aligned} &\frac{d^4u^{(0)}(x)}{dx^4} - \lambda^{(0)}\, u^{(0)}(x) = 0, \quad x \in (0,1) \\ &\frac{d^k u^{(0)}(0)}{dx^k} = \frac{d^{k+1}u^{(0)}(1)}{dx^{k+1}} = 0, \quad k = 0,1, \end{aligned} \tag{2.11}$$

has the solution

$$\begin{aligned} &\lambda_n^{(0)} = (n\pi)^4, \\ &v_n^{(0)}(x) = A_n^{(0)}\Big\{[\cosh(n\pi) + \cos(n\pi)]\,[\sinh(n\pi x) - \sin(n\pi x)] - \\ &- [\sinh(n\pi) + \sin(n\pi)]\,[\cosh(n\pi x) - \cos(n\pi x)]\Big\}, \quad n = 1,2,\dots, \end{aligned}$$

where $A_n^{(0)}$ is a normalizing constant defined below.

The adjoint problem for (2.11) is

$$\frac{d^4 v^{(0)}(x)}{dx^4} - \mu^{(0)}\, v^{(0)}(x) = 0, \quad x \in (0,1)$$
$$\frac{d^k v^{(0)}(0)}{dx^k} = 0, \quad k = 0,1; \quad v^{(0)}(1) = \frac{d^3 v^{(0)}(1)}{dx^3} = 0$$

with the solution

$$\mu^{(0)} = (n\pi)^4,$$
$$v_n^{(0)}(x) = [\cosh(n\pi) - cos\,(n\pi)][\sinh(n\pi x) - \sin(n\pi x)] - \\ - \sinh(n\pi)\,[\cosh(n\pi x) - \cos(n\pi x)],$$

The normalizing condition reads in the non self-adjoint case as

$$\int_0^1 u_n^{(0)}(x) v_n^{(0)}(x) dx = 1,$$

from where we obtain $A_n^{(0)} = (\sinh(n\pi))^{-2}$.

The FD-method requires the solution of the sequence of problems

$$\begin{aligned} &\frac{d^n u_n^{(j+1)}(x)}{dx^n} - (n\pi)^n u_n^{(j+1)}(x) = \sum_{p=0}^{j} \lambda_n^{(j+1-p)} u_n^{(p)}(x) - \\ &- k_0(x) u_n^{(j+1)}(x) = F_n^{(j+1)}(x), \quad x \in (0,1) \\ &\frac{d^k u_n^{(j+1)}(x)}{dx^k} = 0, \quad k = 0,1; \qquad \frac{d^k u_n^{(j+1)}(1)}{dx^k} = 0, \quad k = 1,2. \end{aligned} \tag{2.12}$$

The solvability condition for (2.12) leads to

$$\lambda_n^{(j+1)} = -\sum_{p=1}^{j} \lambda_n^{(j+1-p)} \int_0^1 u_n^{(p)}(x) v_n^{(0)}(x) dx + \int_0^1 k_0(x) u_n^{(j)}(x) v_n^{(0)}(x) dx. \tag{2.13}$$

Using the generalized Green function $G_n(x,\xi)$ we can represent the solution of (2.12) in the form

$$u_n^{(j+1)}(x) = \int_0^1 G_n(x,\xi) F_n^{(j+1)}(\xi) d\xi \tag{2.14}$$

with

$$G_n(x,\xi) = \frac{\cosh(n\pi x) - \cos(n\pi x)}{2(n\pi)^3 \sinh(n\pi)} \left\{-\cos(n\pi(1-\xi)) + \cos(n\pi(1-\xi))\right\},$$

for $x \leq \xi$ and

$$G_n(x,\xi) = \frac{1}{2(n\pi)^3 \sinh(n\pi)} \Big\{ \sinh(n\pi)[\sinh(n\pi(x-\xi)) - \sin(n\pi(x-\xi))] + \\ + (\cosh(n\pi x) - \cos(n\pi x))[-\cosh(n\pi(1-\xi)) + \cos(n\pi(1-\xi))] \Big\},$$

for $\xi \leq x$.

The following estimates hold true

$$|G_n(x,\xi)| \leq \frac{4}{(n\pi)^3(1-e^{-2\pi})}, \qquad x \leq \xi,$$
$$|G_n(x,\xi)| \leq \frac{e^{-\pi} + 18}{4(n\pi)^3(1-e^{-2\pi})}, \qquad \xi \leq x,$$

or more roughly

$$|G_n(x,\xi)| \le \frac{4.52}{(n\pi)^3}.$$

Taking into account this inequality we obtain from (2.14) the estimate

$$\left\|u_n^{(j+1)}\right\| \le \frac{4.52}{(n\pi)^3}\left\|F_n^{(j+1)}\right\| \le \frac{4.52}{(n\pi)^3}\left\{\sum_{p=0}^{j}\left|\lambda_n^{(j+1-p)}\right|\left\|u_n^{(p)}\right\| + \|k_0\|_\infty\left\|u_n^{(j)}\right\|\right\} \tag{2.15}$$

and from (2.13) – the estimate

$$\left|\lambda_n^{(j+1)}\right| \le \left\|v_n^{(0)}\right\|\left\{\sum_{p=0}^{j}\left|\lambda_n^{(j+1-p)}\right|\left\|u_n^{(p)}\right\| + \|k_0\|_\infty\left\|u_n^{(j)}\right\|\right\}. \tag{2.16}$$

Introducing in (2.15)-(2.16) the new variables by

$$\begin{gathered}\frac{\left\|v_n^{(0)}\right\|}{\left[\frac{4.52}{(n\pi)^3}\|k_0\|_\infty\right]^{j+1}}\left\|u_n^{(j+1)}\right\| = u_{j+1},\\ \frac{\left|\lambda_n^{(j+1)}\right|}{\|k_0\|_\infty\left[\frac{4.52}{(n\pi)^3}\|K_0\|_\infty\right]^{j+1}} = \lambda_{j+1},\end{gathered} \tag{2.17}$$

we obtain for the majorant values

$$U_{j+1} \ge u_{j+1}, \qquad \Lambda_{j+1} \ge \lambda_{j+1} \tag{2.18}$$

the following system of equations

$$\begin{aligned} U_{j+1} &= \sum_{p=0}^{j}\Lambda_{j+1-p}U_p + U_j,\\ \Lambda_{j+1} &= \sum_{p=0}^{j}\Lambda_{j+1-p}U_p + U_j, j=0,1..., \qquad U_0 = \left\|u_n^{(0)}\right\|\left\|v_n^{(0)}\right\|.\end{aligned}$$

Using the method of generating functions we obtain analogously to [12]

$$\begin{aligned} U_{j+1} &\le \frac{1}{2}\left(\frac{1+U_0}{\beta}\right)^{j+1}\frac{(2j-1)!!}{(2j+2)!!},\\ \Lambda_{j+1} &\le \frac{1}{2\beta}\left(\frac{1+U_0}{\beta}\right)^{j}\frac{(2j-1)!!}{(2j+2)!!},\end{aligned} \tag{2.19}$$

where

$$\beta = \left[1 + 2U_0 + 2\sqrt{U_0(1+U_0)}\right]^{-1}. \tag{2.20}$$

Taking into account (2.18) and (2.17) equality (2.19) implies

$$\begin{aligned}\left\|u_n^{(j+1)}\right\| &\le \frac{1}{2}\frac{(2j-1)!!}{(2j+2)!!}\frac{1}{\left\|v_n^{(0)}\right\|}(q_n)^{j+1},\\ \left|\lambda_n^{(j+1)}\right| &\le \frac{\|k_0\|_\infty}{2\beta}\frac{(2j-1)!!}{(2j+2)!!}(q_n)^j,\end{aligned}$$

where

$$q_n = \frac{4.52}{(n\pi)^3}\,\|k_0\|_\infty\,\frac{1+U_0}{\beta} = \frac{4.52}{(n\pi)^3}\,\|k_0\|_\infty\,(1+U_0)\left[1+2U_0+2\sqrt{U_0(1+U_0)}\right]. \quad (2.21)$$

Thus, we have proven the following result.

Theorem 2.3 *If $q_n < 1$ then the FD-method for equation (1.3) with the boundary conditions of type $(0,1;1,2)$ is super-exponentially convergent and for the approximation of rank m the following estimates hold*

$$\|u_n - \overset{m}{u}_n\| = \left\|u_n - \sum_{j=0}^{m} u_n^{(j)}\right\| \le \frac{1}{2}\frac{(2m-1)!!}{(2m+2)!!}\frac{1}{\|v_n^{(0)}\|}\frac{(q_n)^{m+1}}{1-q_n},$$
$$|\lambda_n - \overset{m}{\lambda}_n| = \left|\lambda_n - \sum_{j=0}^{m} \lambda_n^{(j)}\right| \le \frac{\|k_0\|_\infty}{2\beta}\frac{(2m-1)!!}{(2m+2)!!}\frac{(q_n)^m}{1-q_n},$$

In order to investigate q_n as function of n we consider $U_0 = U_0(n\pi)$ as function of n:

$$\begin{aligned} U_0 &= \left\|u_n^{(0)}\right\|\left\|v_n^{(0)}\right\| = \frac{1}{\sinh^2(n\pi)}\frac{1}{2n\pi}\times \\ &\times\left[3\sinh(2n\pi) + 2n\pi\sinh^2(n\pi) + 6(-1)^n\sinh(n\pi)\right]^{1/2}\times \\ &\times\left[-\sinh(2n\pi) + 2n\pi\sinh^2(n\pi) + (-1)^n 2\sinh(n\pi)\right]^{1/2} = \\ &= \left[1 + \frac{3\sinh^2(n\pi) + 6(-1)^n\sinh(n\pi)}{2n\pi\sinh^2(n\pi)}\right]^{1/2}\times \\ &\times\left[1 + \frac{-\sinh(2n\pi) + (-1)^n 2\sinh(n\pi)}{2n\pi\sinh^2(n\pi)}\right]^{1/2}. \end{aligned} \quad (2.22)$$

For n odd we have

$$\begin{aligned} \frac{d}{dx}\left(U_0|_{n\pi=x}\right)^2 &= \frac{2\cosh x}{x^3\sinh x}\left[3\coth x + \frac{2x^2}{\sinh x} - \frac{x^2+3}{\sinh x\cosh x} + \frac{2x}{\coth x} - x\right] \le \\ &\le \frac{2\cosh x}{x^3\sinh x}\left[3\coth\pi + \frac{2\pi^2}{\sinh\pi} + \frac{2\pi}{\cosh\pi} - x\right] = \\ &= \frac{2\cosh x}{x^3\sinh x}[5.262... - x] < 0, \qquad \forall x = 3\pi, 5\pi... \end{aligned} \quad (2.23)$$

and for n even it holds

$$\begin{aligned} \frac{d}{dx}\left(U_0|_{n\pi=x}\right)^2 &= \frac{2\cosh x}{x^3\sinh x}\left[-3\coth x + \frac{2x^2}{\sinh x} + \frac{x^2+3}{\sinh x\cosh x} + \frac{2x}{\coth x} + x\right] \le \\ &\le \frac{2\cosh x}{x^3\sinh x}[3\coth\pi - x] = \\ &= \frac{2\cosh x}{x^3\sinh x}[3.011... - x] < 0, \qquad \forall x = 2\pi, 4\pi... \end{aligned} \quad (2.24)$$

Inequalities (2.23)-(2.24) yield that U_0 as function of n decays and

$$U_0|_{n=1} = 1.106701924...$$

From (2.22) one can easily see that U_0 as function of n is decreasing for $n \le 2$, besides we have

$$U_0 > 0, \qquad \lim_{n\to\infty} U_0 = 1, \qquad \max_n U_0 = U_0|_{n=2} = 1.115660759...$$

Hence, returning to (2.21), we see that

$$q_n = \frac{\|k_0\|_\infty}{n^3} = 1.944249575...$$

For the values of n such that $q_n < 1$ the FD-method can be divergent. For such n an other variant of the FD-method should be applied where the coefficients in the front of the low derivatives are approximated by a piece-wise constant functions on a grid with a maximal mesh-size h and the value of q_n can be controlled by h. This variant of our algorithm will be described in a subsequent paper. The assertions of Theorems 1.1, 1.3, 2.1 remain true where the value of q can be controlled by h.

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Received 18.09.2009