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EXACT THREE-POINT DIFFERENCE SCHEMES FOR NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS ON THE SEMIAXIS

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Анотація. Для чисельного розв'язування крайових задач на півпрямій для нелінійних звичайних диференціальних рівнянь другого порядку побудовано та обґрунтовано точну триточкову різницеву схему (ТТРС). За умов існування та єдиності розв'язку крайової задачі доведено існування та єдиність розв'язку ТТРС, а також збіжність методу послідовних наближень для її розв'язування.

Abstract. For the numerical solving of boundary value problems on the semiaxis for second order nonlinear ordinary differential equations the three-point exact difference scheme (EDS) is constructed and proved. On conditions of existence and uniqueness of the solution of boundary value problem the existence and uniqueness of the solution of EDS also convergence of iterative method of successive approximation for its solution is proved.

1. Introduction

The exact three-point difference scheme (EDS) and its algorithmic realization via three-point difference schemes of high order of accuracy for the problem

$$\frac{d^2u}{dx^2} - m^2u = -f(x,u), \quad x \in (0,\infty), \quad m = const > 0,$$
$$u(0) = \mu_1, \quad \lim_{x\to\infty} u(x) = 0$$

are development in the works [3, 4].

In this paper for nonlinear boundary value problem (BVP)

$$\begin{gathered} \frac{d^2u}{dx^2} + m^2\frac{du}{dx} = -\exp(-m^2x)f(x,u), \quad x \in (0,\infty), \quad m = const > 0, \\ u(0) = \mu_1, \quad \lim_{x\to\infty} u(x) = 0 \end{gathered} \tag{1.1}$$

the three-point EDS on the endless irregular grid

$$\hat{\bar{\omega}}_N = \{x_j,\ j = 0,1,\ldots,N,\ x_0 = 0\}$$

with exact nonlinear boundary condition on the right boundary end of the grid x_N has been developed.

Such EDS requires for each node x_j, $j = 0,1,\ldots,N-1$ of the grid $\hat{\omega}_N$ the two Cauchy's problems for nonlinear ordinary differential equations on the intervals $[x_{j-1}, x_j]$ (forward) and $[x_j, x_{j+1}]$ (backward) and nonlinear BVP on the interval $[x_N, \infty)$ to be solved.

Key words. Nonlinear ordinary differential equations, boundary value problem, three-point difference scheme, exact difference scheme.

2. BVP: existence and uniqueness of solutions

Let us introduce sufficient conditions for the existence and uniqueness of a solution of problem (1.1), that flow out of the method of linearization and principle of contraction mapping (see, e.g., [1, 2]).

Let us introduce the function $u^{(0)}(x) = \mu_1 \exp(-m^2 x)$ and the set

$$\Omega(D,\beta) = \left\{ u(x) : u(x) \in C^1[0,\infty),\ \left\| u - u^{(0)} \right\|_{1,\infty,D} \le \beta,\ D \subseteq [0,\infty) \right\},$$

$$\|u\|_{1,\infty,D} = \max\left\{ \|u\|_{0,\infty,D}\,,\ \left\| \frac{du}{dx} \right\|_{0,\infty,D} \right\}, \quad \|u\|_{0,\infty,D} = \max_{x\in D} |u(x)|\,.$$

Theorem 2.1 *Suppose that*

$$\begin{gathered} f_u(x) \equiv f(x,u) \in Q^0[0,\infty), \quad |f(x,u)| \le K, \\ \forall x \in [0,\infty),\ u \in \Omega([0,\infty), r), \quad r = \frac{K}{m^2} \max\left\{ \frac{1}{m^2}, 1 \right\}, \end{gathered} \tag{2.1}$$

$$|f(x,u) - f(x,v)| \le L\,|u - v| \quad \forall x \in [0,\infty),\ u, v \in \Omega([0,\infty), r), \tag{2.2}$$

$$q = \frac{L}{m^2} \max\left\{ \frac{1}{m^2}, 1 \right\} < 1, \tag{2.3}$$

then the BVP (1.1) has a unique solution $u(x) \in \Omega([0,\infty), r)$, that can be found by means of the method of successive approximations

$$\begin{gathered} \frac{d^2 u^{(k)}}{dx^2} + m^2 \frac{du^{(k)}}{dx} = -\exp(-m^2 x) f(x, u^{(k-1)}), \quad x \in (0,\infty), \\ u^{(k)}(0) = \mu_1, \quad \lim_{x\to\infty} u^{(k)}(x) = 0, \quad k = 1, 2, \ldots \end{gathered} \tag{2.4}$$

with the error estimate

$$\left\| u^{(k)} - u \right\|_{1,\infty,[0,\infty)} \le \frac{q^k}{1-q} r. \tag{2.5}$$

Here $Q^0[0,\infty)$ is the class of piecewise continuous functions with a finite number of discontinuity points of the first kind.

Proof. Let us write the problem (1.1) in equivalent integral form

$$u(x) = \mathrm{Re}\ (x, u(\cdot)) = \int_0^\infty G(x,\xi) \exp(-m^2 \xi) f(\xi, u(\xi)) d\xi + u^{(0)}(x), \quad x \ge 0, \tag{2.6}$$

where the Green function $G(x,\xi)$ of the problem (1.1) is

$$G(x,\xi) = \begin{cases} \dfrac{1 - \exp(-m^2 x)}{m^2}, & 0 \le x \le \xi, \\ \dfrac{\exp(m^2 \xi) - 1}{m^2} \exp(-m^2 x), & x \ge \xi. \end{cases}$$

So long as

$$\begin{aligned} |u(x)| &\le K \int_0^\infty G(x,\xi) \exp(-m^2 \xi) d\xi + |\mu_1| \exp(-m^2 x) = \\ &= \frac{K}{m^2} x \exp(-m^2 x) + |\mu_1| \exp(-m^2 x), \end{aligned}$$

function (2.6) satisfies boundary condition if $x \to \infty$.

Let us show that operator (2.6) transforms the set $\Omega([0,\infty),r)$ into itself. In response to equalities

$$\int_0^\infty G(x,\xi)\exp(-m^2\xi)d\xi = \frac{x\exp(-m^2x)}{m^2},$$

$$\int_0^\infty \frac{\partial G(x,\xi)}{\partial x}\exp(-m^2\xi)d\xi = \exp(-m^2x)\left(\frac{1}{m^2}-x\right)$$

we obtain

$$\begin{aligned}
\left\|\mathrm{Re}\,(x,v(\cdot)) - u^{(0)}\right\|_{1,\infty,[0,\infty)} &= \left\|\int_0^\infty G(x,\xi)\exp(-m^2\xi)f(\xi,v(\xi))d\xi\right\|_{1,\infty,[0,\infty)} \le \\
&\le K\left\|\int_0^\infty G(x,\xi)\exp(-m^2\xi)d\xi\right\|_{1,\infty,[0,\infty)} \le \\
&\le K\max\left\{\max_{x\in[0,\infty)}\left|\frac{x\exp(-m^2x)}{m^2}\right|, \max_{x\in[0,\infty)}\left|\exp(-m^2x)\left(\frac{1}{m^2}-x\right)\right|\right\} \le \\
&\le \frac{K}{m^2}\max\left\{\frac{1}{m^2},1\right\} = r \quad \forall v\in\Omega([0,\infty),r).
\end{aligned}$$

Moreover, $\mathrm{Re}\,(x,u(\cdot))$ is a contraction operator on the $\Omega([0,\infty),r)$, because of

$$\begin{aligned}
&\left\|\mathrm{Re}\,(x,u(\cdot)) - \mathrm{Re}\,(x,v(\cdot))\right\|_{1,\infty,[0,\infty)} = \\
&= \left\|\int_0^\infty G(x,\xi)\exp(-m^2\xi)\left[f(\xi,u(\xi)) - f(\xi,v(\xi))\right]d\xi\right\|_{1,\infty,[0,\infty)} \le \\
&\le L\left\|\int_0^\infty G(x,\xi)\exp(-m^2\xi)d\xi\right\|_{1,\infty,[0,\infty)} \|u-v\|_{1,\infty,[0,\infty)} \le \\
&\le \frac{L}{m^2}\max\left\{\frac{1}{m^2},1\right\}\|u-v\|_{1,\infty,[0,\infty)} = \\
&= q\,\|u-v\|_{1,\infty,[0,\infty)} \quad \forall u,v\in\Omega([0,\infty),r).
\end{aligned}$$

Thus, for operator $\mathrm{Re}\,(x,u(\cdot))$ as $q = \frac{L}{m^2}\max\left\{\frac{1}{m^2},1\right\} < 1$ all the conditions of the principle of contraction mapping have been performed, and the therefore the equation (2.6) has unique solution, that can be obtain via the method of successive approximations (2.4) with the error estimate (2.5). □

3. Existence of exact tree-point difference scheme

Let us introduce irregular grid on the interval $[0,\infty)$

$$\hat{\omega}_N = \Big\{x_j\in[0,\infty),\ j=0,1,\dots,N,\ x_0=0, h_j = x_j - x_{j-1} > 0,$$
$$h_1 + h_2 + \ldots + h_N = x_N\Big\}$$

in the way that the points of discontinuity of function $f(x,u)$ should coincide with the nodes of the grid. The set of all points of discontinuity we denote as ρ and suppose that N is so, that $\rho \subseteq \hat{\omega}_N$.

Due to [3, 4] let us restrict the steps h_j of the grid $\hat{\hat{\omega}}_N$ so, that

$$c_1 \le \frac{h_{\max}}{h_{\min}} \le c_2, \tag{3.1}$$

where c_1, c_2 are real constants. In order to achieve the maximum order of convergence of the difference scheme it is necessary that

$$\frac{1}{h_{\max}} \le x_N \le \frac{1}{h_{\min}}. \tag{3.2}$$

From the inequalities $h_{\min}N \le x_N = h_1 + h_2 + \ldots + h_N \le h_{\max}N$ and (3.2) we obtain the correlation:

$$h_{\min} \le \frac{1}{x_N} \le \frac{1}{Nh_{\min}}, \quad \frac{1}{Nh_{\max}} \le \frac{1}{x_N} \le h_{\max}.$$

According to (3.1) we will obtain

$$\frac{h_{\max}}{c_2} \le h_{\min} \le \frac{1}{\sqrt{N}}, \quad c_2 h_{\min} \ge h_{\max} \ge \frac{1}{\sqrt{N}}.$$

Hence

$$\begin{gathered} h_{\max} \le \frac{c_2}{\sqrt{N}}, \quad h_{\min} \ge \frac{1}{c_2\sqrt{N}}, \\ \frac{\sqrt{N}}{c_2} \le h_{\min}N \le x_N \le h_{\max}N \le c_2\sqrt{N}. \end{gathered} \tag{3.3}$$

Thus, we have $h_{\max} \to 0,\ x_N \to \infty$ as $N \to \infty$.

Let us introduce the set of grid functions

$$\Omega(\hat{\hat{\omega}}_N, \beta) = \left\{ v(x),\ x \in \hat{\hat{\omega}}_N : \left\| v - u^{(0)} \right\|^*_{1,\infty,\hat{\omega}_N} \le \beta \right\},$$

where

$$\|y\|^*_{1,\infty,\hat{\omega}_N} = \max\left\{ \|y\|_{0,\infty,\hat{\omega}_N}, \left\| \frac{dy}{dx} \right\|_{0,\infty,\hat{\hat{\omega}}_N} \right\}, \quad \|y\|_{0,\infty,\hat{\omega}_N} = \max_{0 \le j \le N} |y_j|.$$

Let us consider the BVPs

$$\begin{gathered} \frac{d^2 Y^j_\alpha(x,u)}{dx^2} + m^2 \frac{dY^j_\alpha(x,u)}{dx} = -\exp(-m^2 x) f(x, Y^j_\alpha(x,u)), \\ x_{j-2+\alpha} < x < x_{j-1+\alpha}, \\ Y^j_\alpha(x_{j-2+\alpha}, u) = u(x_{j-2+\alpha}), \quad Y^j_\alpha(x_{j-1+\alpha}, u) = u(x_{j-1+\alpha}), \\ j = 2-\alpha,\ 3-\alpha, \ldots, N+1-\alpha, \quad \alpha = 1, 2, \end{gathered} \tag{3.4}$$

$$\begin{gathered} \frac{d^2 Y^N_2(x,u)}{dx^2} + m^2 \frac{dY^N_2(x,u)}{dx} = -\exp(-m^2 x) f(x, Y^N_2(x,u)), \quad x > x_N, \\ Y^N_2(x_N, u) = u(x_N), \quad \lim_{x\to\infty} Y^N_2(x,u) = 0. \end{gathered} \tag{3.5}$$

Lemma 3.1 *Suppose that the assumptions (2.1)-(2.3) are satisfied, then the problems (3.4)-(3.5) have the unique solutions $Y_\alpha^j(x,u)$, $j=2-\alpha,3-\alpha,\ldots,N+1-\alpha$, $\alpha=1,2$, $Y_2^N(x,u)$. Furthermore, the solution of problem (1.1) can be represented in form*

$$\begin{aligned} u(x) =& Y_\alpha^j(x,u), \quad x\in[x_{j-2+\alpha},x_{j-1+\alpha}],\\ & j=2-\alpha,\ 3-\alpha,\ldots,N+1-\alpha, \quad \alpha=1,2,\\ u(x) =& Y_2^N(x,u), \quad x\in[x_N,\infty). \end{aligned} \tag{3.6}$$

Proof. Let us write the BVPs (3.4)-(3.5) in the equivalent form

$$\begin{gathered} Y_\alpha^j(x,u) = \int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}} G^{j-1+\alpha}(x,\xi)\exp(-m^2\xi)f(\xi,Y_\alpha^j(\xi,u))d\xi + \hat{u}(x), \\ x\in[x_{j-2+\alpha},x_{j-1+\alpha}], \quad j=2-\alpha,\ 3-\alpha,\ldots,N+1-\alpha, \quad \alpha=1,2, \end{gathered} \tag{3.7}$$

$$\begin{aligned} \hat{u}(x) =& \frac{\exp(-m^2x_{j-2+\alpha})-\exp(-m^2x)}{\exp(-m^2x_{j-2+\alpha})-\exp(-m^2x_{j-1+\alpha})}u(x_{j-1+\alpha}) + \\ &+ \frac{\exp(-m^2x)-\exp(-m^2x_{j-1+\alpha})}{\exp(-m^2x_{j-2+\alpha})-\exp(-m^2x_{j-1+\alpha})}u(x_{j-2+\alpha}), \end{aligned}$$

$$\begin{aligned} Y_2^N(x,u) =& \int\limits_{x_N}^{\infty} G^\infty(x,\xi)\exp(-m^2\xi)f(\xi,Y_2^N(\xi,u))d\xi + \\ &+ u_N\exp(-m^2(x-x_N)), \quad x\in[x_N,\infty), \end{aligned} \tag{3.8}$$

where

$$G^{j-1+\alpha}(x,\xi) = \begin{cases} \dfrac{\left(\exp(-m^2x_{j-2+\alpha})-\exp(-m^2x)\right)\left(1-\exp(-m^2(x_{j-1+\alpha}-\xi))\right)}{m^2\left(\exp(-m^2x_{j-2+\alpha})-\exp(-m^2x_{j-1+\alpha})\right)}, \\ \qquad\qquad x_{j-2+\alpha}\le x\le\xi, \\ \dfrac{\left(\exp(-m^2x)-\exp(-m^2x_{j-1+\alpha})\right)\left(\exp(m^2(\xi-x_{j-2+\alpha}))-1\right)}{m^2\left(\exp(-m^2x_{j-2+\alpha})-\exp(-m^2x_{j-1+\alpha})\right)}, \\ \qquad\qquad \xi\le x\le x_{j-1+\alpha}, \end{cases}$$

$$G^\infty(x,\xi) = \begin{cases} \dfrac{\exp(-m^2x_N)-\exp(-m^2x)}{m^2\exp(-m^2x_N)}, & x_N\le x\le\xi, \\ \dfrac{\exp(-m^2x)\left(\exp(m^2(\xi-x_N))-1\right)}{m^2\exp(-m^2x_N)}, & x\ge\xi. \end{cases}$$

As $\alpha=1$, we obtain

$$\begin{aligned} \hat{u}(x) =& \frac{\exp(-m^2x_{j-1})-\exp(-m^2x)}{\exp(-m^2x_{j-1})-\exp(-m^2x_j)}\times \\ &\times\left[\int\limits_0^\infty G(x_j,\xi)\exp(-m^2\xi)f(\xi,u)d\xi + \mu_1\exp(-m^2x_j)\right] + \\ &+ \frac{\exp(-m^2x)-\exp(-m^2x_j)}{\exp(-m^2x_{j-1})-\exp(-m^2x_j)}\times \\ &\times\left[\int\limits_0^\infty G(x_{j-1},\xi)\exp(-m^2\xi)f(\xi,u)d\xi + \mu_1\exp(-m^2x_{j-1})\right], \end{aligned}$$

$$u_N(x)\exp(-m^2(x-x_N)) =$$
$$= \exp(-m^2(x-x_N))\left[\int\limits_0^\infty G(x_N,\xi)\exp(-m^2\xi)f(\xi,u)d\xi + \mu_1\exp(-m^2x_N)\right].$$

Since

$$\frac{\exp(-m^2x_{j-1})-\exp(-m^2x)}{\exp(-m^2x_{j-1})-\exp(-m^2x_j)}\exp(-m^2x_j)+$$
$$+\frac{\exp(-m^2x)-\exp(-m^2x_j)}{\exp(-m^2x_{j-1})-\exp(-m^2x_j)}\exp(-m^2x_{j-1}) =$$
$$=\frac{\exp(-m^2(x+x_{j-1}))-\exp(-m^2(x+x_j))}{\exp(-m^2x_{j-1})-\exp(-m^2x_j)} = \exp(-m^2x),$$

we have

$$\hat{u}(x) = \frac{\exp(-m^2x_{j-1})-\exp(-m^2x)}{\exp(-m^2x_{j-1})-\exp(-m^2x_j)}\int\limits_0^\infty G(x_j,\xi)\exp(-m^2\xi)f(\xi,u)d\xi+$$
$$+\frac{\exp(-m^2x)-\exp(-m^2x_j)}{\exp(-m^2x_{j-1})-\exp(-m^2x_j)}\int\limits_0^\infty G(x_{j-1},\xi)\exp(-m^2\xi)f(\xi,u)d\xi + u^{(0)}(x),$$
$$x\in[x_{j-1},x_j],$$
$$u_N(x)\exp(-m^2(x-x_N)) =$$
$$= \exp(-m^2(x-x_N))\int\limits_0^\infty G(x_N,\xi)\exp(-m^2\xi)f(\xi,u)d\xi+$$
$$+\mu_1\exp(-m^2x), \quad x > x_N.$$

Then

$$Y_1^j(x,u) = \frac{\exp(-m^2x_{j-1})-\exp(-m^2x)}{\exp(-m^2x_{j-1})-\exp(-m^2x_j)}\int\limits_0^\infty G(x_j,\xi)\exp(-m^2\xi)f(\xi,u)d\xi+$$
$$+\frac{\exp(-m^2x)-\exp(-m^2x_j)}{\exp(-m^2x_{j-1})-\exp(-m^2x_j)}\int\limits_0^\infty G(x_{j-1},\xi)\exp(-m^2\xi)f(\xi,u)d\xi+$$
$$+\int\limits_{x_{j-1}}^{x_j} G^j(x,\xi)\exp(-m^2\xi)f(\xi,Y_1^j(\xi,u))d\xi + u^{(0)}(x), \quad x\in[x_{j-1},x_j],$$
$$Y_2^N(x,u) = \exp(-m^2(x-x_N))\int\limits_0^\infty G(x_N,\xi)\exp(-m^2\xi)f(\xi,u)d\xi+$$
$$+\int\limits_{x_N}^\infty G^\infty(x,\xi)\exp(-m^2\xi)f(\xi,Y_2^N(\xi,u))d\xi + u^{(0)}(x).$$

Due to equality $Y_2^j(x,u) = Y_1^{j+1}(x,u)$ we have

$$Y_2^j(x,u) = \frac{\exp(-m^2x_j) - \exp(-m^2x)}{\exp(-m^2x_j) - \exp(-m^2x_{j+1})} \int\limits_0^\infty G(x_{j+1},\xi)\exp(-m^2\xi)f(\xi,u)d\xi +$$
$$+ \frac{\exp(-m^2x) - \exp(-m^2x_{j+1})}{\exp(-m^2x_j) - \exp(-m^2x_{j+1})} \int\limits_0^\infty G(x_j,\xi)\exp(-m^2\xi)f(\xi,u)d\xi +$$
$$+ \int\limits_{x_j}^{x_{j+1}} G^{j+1}(x,\xi)\exp(-m^2\xi)f(\xi, Y_2^j(\xi,u))d\xi + u^{(0)}(x).$$

Thus, the question of existence and uniqueness of solution of problem (3.7)-(3.8) is equivalent to analogous problem for the equations

$$\begin{gathered}U_\alpha^j(x) = \Im_\alpha^j(x,u,U_\alpha^j) = \\
= \frac{\exp(-m^2x_{j-2+\alpha}) - \exp(-m^2x)}{\exp(-m^2x_{j-2+\alpha}) - \exp(-m^2x_{j-1+\alpha})} \int\limits_0^\infty G(x_{j-1+\alpha},\xi)\exp(-m^2\xi)f(\xi,u)d\xi + \\
+ \frac{\exp(-m^2x) - \exp(-m^2x_{j-1+\alpha})}{\exp(-m^2x_{j-2+\alpha}) - \exp(-m^2x_{j-1+\alpha})} \int\limits_0^\infty G(x_{j-2+\alpha},\xi)\exp(-m^2\xi)f(\xi,u)d\xi + \\
+ \int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}} G^{j-1+\alpha}(x,\xi)\exp(-m^2\xi)f(\xi, U_\alpha^j(\xi,u))d\xi + u^{(0)}(x), \\
x \in [x_{j-2+\alpha}, x_{j-1+\alpha}], \quad j = 2-\alpha,\ 3-\alpha, \ldots,\ N+1-\alpha, \quad \alpha = 1,2,\end{gathered} \tag{3.9}$$

$$\begin{gathered}U_2^N(x) = \Im_2^N(x,u,U_2^N) = \exp(-m^2(x-x_N)) \int\limits_0^\infty G(x_N,\xi)\exp(-m^2\xi)f(\xi,u)d\xi + \\
+ \int\limits_{x_N}^\infty G^\infty(x,\xi)\exp(-m^2\xi)f(\xi, U_2^N(\xi,u))d\xi + u^{(0)}(x).\end{gathered} \tag{3.10}$$

Let us show that the operators $\Im_1^j(x,u,U_1^j)$, $\Im_2^N(x,u,U_2^N)$ transform accordingly the sets $\Omega([x_{j-1},x_j],r)$, $\Omega([x_N,\infty),r)$ into themselves. Suppose $U_1^j(x) \in \Omega([x_{j-1},x_j],r)$, $U_2^\infty(x) \in \Omega([x_N,\infty),r)$. Then

$$\begin{gathered}\left|\Im_1^j(x,u,U_1^j) - u^{(0)}(x)\right| \le \\
\le K\left\{\frac{\exp(-m^2x) - \exp(-m^2x_j)}{\exp(-m^2x_{j-1}) - \exp(-m^2x_j)} \int\limits_0^\infty G(x_{j-1},\xi)\exp(-m^2\xi)d\xi + \right. \\
+ \frac{\exp(-m^2x_{j-1}) - \exp(-m^2x)}{\exp(-m^2x_{j-1}) - \exp(-m^2x_j)} \int\limits_0^\infty G(x_j,\xi)\exp(-m^2\xi)d\xi + \\
\left. + \int\limits_{x_{j-1}}^{x_j} G^j(x,\xi)\exp(-m^2\xi)d\xi\right\} = \frac{K}{m^2} x\exp(-m^2x),\end{gathered}$$

$$\left|\Im_2^N(x,u,U_2^N)-u^{(0)}(x)\right| \le K\left\{\exp(-m^2(x-x_N))\int\limits_0^\infty G(x_N,\xi)\exp(-m^2\xi)d\xi+\right.$$
$$\left.+\int\limits_{x_N}^\infty G^\infty(x,\xi)\exp(-m^2\xi)d\xi\right\}=\frac{K}{m^2}x\exp(-m^2x).$$

Besides

$$\left\|\Im_\alpha^j(x,u,U_\alpha^j)-\Im_\alpha^j(x,u,\tilde U_\alpha^j)\right\|_{1,\infty,[x_{j-2+\alpha},x_{j-1+\alpha}]}\le$$
$$\le L\left\|\int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}} G^{j-1+\alpha}(x,\xi)\exp(-m^2\xi)d\xi\right\|_{1,\infty,[0,\infty)}\left\|U_\alpha^j-\tilde U_\alpha^j\right\|_{1,\infty,[0,\infty)}.$$

Since

$$\int\limits_{x_{j-2+\alpha}}^{x_{j-1+\alpha}} G^{j-1+\alpha}(x,\xi)\exp(-m^2\xi)d\xi=\frac{\exp(-m^2x)-\exp(-m^2x_{j-1+\alpha})}{m^2\left(\exp(-m^2x_{j-2+\alpha})-\exp(-m^2x_{j-1+\alpha})\right)}\times$$
$$\times\int\limits_{x_{j-2+\alpha}}^{x}\left(\exp(-m^2x_{j-2+\alpha})-\exp(-m^2\xi)\right)d\xi+$$
$$+\frac{\exp(-m^2x_{j-2+\alpha})-\exp(-m^2x)}{m^2\left(\exp(-m^2x_{j-2+\alpha})-\exp(-m^2x_{j-1+\alpha})\right)}\times$$
$$\times\int\limits_{x}^{x_{j-1+\alpha}}\left(\exp(-m^2\xi)-\exp(-m^2x_{j-1+\alpha})\right)d\xi=$$
$$=\frac{x\exp(-m^2x)}{m^2}-x_{j-1+\alpha}\frac{\exp(-m^2x)-\exp(-m^2x_{j-2+\alpha})}{m^2\left(\exp(-m^2x_{j-2+\alpha})-\exp(-m^2x_{j-1+\alpha})\right)}-$$
$$-x_{j-2+\alpha}\frac{\exp(-m^2x_{j-1+\alpha})-\exp(-m^2x)}{m^2\left(\exp(-m^2x_{j-2+\alpha})-\exp(-m^2x_{j-1+\alpha})\right)}\le$$
$$\le\frac{x\exp(-m^2x)}{m^2},$$

$$\int\limits_{x_N}^\infty G^\infty(x,\xi)\exp(-m^2\xi)d\xi=\frac{\exp(-m^2x)}{m^2\exp(-m^2x_N)}\int\limits_{x_N}^{x}\left(\exp(-m^2x_N)-\exp(-m^2\xi)\right)d\xi+$$
$$+\frac{\exp(-m^2x_N)-\exp(-m^2x)}{m^2\exp(-m^2x_N)}\int\limits_x^\infty\exp(-m^2\xi)d\xi=$$
$$=\frac{\exp(-m^2x)\left(x-x_N\right)}{m^2}\le\frac{x\exp(-m^2x)}{m^2},$$

we have the estimate

$$\left\|\Im_\alpha^j(x,u,U_\alpha^j)-\Im_\alpha^j(x,u,\tilde U_\alpha^j)\right\|_{1,\infty,[x_{j-2+\alpha},x_{j-1+\alpha}]}\le$$
$$\le\frac{L}{m^2}\max\left\{\frac{1}{m^2},1\right\}\left\|U_\alpha^j-\tilde U_\alpha^j\right\|_{1,\infty,[x_{j-2+\alpha},x_{j-1+\alpha}]}=$$
$$=q\left\|U_\alpha^j-\tilde U_\alpha^j\right\|_{1,\infty,[x_{j-2+\alpha},x_{j-1+\alpha}]},$$

$$\left\| \Im_2^N(x,u,U_2^N) - \Im_2^N(x,u,\tilde{U}_2^N) \right\|_{1,\infty,[x_N,\infty)} \le q \left\| U_2^N - \tilde{U}_2^N \right\|_{1,\infty,[x_N,\infty)},$$

where $q = \dfrac{L}{m^2} \max\left\{ \dfrac{1}{m^2}, 1 \right\} < 1$.

Therefore, for the operators (3.9)-(3.10) in the areas $\Omega\left([x_{j-2+\alpha}, x_{j-1+\alpha}], r\right)$ and $\Omega\left([x_N,\infty), r\right)$ accordingly all the conditions of compressibility of reflections have implemented, and thus the problems (3.4) and (3.5) have unique solution. □

Theorem 3.2 *Suppose that the assumptions of Theorem 2.1 are satisfied. Then, for problem (1.1), there exist an EDS of the form*

$$(au_{\bar{x}})_{\hat{x},j} = -\hat{T}^{x_j}(f(\xi,u(\xi))), \qquad j = 1,2,\ldots,N-1, \tag{3.11}$$

$$u_0 = \mu_1, \quad -a_N u_{\bar{x},N} = \beta_2 u_N - \hat{\hat{T}}^{x_N}(f(\xi,u(\xi))), \tag{3.12}$$

where

$$u_{\bar{x},j} = \frac{u_j - u_{j-1}}{h_j}, \qquad u_{\hat{x},j} = \frac{u_{j+1} - u_j}{\hbar_j}, \qquad \hbar_j = \frac{h_j + h_{j+1}}{2},$$

$$a_j = \frac{m^2 h_j}{\exp(-m^2 x_{j-1}) - \exp(-m^2 x_j)}, \; j = 1,2,\ldots,N, \; \beta_2 = m^2 \exp(m^2 x_N),$$

$$\begin{aligned}
\hat{T}^{x_j}(f(\xi,u(\xi))) =& \frac{1}{\hbar_j \left(\exp(-m^2 x_j) - \exp(-m^2 x_{j+1})\right)} \times \\
&\times \int\limits_{x_j}^{x_{j+1}} \left(\exp(-m^2\xi) - \exp(-m^2 x_{j+1})\right) f(\xi,u(\xi)) d\xi + \\
&+ \frac{1}{\hbar_j \left(\exp(-m^2 x_{j-1}) - \exp(-m^2 x_j)\right)} \times \\
&\times \int\limits_{x_{j-1}}^{x_j} \left(\exp(-m^2 x_{j-1}) - \exp(-m^2\xi)\right) f(\xi,u(\xi)) d\xi, \\
& \qquad\qquad\qquad j = 1,2,\ldots,N-1,
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
\hat{\hat{T}}^{x_N}(f(\xi,u(\xi))) =& \exp(m^2 x_N) \int\limits_{x_N}^{\infty} \exp(-m^2\xi) f(\xi,u(\xi)) d\xi + \\
&+ \frac{1}{\exp(-m^2 x_{N-1}) - \exp(-m^2 x_N)} \int\limits_{x_{N-1}}^{x_N} \left(\exp(-m^2 x_{N-1}) - \exp(-m^2\xi)\right) f(\xi,u(\xi)) d\xi.
\end{aligned}$$

The function $u(\xi)$ on the right-hand side of (3.11)-(3.12) is given by (3.6) and depends only on $u_0, u_1, \ldots, u_N$.

Proof. The EDS for the problem

$$\begin{gathered}
\frac{d^2\tilde{u}}{dx^2} + m^2 \frac{d\tilde{u}}{dx} = -\exp(-m^2 x) f(x,\tilde{u}), \quad x \in (0, x_{N+1}), \\
\tilde{u}(0) = \mu_1, \qquad \tilde{u}(x_{N+1}) = 0
\end{gathered}$$

holds

$$\begin{gathered}
(a\tilde{u}_{\bar{x}})_{\hat{x}} = -\hat{T}^x(f(\xi,\tilde{u}(\xi))), \qquad x \in \omega_N^+, \\
\tilde{u}(0) = \mu_1, \qquad \tilde{u}(x_{N+1}) = 0,
\end{gathered} \tag{3.14}$$

$$a(x_j) = \frac{m^2 h_j}{\exp(-m^2 x_{j-1}) - \exp(-m^2 x_j)}.$$

Let us pass in (3.14) to the boundary as $x_{N+1} \to \infty$. Multiply the both parts of the equation

$$(a\tilde{u}_{\bar{x}})_{\hat{x},N} = -\hat{T}^{x_N}(f(\xi, \tilde{u}(\xi)))$$

to $\hbar_N = \frac{1}{2}(h_{N+1} + h_N)$, $h_{N+1} = x_{N+1} - x_N$ and pass to the boundary as $x_{N+1} \to \infty$, then we obtain (3.11) and (3.12). □

The existence of solution of nonlinear three-point EDS (3.11)-(3.12) has proved in the Theorem 3.2. And the following Lemma proves uniqueness of (3.11)-(3.12).

Lemma 3.3 *Assume that the conditions (2.1)-(2.3) are satisfied. Then, there exists a $h_0 > 0$ such that for all $|h| \le h_0$ and all functions $u \in \Omega(\hat{\bar{\omega}}_N, r)$ the EDS (3.11)-(3.12) have a unique solution which can be obtained by means of the method of successive approximations*

$$\begin{gathered}
(au^{(k)}_{\bar{x}})_{\hat{x},j} = -\hat{T}^{x_j}(f(\xi, u^{(k-1)}(\xi))), \qquad j = 1, 2, \dots, N-1, \\
u^{(k)}_0 = \mu_1, \quad -a(x_N)u^{(k)}_{\bar{x},N} = \beta_2 u^{(k)}_N - \hat{\hat{T}}^{x_N}(f(\xi, u^{(k-1)}(\xi))),
\end{gathered} \tag{3.15}$$

$$\begin{gathered}
u^{(k)}(x) = Y^j_\alpha(x, u^{(k)}), \quad x \in [x_{j-2+\alpha}, x_{j-1+\alpha}], \\
j = 2-\alpha,\ 3-\alpha, \dots, N+1-\alpha, \quad \alpha = 1, 2, \\
u^{(k)}(x) = Y^N_2(x, u^{(k)}), \quad x \in [x_N, \infty), \\
k = 1, 2, \dots, \quad u^{(0)}(x) = \mu_1 \exp(-mx)
\end{gathered}$$

with the estimate error

$$\left\| u^{(k)} - u \right\|^*_{1,\infty,\hat{\bar{\omega}}_N} = \max\left\{ \left\| u^{(k)} - u \right\|_{0,\infty,\hat{\bar{\omega}}_N}, \left\| \frac{du^{(k)}}{dx} - \frac{du}{dx} \right\|_{0,\infty,\hat{\bar{\omega}}_N} \right\} \le M q_1^k, \tag{3.16}$$

where $q_1 = q + M_1 |h| < 1,\ M,\ M_1$ are constants.

Proof. Let us use the principle of contraction mapping. The solution of problem (3.14) we writhe as

$$\begin{aligned}
\tilde{u}(x) = &\sum_{i=1}^{N} \hbar_i \tilde{G}^h(x, x_i) \exp(-m^2 x_i)\hat{T}^{x_i}\left(f(\eta, \tilde{u}(\eta))\right) + \\
&+ \frac{\exp(-m^2 x) - \exp(-m^2 x_{N+1})}{1 - \exp(-m^2 x_{N+1})}\mu_1 \quad \forall x \in \hat{\bar{\omega}}_N,
\end{aligned} \tag{3.17}$$

where $\tilde{G}^h(x, \xi)$ is a Green function of the problem (3.14), i.e.,

$$\tilde{G}^h(x, \xi) = \begin{cases} \dfrac{\left(1 - \exp(-m^2(x_{N+1} - \xi))\right)\left(1 - \exp(-m^2 x)\right)}{m^2\left(1 - \exp(-m^2 x_{N+1})\right)}, & 0 \le x \le \xi, \\[2ex] \dfrac{\left(1 - \exp(m^2 \xi)\right)\left(\exp(-m^2 x_{N+1}) - \exp(-m^2 x)\right)}{m^2\left(1 - \exp(-m^2 x_{N+1})\right)}, & \xi \le x \le x_{N+1}. \end{cases}$$

Let us pass in the equality (3.17) go to the boundary as $x_{N+1} \to \infty$ and obtain the solution of the problem (3.11)-(3.12)

$$\begin{aligned}
u(x) = \mathrm{Re}\,{}_h(x, u) = &\sum_{i=1}^{N-1} \hbar_i G(x, x_i) \exp(-m^2 x_i)\hat{T}^{x_i}(f(\eta, u(\eta))) + u^{(0)}(x) + \\
&+ G(x, x_N) \exp(-m^2 x_N)\hat{\hat{T}}^{x_N}(f(\eta, u(\eta))) \quad \forall x \in \hat{\bar{\omega}}_N.
\end{aligned}$$

Here $G(x,\xi)$ is the Green function of the problem (1.1).

Note, that the operators $\hat{T}^{x_i}, i=1,2,\ldots,N-1,\ \hat{\hat{T}}^{x_N}$ command the property: $\forall Q(\xi)\in H_h,\ w(\eta)\in L_1(0,\infty)$ (H_h is s space of grid functions $v(x),\ x\in\hat{\omega}_N$ such, that $v(0)=0$) we can write the identity:

$$\sum_{i=1}^{N-1}\hbar_i\hat{T}^{x_i}(w(\eta))\exp(-m^2x_i)Q(x_i)+\hat{\hat{T}}^{x_N}(w(\eta))\exp(-m^2x_N)Q(x_N)=$$
$$=\sum_{i=1}^{N}\frac{Q(x_i)\exp(-m^2x_i)}{\exp(-m^2x_{i-1})-\exp(-m^2x_i)}\int\limits_{x_{i-1}}^{x_i}\big(\exp(-m^2x_{i-1})-\exp(-m^2\eta)\big)\,w(\eta)d\eta+$$
$$+\sum_{i=1}^{N-1}\frac{Q(x_i)\exp(-m^2x_i)}{\exp(-m^2x_i)-\exp(-m^2x_{i+1})}\int\limits_{x_i}^{x_{i+1}}\big(\exp(-m^2\eta)-\exp(-m^2x_{i+1})\big)\,w(\eta)d\eta+$$
$$+Q(x_N)\exp(-m^2x_N)\int\limits_{x_N}^{\infty}\exp(m^2x_N)\exp(-m^2\eta)w(\eta)d\eta=$$

$$=\sum_{i=1}^{N}\int\limits_{x_{i-1}}^{x_i}\left\{\frac{Q(x_i)\exp(-m^2x_i)\big(\exp(-m^2x_{i-1})-\exp(-m^2\eta)\big)}{\exp(-m^2x_{i-1})-\exp(-m^2x_i)}+\right.$$
$$\left.+\frac{Q(x_{i-1})\exp(-m^2x_{i-1})\big(\exp(-m^2\eta)-\exp(-m^2x_i)\big)}{\exp(-m^2x_{i-1})-\exp(-m^2x_i)}\right\}w(\eta)d\eta+ \tag{3.18}$$
$$+Q(x_N)\int\limits_{x_N}^{\infty}\exp(-m^2\eta)w(\eta)d\eta.$$

According to (3.18), we obtain

$$\operatorname{Re}{}_h(x,u)=\sum_{i=1}^{N}\int\limits_{x_{i-1}}^{x_i}\left\{\frac{G(x,x_i)\exp(-m^2x_i)\big(\exp(-m^2x_{i-1})-\exp(-m^2\eta)\big)}{\exp(-m^2x_{i-1})-\exp(-m^2x_i)}+\right.$$
$$\left.+\frac{G(x,x_{i-1})\exp(-m^2x_{i-1})\big(\exp(-m^2\eta)-\exp(-m^2x_i)\big)}{\exp(-m^2x_{i-1})-\exp(-m^2x_i)}\right\}f\left(\eta,u(\eta)\right)d\eta+$$
$$+\frac{1-\exp(-m^2x)}{m^2}\int\limits_{x_N}^{\infty}\exp(-m^2\eta)f\left(\eta,u(\eta)\right)d\eta+u^{(0)}(x)=$$
$$=\sum_{i=1}^{N}\int\limits_{x_{i-1}}^{x_i}G(x,\eta)\exp(-m^2\eta)f(\eta,u(\eta))d\eta+$$
$$+\int\limits_{x_N}^{\infty}G(x,\eta)\exp(-m^2\eta)f(\eta,u(\eta))d\eta+u^{(0)}(x)=$$

$$=\int\limits_{0}^{\infty}G(x,\eta)\exp(-m^2\eta)f(\eta,u(\eta))d\eta+u^{(0)}(x),\quad x\in\hat{\omega}_N, \tag{3.19}$$

where

$$u(\eta) = Y_1^i(\eta, u), \quad \eta \in [x_{i-1}, x_i], \quad i = 1, 2, \ldots, N,$$
$$u(\eta) = Y_2^N(\eta, u), \quad \eta \in [x_N, \infty).$$

Let us investigate the properties of the operator $\mathrm{Re}\,_h(x, u)$. The operator (3.19) transforms the set $\Omega\left(\hat{\bar{\omega}}_N, r\right)$ into itself. Suppose $v \in \Omega\left(\hat{\bar{\omega}}_N, r\right)$, then

$$v(x) = Y_1^i(x, v) \in \Omega([x_{j-1}, x_j], r), \quad v(x) = Y_2^N(x, v) \in \Omega([x_N, \infty), r).$$

As

$$\int_0^\infty G(x, \eta) \exp(-m^2\eta) d\eta = \frac{x \exp(-m^2 x)}{m^2},$$
$$\int_0^\infty G_{\bar{x}}(x, \eta) \exp(-m^2\eta) d\eta = \frac{\exp(-m^2 x)\left(1 - m^2 x\right)}{m^2} - \frac{h(x)}{2} \exp(-m^2\xi)\left(m^2\xi - 2\right) \leq$$
$$\leq \frac{\exp(-m^2 x)\left(1 - m^2 x\right)}{m^2} + M\,|h|\,,$$

then

$$\left\|\mathrm{Re}\,_h(x, v) - u^{(0)}\right\|_{1,\infty,\hat{\bar{\omega}}_N} \leq K \left\| \int_0^\infty G(x, \eta) \exp(-m^2\eta) d\eta \right\|_{1,\infty,\hat{\bar{\omega}}_N} \leq$$
$$\leq r + M\,|h| \quad \forall v \in \Omega(\hat{\bar{\omega}}_N, r).$$

Moreover

$$\|\mathrm{Re}\,_h(x, u) - \mathrm{Re}\,_h(x, v)\|_{1,\infty,\hat{\bar{\omega}}_N} \leq (q + M\,|h|)\,\|u - v\|_{0,\infty,[0,\infty)} \quad \forall u, v \in \Omega(\hat{\bar{\omega}}_N, r). \tag{3.20}$$

Let us shown, that

$$\|u - v\|_{0,\infty,[0,\infty)} \leq (1 + M\,|h|)\,\|u - v\|_{0,\infty,\hat{\bar{\omega}}_N}\,. \tag{3.21}$$

For this purpose we will consider the BVPs

$$\frac{d^2u}{dx^2} + m^2\frac{du}{dx} = -\exp(-m^2 x) f(x, u), \quad x \in (x_{j-1}, x_j),$$
$$u(x_{j-1}) = u_{j-1}, \quad u(x_j) = u_j, \quad j = 1, 2, \ldots, N,$$
$$\frac{d^2u}{dx^2} + m^2\frac{du}{dx} = -\exp(-m^2 x) f(x, u), \quad x \in (x_N, \infty),$$
$$u(x_N) = u_N, \quad \lim_{x\to\infty} u(x) = 0.$$

This solutions should be written as

$$u(x) = \int_{x_{j-1}}^{x_j} G^j(x, \xi) \exp(-m^2\xi) f(\xi, u(\xi)) d\xi + \hat{u}(x), \quad x_{j-1} \leq x \leq x_j, \quad j = 1, 2, \ldots, N,$$
$$u(x) = \int_{x_N}^\infty G^\infty(x, \xi) \exp(-m^2\xi) f(\xi, u(\xi)) d\xi + u_N \exp(-m^2(x - x_N)), \quad x \geq x_N,$$

where

$$\hat{u}(x) = \frac{u(x_j)\left(\exp(-m^2 x_{j-1}) - \exp(-m^2 x)\right) + u(x_{j-1})\left(\exp(-m^2 x) - \exp(-m^2 x_j)\right)}{\exp(-m^2 x_{j-1}) - \exp(-m^2 x_j)},$$
$$x \in [x_{j-1}, x_j], \quad j = 1, 2, \ldots, N,$$

$$G^j(x,\xi) = \begin{cases} \dfrac{\left(\exp(-m^2 x_{j-1}) - \exp(-m^2 x)\right)\left(1 - \exp(m^2(\xi - x_j))\right)}{m^2\left(\exp(-m^2 x_{j-1}) - \exp(-m^2 x_j)\right)}, & x_{j-1} \le x \le \xi, \\ \dfrac{\left(\exp(-m^2 x) - \exp(-m^2 x_j)\right)\left(\exp(-m^2(x_{j-1} - \xi)) - 1\right)}{m^2\left(\exp(-m^2 x_{j-1}) - \exp(-m^2 x_j)\right)}, & \xi \le x \le x_j, \end{cases}$$
$$j = 1, 2, \ldots, N,$$

$$G^\infty(x,\xi) = \begin{cases} \dfrac{\exp(-m^2 x_N) - \exp(-m^2 x)}{m^2 \exp(-m^2 x_N)}, & x_N \le x \le \xi, \\ \dfrac{\exp(-m^2 x)\left(\exp(-m^2(x_N - \xi)) - 1\right)}{m^2 \exp(-m^2 x_N)}, & x \ge \xi. \end{cases}$$

Due to Lipschitz condition

$$\|u - v\|_{0,\infty,[x_{j-1},x_j]} \le L \left\| \int_{x_{j-1}}^{x_j} \exp(-m^2\xi) G^j(x,\xi) d\xi \right\|_{1,\infty,[x_{j-1},x_j]} \|u - v\|_{0,\infty,[x_{j-1},x_j]} +$$
$$+ \|\hat{u} - \hat{v}\|_{0,\infty,[x_{j-1},x_j]} \le \frac{L}{m^2} \left\| x \exp(-m^2 x) \right\|_{0,\infty,[x_{j-1},x_j]} \|u - v\|_{0,\infty,[x_{j-1},x_j]} +$$
$$+ \|u - v\|_{0,\infty,\hat{\bar{\omega}}_N} \le \|u - v\|_{0,\infty,\hat{\bar{\omega}}_N} + |h|\, q \, \|u - v\|_{0,\infty,[x_{j-1},x_j]}, \quad j = 1, 2, \ldots, N,$$

$$\|u - v\|_{0,\infty,[x_N,\infty)} \le$$
$$\le L \|u - v\|_{0,\infty,[x_N,\infty)} \left\| \int_{x_N}^{\infty} \exp(-m^2\xi) G^\infty(x,\xi) d\xi \right\|_{0,\infty[x_N,\infty)} + |u(x_N) - v(x_N)| =$$
$$= \frac{L}{m^2} \left\| (x - x_N) \exp(-m^2 x) \right\|_{0,\infty,[x_N,\infty)} \|u - v\|_{0,\infty,[x_N,\infty)} + \|u - v\|_{0,\infty,\hat{\bar{\omega}}_N} \le$$
$$\le \|u - v\|_{0,\infty,\hat{\bar{\omega}}_N} + |h|\, q \, \|u - v\|_{0,\infty,[x_N,\infty)}.$$

Hence, we obtain the estimates

$$\|u - v\|_{0,\infty,[x_{j-1},x_j]} \le \frac{1}{1 - |h|\, q} \|u - v\|_{0,\infty,\hat{\bar{\omega}}_N} \le (1 + |h|\, M_2) \|u - v\|_{0,\infty,\hat{\bar{\omega}}_N},$$
$$\|u - v\|_{0,\infty,[x_{N-1},\infty)} \le \frac{1}{1 - |h|\, q} \|u - v\|_{0,\infty,\hat{\bar{\omega}}_N} \le (1 + |h|\, M_3) \|u - v\|_{0,\infty,\hat{\bar{\omega}}_N},$$

that follows the inequality (3.21).

As long as

$$\left.\frac{du}{dx}\right|_{x=x_0} = a_1 u_{\bar{x},1} + \frac{1}{1-\exp(-m^2 x_1)} \int\limits_0^{x_1} \left(\exp(-m^2\xi) - \exp(-m^2 x_1)\right) f\left(\xi, u(\xi)\right) d\xi,$$

$$\left.\frac{du}{dx}\right|_{x=x_j} = a_j \exp(m^2 x_j) u_{\bar{x},j} + $$

$$+ \frac{1}{1-\exp(m^2 h_j)} \int\limits_{x_{j-1}}^{x_j} \left(\exp(-m^2 x_{j-1}) - \exp(-m^2\xi)\right) f\left(\xi, u(\xi)\right) d\xi,$$

$$j = 1, 2, \ldots, N,$$

then, taking into account (3.21) we will have

$$\left\|\frac{du}{dx} - \frac{dv}{dx}\right\|_{0,\infty,\hat{\bar{\omega}}_N} \le M_1 \left\|u_{\bar{x}} - v_{\bar{x}}\right\|_{0,\infty,\hat{\bar{\omega}}_N} + L\left\|u-v\right\|_{0,\infty,[0,\infty)} \le \le M\left\|u-v\right\|_{1,\infty,\hat{\bar{\omega}}_N}. \tag{3.22}$$

Taking into account (3.21) from the estimate (3.20) we obtain

$$\left\|\text{Re}\ _h(x,u) - \text{Re}\ _h(x,v)\right\|_{1,\infty,\hat{\bar{\omega}}_N} \le (q + M_1|h|)\left\|u-v\right\|_{1,\infty,\hat{\bar{\omega}}_N} = = q_1\left\|u-v\right\|_{1,\infty,\hat{\bar{\omega}}_N}.$$

Because of (2.3) $q<1$, then $q_1<1$ as h_0 is sufficiently little and operator (3.19) for all $u, v \in \Omega(\hat{\bar{\omega}}_N, r)$ implements the contraction mapping. Thus, according to the principle of contraction mapping and if h_0 is enough little the EDS (3.11)-(3.12) has a unique solution, that can be obtained via the method of successive approximations (3.15 with the error estimate

$$\left\|u^{(k)} - u\right\|_{1,\infty,\hat{\bar{\omega}}_N} \le \frac{q_1^k}{1-q_1} r. \tag{3.23}$$

Moreover, due to (3.22)-(3.23)

$$\left\|\frac{du^{(k)}}{dx} - \frac{du}{dx}\right\|_{0,\infty,\hat{\bar{\omega}}_N} \le M_1\left\|u^{(k)} - u\right\|_{1,\infty,\hat{\bar{\omega}}_N} \le M q_1^k.$$

So, the estimate (3.16) is true. □

Lemma 3.4 *Assume the existence of the constant $\Delta > 0$ and it is so, that the conditions (2.1) and (2.2) come true in the area $\Omega([0,1],\ r+\Delta)$. Then there exists such $h_0 > 0$, that as $|h| \le h_0$ and for all $(v_j)_{j=0}^N \in \Omega\left(\hat{\bar{\omega}}_N, r\right)$ the problems*

$$\begin{gathered} \frac{d^2 Y_\alpha^j(x,v)}{dx^2} + m^2\frac{dY_\alpha^j(x,v)}{dx} = -\exp(-m^2 x) f(x, Y_\alpha^j(x,v)), \\ x_{j-2+\alpha} < x < x_{j-1+\alpha}, \\ Y_\alpha^j(x_{j+(-1)^\alpha}, v) = v(x_{j+(-1)^\alpha}), \quad \frac{dY_\alpha^j(x_{j+(-1)^\alpha}, v)}{dx} = \left.\frac{dv}{dx}\right|_{x=x_{j+(-1)^\alpha}}, \\ j = 2-\alpha, 3-\alpha, \ldots, N+1-\alpha, \quad \alpha = 1,2, \end{gathered} \tag{3.24}$$

$$\begin{gathered} \frac{d^2 Y_2^N(x,v)}{dx^2} + m^2\frac{dY_2^N(x,v)}{dx} = -\exp(-m^2 x) f(x, Y_2^N(x,v)), \qquad x > x_N, \\ Y_2^N(x_N, v) = v(x_N), \quad \lim_{x\to\infty} Y_2^N(x,v) = 0 \end{gathered} \tag{3.25}$$

will have a unique solution.

Proof. The problems (3.24)-(3.25) are equivalent to the operator equations

$$U_\alpha^j(x) = \mathrm{Re}\,{}_\alpha^j(x, v, U_\alpha^j) = -\frac{1}{m^2} \int\limits_{x_{j+(-1)^\alpha}}^{x} \big(\exp(-m^2\xi) - \exp(-m^2x)\big)\, f(\xi, U_\alpha^j) d\xi +$$
$$+ v_{j+(-1)^\alpha} + \frac{1-\exp(-m^2(x - x_{j+(-1)^\alpha}))}{m^2} \left.\frac{dv}{dx}\right|_{x=x_{j+(-1)^\alpha}},$$
$$x \in [x_{j-2+\alpha}, x_{j-1+\alpha}], \quad j = 2-\alpha, 3-\alpha, \ldots, N+1-\alpha, \quad \alpha = 1, 2,$$
$$U_2^N(x) = \mathrm{Re}\,{}_2^N(x, v, U_2^N) = \int\limits_{x_N}^{\infty} G^\infty(x,\xi) \exp(-m^2\xi) f(\xi, U_2^N(\xi, v)) d\xi +$$
$$+ v_N \exp(-m^2(x - x_N)), \qquad x \in [x_N, \infty),$$

where

$$G^\infty(x,\xi) = \begin{cases} \dfrac{\exp(-m^2x_N) - \exp(-m^2x)}{m^2\exp(-m^2x_N)}, & x_N \le x \le \xi, \\[2mm] \dfrac{\exp(-m^2x)\big(\exp(-m^2(x_N - \xi)) - 1\big)}{m^2\exp(-m^2x_N)}, & x \ge \xi. \end{cases}$$

Let us investigate the properties of operators $\mathrm{Re}\,{}_\alpha^j(x, v, U_\alpha^j)$, $\alpha = 1, 2$, $\mathrm{Re}\,{}_2^N(x, v, U_2^N)$. Note, that

$$u^{(0)}(x) = \mu_1 \exp(-m^2x) = u_{j+(-1)^\alpha}^{(0)} + \frac{1-\exp(-m^2(x - x_{j+(-1)^\alpha}))}{m^2} \left.\frac{du^{(0)}}{dx}\right|_{x=x_{j+(-1)^\alpha}},$$
$$u^{(0)}(x) = u_N^{(0)} \exp(-m^2(x - x_N)).$$

Suppose, $U_\alpha^j \in \Omega\left([x_{j-2+\alpha}, x_{j-1+\alpha}],\ r+\Delta\right)$, $U_2^N \in \Omega\left([x_N, \infty),\ r+\Delta\right)$, then

$$\left\|\mathrm{Re}\,{}_\alpha^j(x, v, U_\alpha^j) - u^{(0)}\right\|_{1,\infty,[x_{j-2+\alpha}, x_{j-1+\alpha}]} \le \Bigg\| u_{j+(-1)^\alpha}^{(0)} - v_{j+(-1)^\alpha} +$$
$$+ \frac{1-\exp(-m^2(x - x_{j+(-1)^\alpha}))}{m^2} \left(\left.\frac{du^{(0)}}{dx}\right|_{x=x_{j+(-1)^\alpha}} - \left.\frac{dv}{dx}\right|_{x=x_{j+(-1)^\alpha}} \right) +$$
$$+ \frac{K}{m^2} \int\limits_{x_{j+(-1)^\alpha}}^{x} \big(\exp(-m^2\xi) - \exp(-m^2x)\big) d\xi \Bigg\|_{1,\infty,[x_{j-2+\alpha}, x_{j-1+\alpha}]} \le$$
$$\le r \Bigg\| 1 + \frac{1-\exp(-m^2(x - x_{j+(-1)^\alpha}))}{m^2} +$$
$$+ \exp(-m^2 x_{j+(-1)^\alpha}) \exp(-m^2x) -$$
$$- m^2\exp(-m^2x)\left(x - x_{j+(-1)^\alpha}\right) \Bigg\|_{1,\infty,[x_{j-2+\alpha}, x_{j-1+\alpha}]} =$$
$$= r \Bigg\| 1 + \frac{1-\exp(-m^2(x - x_{j+(-1)^\alpha}))}{m^2} +$$
$$+ m^4 \frac{(x - x_{j+(-1)^\alpha})}{2} \exp(-m^2\xi) \Bigg\|_{1,\infty,[x_{j-2+\alpha}, x_{j-1+\alpha}]} \le$$
$$\le r + \Delta, \qquad \xi = \theta x_{j+(-1)^\alpha} + (1-\theta)x, \quad 0 < \theta < 1,$$

$$\left\|\mathrm{Re}\,{}_2^N(x,v,U_2^N)-u^{(0)}\right\|_{1,\infty,[x_N,\infty)} \le \left\| \left|v_N - u_N^{(0)}\right| \exp(-m^2(x-x_N))+ + K\int\limits_{x_N}^{\infty} G^{\infty}(x,\xi)\exp(-m^2\xi)d\xi \right\|_{1,\infty,[x_N,\infty)} \le$$

$$\le r\left\|\exp(-m^2(x-x_N)) + m^2(x-x_N)\exp(-m^2x)\right\|_{1,\infty,[x_N,\infty)} \le$$
$$\le r\left\|\exp(-m^2(x-x_N))\left(1+m^2(x-x_N)\right)\right\|_{1,\infty,[x_N,\infty)} \le$$
$$\le r\left\|\exp(-m^2(x-x_N))\exp(m^2(x-x_N))\right\|_{1,\infty,[x_N,\infty)} = r, \quad \forall v\in\Omega(\hat{\hat{\omega}}_N, r),$$

i.e. the operators $\mathrm{Re}\,{}_\alpha^j(x,v,U_\alpha^j)$, $\alpha = 1,2$, $\mathrm{Re}\,{}_2^N(x,v,U_2^N)$ transform the sets

$$\Omega\left([x_{j-2+\alpha}, x_{j-1+\alpha}],\ r+\Delta\right), \quad \Omega\left([x_N,\infty),\ r+\Delta\right)$$

into themselves accordingly.

Moreover

$$\left\|\mathrm{Re}\,{}_\alpha^j(x,v,U_\alpha^j) - \mathrm{Re}\,{}_\alpha^j(x,v,\tilde{U}_\alpha^j)\right\|_{1,\infty,[x_{j-2+\alpha},x_{j-1+\alpha}]} \le$$
$$\le \frac{L}{m^2}\left\|U_\alpha^j - \tilde{U}_\alpha^j\right\|_{1,\infty,[x_{j-2+\alpha},x_{j-1+\alpha}]} \times$$
$$\times \left\| \int\limits_{x_{j+(-1)^\alpha}}^{x} \left(\exp(-m^2\xi) - \exp(-m^2x)\right)d\xi \right\|_{1,\infty,[x_{j-2+\alpha},x_{j-1+\alpha}]} \le$$

$$\le q\,|h| \left\| \frac{(x-x_{j+(-1)^\alpha})^2}{2}\exp(-m^2\xi)\right\|_{1,\infty,[x_{j-2+\alpha},x_{j-1+\alpha}]} \left\|U_\alpha^j - \tilde{U}_\alpha^j\right\|_{1,\infty,[x_{j-2+\alpha},x_{j-1+\alpha}]} \le$$
$$\le q\,|h| \left\|U_\alpha^j - \tilde{U}_\alpha^j\right\|_{1,\infty,[x_{j-2+\alpha},x_{j-1+\alpha}]}, \quad \xi = \theta x_{j+(-1)^\alpha} + (1-\theta)x, \quad 0<\theta<1,$$

$$\left\|\mathrm{Re}\,{}_2^N(x,v,U_2^N) - \mathrm{Re}\,{}_2^N(x,v,\tilde{U}_2^N)\right\|_{1,\infty,[x_N,\infty)} \le$$
$$\le L \left\|\int\limits_{x_N}^{\infty} G^{\infty}(x,\xi)\exp(-m^2\xi)d\xi\right\|_{1,\infty,[x_N,\infty)} \left\|U_2^N - \tilde{U}_2^N\right\|_{1,\infty,[x_N,\infty)} \le$$
$$\le q\left\|(x-x_N)\exp(-m^2x)\right\|_{1,\infty,[x_N,\infty)} \left\|U_2^N - \tilde{U}_2^N\right\|_{1,\infty,[x_N,\infty)} \le$$
$$\le q\left\|U_2^N - \tilde{U}_2^N\right\|_{1,\infty,[x_N,\infty)}.$$

As long as from (2.3) flows out that $q<1$ and $\tilde{q} = q\,|h| < 1$ as h_0 is enough little, the operators $\mathrm{Re}\,{}_\alpha^j(x,v,U_\alpha^j)$, $\alpha = 1,2$, $\mathrm{Re}\,{}_2^N(x,v,U_2^N)$ implement the contraction mapping. Therefore, due to the principle of contraction mapping, as h_0 is sufficiently little as problems (3.24)-(3.25) will have a unique solution. □

Since

$$\int\limits_{x_j}^{x_{j+(-1)^\alpha}} \left(\exp(-m^2\xi) - \exp(-m^2x_{j+(-1)^\alpha})\right) f(\xi,u)d\xi =$$
$$= \left(1-\exp(-m^2(x_{j+(-1)^\alpha} - x_j))\right) Z_\alpha^j(x_j,u) + m^2\left(Y_\alpha^j(x_j,u) - u_{j+(-1)^\alpha}\right), \quad \alpha = 1,2,$$

$$\int\limits_{x_N}^{\infty} \exp(-m^2\xi) f(\xi, Y_2^N(\xi,u)) d\xi = Z_2^N(x_N,u) + m^2 u_N,$$

then

$$\varphi(x_j,u) = \hat{T}^{x_j}(f(\xi,u(\xi))) =$$
$$= \frac{1}{\hbar_j}\Bigg[\exp(-m^2 x_j)\left(Z_2^j(x_j,u) - Z_1^j(x_j,u)\right) +$$
$$+ m^2 \frac{Y_1^j(x_j,u) - u_{j-1}}{\exp(-m^2 x_{j-1}) - \exp(-m^2 x_j)} + m^2 \frac{Y_2^j(x_j,u) - u_{j+1}}{\exp(-m^2 x_j) - \exp(-m^2 x_{j+1})}\Bigg],$$
$$\alpha = 1,2,$$

$$\mu_2(x_N,u) = \hat{\hat{T}}^{x_N}(f(\xi,u(\xi))) =$$
$$= \exp(m^2 x_N)\left(Z_2^N(x_N,u) - Z_1^N(x_N,u)\right) +$$
$$+ m^2 \exp(m^2 x_N) u_N + \frac{Y_1^N(x_N,u) - u_{N-1}}{\exp(-m^2 x_{N-1}) - \exp(-m^2 x_N)},$$

where $Y_\alpha^j(x_j,u)$, $Z_\alpha^j(x_j,u)$, $\alpha = 1,2$ – are the solutions of Cauchy problems

$$\frac{dY_\alpha^j(x,u)}{dx} = Z_\alpha^j(x,u),$$

$$\frac{dZ_\alpha^j(x,u)}{dx} + m^2 Z_\alpha^j(x,u) = -\exp(-m^2 x) f(x, Y_\alpha^j(x,u)),$$

$$x_{j-2+\alpha} < x < x_{j-1+\alpha}, \tag{3.26}$$

$$Y_\alpha^j(x_{j+(-1)^\alpha},u) = u_{j+(-1)^\alpha}, \quad Z_\alpha^j(x_{j+(-1)^\alpha},u) = \left.\frac{du}{dx}\right|_{x=x_{j+(-1)^\alpha}},$$

$$j = 2-\alpha, 3-\alpha, \ldots, N+1-\alpha, \quad \alpha = 1,2,$$

and $Y_2^N(x_N,u)$, $Z_2^N(x_N,u)$ – are the solutions of the BVP

$$\frac{dY_2^N(x,u)}{dx} = Z_2^N(x,u),$$

$$\frac{dZ_2^N(x,u)}{dx} + m^2 Z_2^N(x,u) = -\exp(-m^2 x) f(x, Y_2^N(x,u)), \qquad x > x_N, \tag{3.27}$$

$$Y_2^N(x_N,u) = u_N, \quad \lim_{x\to\infty} Y_2^N(x,u) = 0.$$

If the problems (3.26)-(3.27) solved numerically, it could be possible to develop the truncated difference scheme.

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