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SOME COMPUTATIONAL ASPECTS OF ALGORITHMS FOR SOLVING NONLINEAR TWO-PARAMETER EIGENVALUE PROBLEMS

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Анотація. Розглядаються обчислювальні аспекти використання ефективної чисельної процедури обчислення похідних детермінанта матриці у алгоритмах знаходження кривих власних значень та точок біфуркації нелінійних двопараметричних спектральних задач. Наведено чисельні результати.

Abstract. The computational aspects of the use of efficient numerical procedure to calculate derivatives of matrix determinant in the algorithms of finding the eigenvalue curves and bifurcation points of nonlinear two-parameter eigenvalue problems are considered. Some numerical examples are given.

1. Introduction

The multiparameter eigenvalue problem

$$T(\lambda)u \equiv T(\lambda_1, \lambda_2, \ldots, \lambda_m)u = 0$$

with the operator-function $T(\lambda): \ R^m \to L(H)$ ($L(H)$ is a set of the linear operators of some Hilbert space H), which nonlinearly depends on a several spectral parameters $\lambda_1, \lambda_2, \ldots, \lambda_m$ arises in many fields of analysis and mathematical physics. Such problems play important role when one investigates, in particular, the branching of solutions of nonlinear integral equations of Hammerstein type. So, when it is necessary to find the solutions of nonlinear integral equations, which we obtain as a result of solving of synthesis problem [1], two-parameter eigenvalue problems with nonlinear spectral parameters which are included analytically in the kernel of a linearized operator, arise. These problems are not enough investigated both from the theoretical point of view, and from the point of view of construction of numerical methods of their solution.

Essential difference already of two-parameter problems

$$T(\lambda, \mu)u = 0 \tag{1.1}$$

from the one-parameter problems consists in that even a linear two-parameter problem

$$Au - \lambda Bu - \mu Cu = 0 \tag{1.2}$$

where A, B, C are some linear operators, can not generally have solutions or, conversely, to have their as a continuum set which in the case of real parameters is the eigenvalue curves [2].

At present there are many open questions related to this problem, for example, such as the existence of solutions and their number and multiple, and also development of numerical methods for solving such eigenvalue problems.

Key words. Nonlinear two-parameter eigenvalue problem, derivatives of matrix determinant, numerical algorithm, eigenvalue curves, bifurcation points.

There are few approaches to numerical solution of the problems (1.1)-(1.2). For separate differential equations some authors [3-5] use a set of transformations and obtain the transcendental equations for calculation of the eigenvalue curves of the equation (1.2), which can be obtained by numerical methods (for examples, as in [4]).

Other approach, which gives an approximate solution of the problem (1.2), uses the perturbation theory [6]. But this method can give good approximations to the solution of the problem (1.2) only for a small region of parameter.

One more approach is offered in the work [7]. After calculation of simple eigenvalue λ_1 and corresponding eigenvector of the matrix problem (1.2) for some value of the parameter μ_1 by one of standard methods, by means of the methods of continuation over the parameter, using the theorem of implicit function, the eigenvalue curve, on which the eigenvalue lies, is calculated. The theorem of implicit function is used also in the paper [9]. But at such approach, if the eigenvalue curve at some point has an algebraic singularities – the limit point or the bifurcation point then in the neighborhood of such points an algorithm stops to work. For this case in [7], and also in [8] the bifurcation equations for calculation of branching of simple eigenvalue curves are obtained.

Another approach this is reduction of the problem (1.2) to the successive solution of one-parameter problems, assigning a certain value to one of parameters. Such approach for matrix problems with usage of the numerical procedure for calculation of derivatives of matrix determinant allows to define the quantity of eigenvalues and their multiple in some interval of change of parameter, and consequently, to build all curves which belong to the region of change of spectral parameters.

Since by discretization of differential or integral operators the problem can be reduced to a finite-dimensional problem, then further we will consider the matrix two-parameter eigenvalue problem

$$\mathbf{T}(\lambda, \mu)\mathbf{u} = 0 \tag{1.3}$$

with the square matrix $\mathbf{T}(\lambda, \mu)$ of dimension $n \times n$, the elements of which nonlinearly depend on the parameters λ and μ, $\mathbf{u} \in R^n$. The problem is to find such values of parameters λ and μ, for which the problem (1.3) would have nontrivial solution. Obviously, in order that the problem (1.3) should have different from zero solution, it is necessary, that

$$f(\lambda, \mu) \equiv \det \mathbf{T}(\lambda, \mu) = 0, \tag{1.4}$$

i.e. the eigenvalues of the problem (1.3), are zeros of the function $f(\lambda, \mu)$.

2. Algorithm of finding eigenvalue curves

We shall replace in the problem (1.3), for example, the parameter μ by the expression $\mu = \alpha\lambda + \beta$ and we shall consider the appropriate one-parameter problem

$$\mathbf{T}(\lambda)u \equiv \mathbf{T}(\lambda,\, \alpha,\, \beta)u = 0 \tag{2.1}$$

for the given fixed value of values α and β. Here $\mathbf{T}(\lambda)$ is the real matrix, the elements of which nonlinearly depend on the parameter λ. The equation (1.4) appropriate to (2.1), has a form

$$f(\lambda) \equiv \det\, \mathbf{T}(\lambda) = 0 \tag{2.2}$$

It is obvious that if λ is the solution of the equation (2.2) then (λ, $\mu = \alpha\lambda + \beta$) is the eigenvalue of the problem (1.3). Therefore, solving the equation (2.2) for a sequence of different values α and β, we obtain the single-valued part of eigenvalue curve $\mu(\lambda)$ of the problem (1.3).

Using such an approach, we can define the quantity of zeros (m) of function, and consequently, the quantity of eigenvalues of the problem (2.1), which belong to the given interval of change of the value of parameter λ, for example, $\lambda \in [\lambda_{c_k}, \lambda_{d_k}] \subset R$ and to calculate them (taking into account its multiple).

If to continue λ in the region of complex variables then formulae which give the solution of this problem are known:

$$m = s_0 = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{f'(\lambda)}{f(\lambda)} d\lambda, \tag{2.3}$$

$$\sum_{j=1}^{m} (\lambda_j)^k = s_k\,, \quad k = 1,\ \dots\ , m, \tag{2.4}$$

where

$$s_k = \frac{1}{2\pi i} \int\limits_{\Gamma} \lambda^k \frac{f'(\lambda)}{f(\lambda)} d\lambda\,, \quad k = 0, 1,\ \dots\ , m, \tag{2.5}$$

and Γ is the boundary of some region G of a complex plane, which does not contain zeros of the function $f(\lambda)$.

Thus, putting the interval $[\lambda_{c_k}, \lambda_{d_k}]$ in the region G, for example, the circle with the point $r_{0_t} = (\lambda_{c_t} + \lambda_{d_t})/2$ as centre and the radius $\rho_t = (\lambda_{d_t} - \lambda_{c_t})/2$, from the system (2.4) we find all eigenvalues of the problem (2.1), which belong to the region G, that is, to the given interval $[\lambda_{c_k}, \lambda_{d_k}]$. Integrals in the formulae (2.3) and (2.5) are replaced by an approximated quadrature formula, for example, the rectangular formula, at the points N_Γ on Γ, and as Γ is a circle, for $s_k\,,\ \ k = 0, 1, 2, ...,$ we obtain relations

$$s_k = \frac{1}{N_\Gamma} \sum_{j=1}^{N_\Gamma} (\lambda_j)^k \rho_t \exp\left(i \frac{2\pi j}{N_\Gamma}\right) \frac{f'(\lambda)}{f(\lambda)}, \tag{2.6}$$

where $\lambda_j = r_{0_t} + \rho_t \exp\left(i\frac{2\pi j}{N_\Gamma}\right)$, and system (2.4) we solve by Newton's method, selecting the initial approximations on the boundary Γ of the region G, i.e.

$$\lambda_j^{(0)} = r_{0_t} + \rho_t \exp\left(i \frac{2\pi j}{m}\right), \quad j = 1, 2,\ \dots\ , m.$$

A principle of argument (2.3) and the formulas such as a principle of argument (2.4)-(2.5) repeatedly were applied at solution of different spectral problems (see, for example, [10-13]), but the feature of algorithm, which is offered further, consists in calculation the values of the function $f(\lambda)$ and its derivatives on the basis of LU-decomposition of matrix $T(\lambda)$:

$$f(\lambda_l) = (-1)^q \prod_{i=1}^{N} u_{ii}, \qquad f'(\lambda_l) = (-1)^q \sum_{k=1}^{N} v_{kk} \prod_{i=1, i\neq k}^{N} u_{ii},$$

$$f''(\lambda_l) = (-1)^q \sum_{k=1}^{N} w_{kk} \prod_{i=1, i\neq k}^{N} u_{ii} + (-1)^q \sum_{k=1}^{N} v_{kk} \left(\sum_{j=1, j\neq k}^{N} v_{jj} \prod_{i=1, i\neq k, i\neq j}^{N} u_{ii} \right), \tag{2.7}$$

where $u_{ii},\ v_{ii},\ w_{ii}$ are the diagonal elements of matrices $\mathbf{U},\ \mathbf{V},\ \mathbf{W}$ in the decompositions

$$\begin{array}{l} \mathbf{PT} = \mathbf{LU} \\ \mathbf{PB} = \mathbf{MU} + \mathbf{LV} \\ \mathbf{PC} = \mathbf{NU} + 2\mathbf{MV} + \mathbf{LW} \end{array}, \tag{2.8}$$

for the fixed value λ_l. Here $\mathbf{P}$ is the permutation matrix, moreover, $\det \mathbf{P} = (-1)^q$, q is the number of permutations (for example, a rows), $\mathbf{B} = [\mathbf{T}(\lambda)]'_\lambda\big|_{\lambda=\lambda_l}$, $\mathbf{C} = [\mathbf{T}(\lambda)]''_\lambda\big|_{\lambda=\lambda_l}$.

The elements of matrices from decompositions (2.8) can be calculated by the recurrence relations

$$r = 1,\ 2,\ \dots,\ n,$$

$$u_{rk} = t_{rk} - \sum_{j=1}^{r-1} l_{rj} u_{jk}\ , \quad k = r,\ \dots,\ n,$$

$$l_{ir} = \left(t_{ir} - \sum_{j=1}^{r-1} l_{ij} u_{jr} \right) / u_{rr}\ , \quad i = r+1,\ \dots,\ n,$$

$$v_{rk} = b_{rk} - \sum_{j=1}^{r-1} (m_{rj} u_{jk} + l_{rj} v_{jk})\ , \quad k = r,\ \dots,\ n,$$

$$m_{ir} = \left[b_{ir} - \sum_{j=1}^{r-1} (m_{ij} u_{jr} + l_{ij} v_{jr}) - l_{ir} v_{rr} \right] / u_{rr}\ , \quad i = r+1,\ \dots,\ n,$$

$$w_{rk} = c_{rk} - \sum_{j=1}^{r-1} (n_{rj} u_{jk} + 2 m_{rj} v_{jk} + l_{rj} w_{jk})\ , \quad k = r,\ \dots,\ n,$$

$$n_{ir} = \left[c_{ir} - \sum_{j=1}^{r-1} (n_{ij} u_{jr} + 2 m_{ij} v_{jr} + l_{ij} w_{jr}) - 2 m_{ir} v_{rr} - l_{ir} w_{rr} \right] / u_{rr}\ ,\ i = r+1,\ \dots,\ n.$$

Thus, *the algorithm of finding the eigenvalue curves* of two-parameter spectral problem consists of such steps.

Step 1. We define the interval $\Lambda = [\lambda_c, \lambda_d]$, in which we shall seek the eigenvalues of the problem (2.1). It can be one interval or sequence of intervals $\Lambda_t = [\lambda_c, \lambda_d]$ such, that $\Lambda = \cup \Lambda_t$. For this purpose the interval Λ_t is put in a circle (region G), setting centre of a circle $r_{0_t} = (\lambda_{c_t} + \lambda_{d_t})/2$ and its radius $\rho_t = (\lambda_{d_t} - \lambda_{c_t})/2$, and also the number of points of splitting N_Γ of boundary Γ of the region G, that is a circle.

Step 2. We define the value of parameter $\mu = \alpha_k \lambda + \beta_k$, assigning the next value to the sizes α_k and β_k.

Step 3. Using decomposition (2.8) for complex λ, we compute the quantity of eigenvalues which are found in the definition at *Step 1* of the region G by the formula (2.6), which with consideration of (2.7) for $k = 0$ accepts the form

$$m = s_0 = \frac{1}{N_\Gamma} \sum_{j=1}^{N_\Gamma} \rho_t \exp\left(i \frac{2\pi j}{N_\Gamma} \right) \sum_{r=1}^{n} \frac{v_{rr}}{u_{rr}}.$$

Approximate values are found by solving the system of equations (2.4), previously calculating the right parts of the system by the formula (2.6), which with regard for (2.7) has the form

$$s_k = \frac{1}{N_\Gamma} \sum_{j=1}^{N_\Gamma} \left((\lambda_j)^k \rho_t \exp\left(i \frac{2\pi j}{N_\Gamma} \right) \sum_{r=1}^{n} \frac{v_{rr}}{u_{rr}} \right), \quad k = 1, 2, \dots, m.$$

Step 4. Using decomposition (2.8) for real λ, we specify all eigenvalues which get into the given region, using for this purpose Newton's method which taking into account of (2.7) will accept the form

$$\lambda_{\ell+1} = \lambda_\ell - 1 / \left(\sum_{r=1}^{n} \frac{v_{rr}}{u_{rr}} \right), \quad \ell = 0, 1, 2, \dots,$$

or one of its bilateral analogues [14], for example

$$\begin{aligned} \lambda_{2\ell+1} &= \lambda_{2\ell} - \left(\sum_{k=1}^{n} \frac{v_{kk}}{u_{kk}}\right) / \left(\sum_{k=1}^{n} \left(\left(\frac{v_{kk}}{u_{kk}}\right)^2 - \frac{w_{kk}}{u_{kk}}\right)\right), \\ \lambda_{2\ell+2} &= \lambda_{2\ell+1} - 1/\left(\sum_{k=1}^{n} \frac{\bar{v}_{kk}}{\bar{u}_{kk}}\right), \end{aligned} \qquad \ell = 0, 1, 2, \ldots,$$

where $u_{kk},\ v_{kk},\ w_{kk}$ are the elements of matrices $\mathbf{U},\ \mathbf{V},\ \mathbf{W}$ in decomposition (2.8) for the fixed value $\lambda = \lambda_{2\ell}$ and $\bar{u}_{kk},\ \bar{v}_{kk}$ are the elements of matrices $\mathbf{U},\ \mathbf{V}$ in decomposition (2.8) for the fixed value $\lambda = \lambda_{2\ell+1}$. As initial approximations we take the values obtained at *Step 3*.

Step 5. We go to *Step 2*.

Step 6. If necessary we update the region G by change of its center and/or its radius and go to *Step 2,* otherwise go to *Step 7*.

Step 7. End.

The application of modification of algorithm to the linear two-parameter eigenvalue problems was considered in the work [15].

3. Algorithm of finding bifurcation points of eigenvalue curves

Let now

$$M = \{(\lambda, \mu): \ f(\lambda, \mu) = 0, f'_\lambda(\lambda, \mu) = 0, f'_\mu(\lambda, \mu) = 0\}$$

be a finite set of points (λ, μ) that $M \neq \emptyset$. Then, according to [16], there are bifurcation points of the spectral problem (1.3), i.e. the bifurcation points are solutions of the system of two nonlinear algebraic equations

$$\begin{aligned} [f(\lambda, \mu)]'_\lambda &\equiv [\det\, T_n(\lambda, \mu)]'_\lambda = 0, \\ [f(\lambda, \mu)]'_\mu &\equiv [\det\, T_n(\lambda, \mu)]'_\mu = 0, \end{aligned} \tag{3.1}$$

satisfying equation (1.4). But such approach was rarely used, as it requires the calculation of derivatives of a matrix determinant.

We will mark that having some approximation $(\lambda_m,\ \mu_m)$ to solution of the system (3.1), for its solving it is possible to apply as in [17] the iterative process of Newton's method

$$\begin{aligned} \lambda_{m+1} &= \lambda_m + \Delta\lambda_m, \\ \mu_{m+1} &= \mu_m + \Delta\mu_m, \end{aligned} \qquad m = 0,\ 1,\ \ldots, \tag{3.2}$$

where deviations $\Delta\lambda_k$ and $\Delta\mu_k$ are the solutions of the system of two linear equations

$$\begin{aligned} [f(\lambda_m, \mu_m)]''_{\lambda\lambda}\, \Delta\lambda_m + [f(\lambda_m, \mu_m)]''_{\lambda\mu}\, \Delta\mu_m &= -\, [f(\lambda_m, \mu_m)]'_\lambda \\ [f(\lambda_m, \mu_m)]''_{\mu\lambda}\, \Delta\lambda_m + [f(\lambda_m, \mu_m)]''_{\mu\mu}\, \Delta\mu_m &= -\, [f(\lambda_m, \mu_m)]'_\mu \end{aligned}\ . \tag{3.3}$$

In what follows we shall assume, that the determinant of matrix of the system (3.3), the elements of which are defined at the point $(\lambda_m,\ \mu_m)$, is different from zero.

Consequently, at every step of iterative process it is needed to calculate the value of function $f(\lambda, \mu) = \det T_n(\lambda, \mu)$ and its partial derivatives (the first and the second) only for the fixed values of parameters λ and μ. It can be made by the numerical procedure, using LU-decomposition of the matrix, namely:

$$f'_\lambda(\lambda_m, \mu_m) = (-1)^q \sum_{k=1}^{n} v^1_{kk} \prod_{i=1, i\neq k}^{n} u_{ii}, \qquad f'_\mu(\lambda_m, \mu_m) = (-1)^q \sum_{k=1}^{n} v^2_{kk} \prod_{i=1, i\neq k}^{n} u_{ii},$$

$$f''_{\lambda\lambda}(\lambda_m,\mu_m) = \\ = (-1)^q \sum_{k=1}^{n} w_{kk}^{1,1} \prod_{i=1, i\neq k}^{n} u_{ii} + (-1)^q \sum_{k=1}^{n} v_{kk}^{1} \left(\sum_{j=1, j\neq k}^{n} v_{jj}^{1} \prod_{i=1, i\neq k, i\neq j}^{n} u_{ii} \right),$$

$$f''_{\mu\mu}(\lambda_m,\mu_m) = \\ = (-1)^q \sum_{k=1}^{n} w_{kk}^{2,2} \prod_{i=1, i\neq k}^{n} u_{ii} + (-1)^q \sum_{k=1}^{n} v_{kk}^{2} \left(\sum_{j=1, j\neq k}^{n} v_{jj}^{2} \prod_{i=1, i\neq k, i\neq j}^{n} u_{ii} \right), \tag{3.4}$$

$$f''_{\lambda\mu}(\lambda_m,\mu_m) = f''_{\mu\lambda}(\lambda_m,\mu_m) = \\ = (-1)^q \sum_{k=1}^{n} w_{kk}^{1,2} \prod_{i=1, i\neq k}^{n} u_{ii} + (-1)^q \sum_{k=1}^{n} v_{kk}^{1} \left(\sum_{j=1, j\neq k}^{n} v_{jj}^{2} \prod_{i=1, i\neq k, i\neq j}^{n} u_{ii} \right),$$

where u_{ii}, v_{ii}^1, v_{ii}^2, $w_{ii}^{1,1}$, $w_{ii}^{2,2}$, $w_{ii}^{1,2}$ are the diagonal elements of matrices $\mathbf{U}$, $\mathbf{V}^1$, $\mathbf{V}^2$, $\mathbf{W}^{1,1}$, $\mathbf{W}^{2,2}$, $\mathbf{W}^{1,2}$ in the decompositions

$$\begin{aligned} &\mathbf{PT} = \mathbf{LU}, \\ &\mathbf{PB}^1 = \mathbf{M}^1\mathbf{U} + \mathbf{LV}^1, \\ &\mathbf{PB}^2 = \mathbf{M}^2\mathbf{U} + \mathbf{LV}^2, \\ &\mathbf{PC}^{1,1} = \mathbf{N}^{1,1}\mathbf{U} + 2\mathbf{M}^1\mathbf{V}^1 + \mathbf{LW}^{1,1}, \\ &\mathbf{PC}^{2,2} = \mathbf{N}^{2,2}\mathbf{U} + 2\mathbf{M}^2\mathbf{V}^2 + \mathbf{LW}^{2,2}, \\ &\mathbf{PC}^{1,2} = \mathbf{N}^{1,2}\mathbf{U} + \mathbf{M}^1\mathbf{V}^2 + \mathbf{M}^2\mathbf{V}^1 + \mathbf{LW}^{1,2}, \end{aligned} \tag{3.5}$$

for the fixed values λ_m and μ_m. Here $\mathbf{P}$ is the permutation matrix, moreover, $\det \mathbf{P} = (-1)^q$, q is the number of permutations,

$$\mathbf{B}^1 = [\mathbf{T}(\lambda,\mu)]'_{\lambda}\Big|_{\substack{\lambda=\lambda_m \\ \mu=\mu_m}}, \quad \mathbf{B}^2 = [\mathbf{T}(\lambda,\mu)]'_{\mu}\Big|_{\substack{\lambda=\lambda_m \\ \mu=\mu_m}},$$

$$\mathbf{C}^{1,1} = [\mathbf{T}(\lambda,\mu)]''_{\lambda\lambda}\Big|_{\substack{\lambda=\lambda_m \\ \mu=\mu_m}}, \quad \mathbf{C}^{2,2} = [\mathbf{T}(\lambda,\mu)]''_{\mu\mu}\Big|_{\substack{\lambda=\lambda_m \\ \mu=\mu_m}},$$

$$\mathbf{C}^{1,2} = [\mathbf{T}(\lambda,\mu)]''_{\lambda\mu}\Big|_{\substack{\lambda=\lambda_m \\ \mu=\mu_m}}.$$

The elements of matrices from decompositions (3.5) can be immediately calculated by the recurrence formulae

$$r = 1,\ 2,\ \ldots,\ n,$$

$$u_{rk} = t_{rk} - \sum_{j=1}^{r-1} l_{rj} u_{jk}, \quad k = r,\ \ldots,\ n,$$

$$l_{ir} = \left(t_{ir} - \sum_{j=1}^{r-1} l_{ij} u_{jr} \right) / u_{rr}, \quad i = r+1,\ \ldots,\ n,$$

$$v_{rk}^1 = b_{rk}^1 - \sum_{j=1}^{r-1} (m_{rj}^1 u_{jk} + l_{rj} v_{jk}^1), \quad k = r,\ \ldots,\ n,$$

$$m_{ir}^1 = \left[b_{ir}^1 - \sum_{j=1}^{r-1} (m_{ij}^1 u_{jr} + l_{ij} v_{jr}^1) - l_{ir} v_{rr}^1 \right] / u_{rr} \ , \quad i = r+1, \ \dots \ , \ n,$$

$$v_{rk}^2 = b_{rk}^2 - \sum_{j=1}^{r-1} (m_{rj}^2 u_{jk} + l_{rj} v_{jk}^2) \ , \quad k = r, \ \dots \ , \ n,$$

$$m_{ir}^2 = \left[b_{ir}^2 - \sum_{j=1}^{r-1} (m_{ij}^2 u_{jr} + l_{ij} v_{jr}^2) - l_{ir} v_{rr}^2 \right] / u_{rr} \ , \quad i = r+1, \ \dots \ , \ n,$$

$$w_{rk}^{1,1} = c_{rk}^{1,1} - \sum_{j=1}^{r-1} (n_{rj}^{1,1} u_{jk} + 2m_{rj}^1 v_{jk}^1 + l_{rj} w_{jk}^{1,1}) \ , \quad k = r, \ \dots \ , \ n,$$

$$n_{ir}^{1,1} = \left[c_{ir}^{1,1} - \sum_{j=1}^{r-1} (n_{ij}^{1,1} u_{jr} + 2m_{ij}^1 v_{jr}^1 + l_{ij} w_{jr}^{1,1}) - 2m_{ir}^1 v_{rr}^1 - l_{ir} w_{rr}^{1,1} \right] / u_{rr}, \quad i = r+1, \ \dots \ , \ n,$$

$$w_{rk}^{2,2} = c_{rk}^{2,2} - \sum_{j=1}^{r-1} (n_{rj}^{2,2} u_{jk} + 2m_{rj}^2 v_{jk}^2 + l_{rj} w_{jk}^{2,2}) \ , \quad k = r, \ \dots \ , \ n,$$

$$n_{ir}^{2,2} = \left[c_{ir}^{2,2} - \sum_{j=1}^{r-1} (n_{ij}^{2,2} u_{jr} + 2m_{ij}^2 v_{jr}^2 + l_{ij} w_{jr}^{2,2}) - 2m_{ir}^2 v_{rr}^2 - l_{ir} w_{rr}^{2,2} \right] / u_{rr}, \quad i = r+1, \ \dots \ , \ n,$$

$$w_{rk}^{1,2} = c_{rk}^{1,2} - \sum_{j=1}^{r-1} (n_{rj}^{1,2} u_{jk} + m_{rj}^1 v_{jk}^2 + m_{rj}^2 v_{jk}^1 + l_{rj} w_{jk}^{1,2}) \ , \quad k = r, \ \dots \ , \ n,$$

$$n_{ir}^{1,2} = \left[c_{ir}^{1,2} - \sum_{j=1}^{r-1} (n_{ij}^{1,2} u_{jr} + m_{ij}^1 v_{jr}^2 + m_{ij}^2 v_{jr}^1 + l_{ij} w_{jr}^{1,2}) - m_{ir}^1 v_{rr}^2 - m_{ir}^2 v_{rr}^1 - l_{ir} w_{rr}^{1,2} \right] \Big/ u_{rr}, \quad i = r+1, \ \dots \ , \ n.$$

Thus, *the algorithm of finding the bifurcation points of the eigenvalue curves* of two-parameter spectral problem consists of such steps.

Step 1. Set exactness of calculations: from the parameters – ε_p and from the function $f(\lambda, \mu)$ – ε_f.

Step 2. Start with the initial approximation of the bifurcation point (λ_0, μ_0).

Step 3. **for** $m = 0, 1, 2, \ \dots$ until convergence **do**

Step 4. Calculate the matrices $\mathbf{T}, \mathbf{B}^1, \mathbf{B}^2, \mathbf{C}^{1,1}, \mathbf{C}^{2,2}, \mathbf{C}^{1,2}$.

Step 5. By the formulas (3.4)-(3.5) calculate $f'_\lambda(\lambda_m, \mu_m)$, $f'_\mu(\lambda_m, \mu_m)$, $f''_{\lambda\lambda}(\lambda_m, \mu_m)$, $f''_{\mu\mu}(\lambda_m, \mu_m)$, $f''_{\lambda\mu}(\lambda_m, \mu_m)$.

Step 6. According to the calculated coefficients we create the matrix of the system (3.3) and we solve it relative to $\Delta\lambda_m$, $\Delta\mu_m$.

Step 7. Calculate next approximation to λ and μ by the formula (3.2).

Step 8. **end for** m.

Step 9. **if** $|f(\lambda_m, \mu_m)| \le \varepsilon_f$ **then go to *Step 11***.

Step 10. **else** Set another initial approximation of the bifurcation point (λ_0, μ_0) and **go to Step 3.**

Step 11. End.

4. Numerical examples

We will consider the nonlinear two-parameter eigenvalue problem

$$(T(\lambda,\mu)-I)\,u(\xi_1,\xi_2,\lambda,\mu)=0 \tag{4.1}$$

for the integral equation with the operator

$$T(\lambda,\mu)\,u(\xi_1,\,\xi_2,\,\lambda,\,\mu)\equiv\iint\limits_{\Omega}F(\xi_1',\,\xi_2')K(\xi_1,\,\xi_2,\,\xi_1',\,\xi_2',\,\lambda,\,\mu)\frac{u(\xi_1',\,\xi_2',\,\lambda,\,\mu)}{f_0(\xi_1',\,\xi_2',\,\lambda,\,\mu)}\,d\xi_1'd\xi_2'$$

which arises in the radiating systems synthesis theory at finding lines of possible branching of solutions of nonlinear integral equation [1]

$$f(\xi_1,\xi_2)=\iint\limits_{\Omega}F(\xi_1',\xi_2')K(\xi_1,\xi_2,\xi_1',\xi_2',\lambda,\mu)e^{i\,\arg\,f(\xi_1',\xi_2')}d\xi_1'd\xi_2'.$$

Here $F(\xi_1,\xi_2)$ is continuous in the region $\Omega\,:\,\{\,|\xi_1|\,\leq\,1,\;|\xi_2|\,\leq\,1\}$ real and positive function,

$$K(\xi_1,\xi_2,\xi_1',\xi_2',\lambda,\mu)\;=\;\sum_{n=-M_1}^{M_1}\;\sum_{m=-M_2}^{M_2}\;e^{i\,[\lambda n(\xi_1-\xi_1')+\mu m(\xi_2-\xi_2')]}\,,$$

$$f_0(\xi_1,\xi_2,\lambda,\mu)=\iint\limits_{\Omega}F(\xi_1',\xi_2')K(\xi_1,\xi_2,\xi_1',\xi_2',\lambda,\mu)d\xi_1'd\xi_2'.$$

It is easy to make sure, that $f_0(\xi_1,\xi_2,\lambda,\mu)$ is the eigenfunction of our problem for any values of parameters λ and μ $\quad(\lambda>0,\mu>0)$. The problem is to find those values of parameters λ and μ, for which the problem (4.1) has the solution distinct from $f_0(\xi_1,\xi_2,\lambda,\mu)$.

For this purpose we shall eliminate this function from the kernel, beforehand reducing the problem (4.1) to the selfajoint form, that is, fulfilling replacement

$$\varphi(\xi_1,\xi_2,\lambda,\mu)=\sqrt{w(\xi_1,\xi_2,\lambda,\mu)}\;u(\xi_1,\xi_2,\lambda,\mu),$$

where $w(\xi_1,\xi_2,\lambda,\mu)=F(\xi_1,\xi_2)/f_0(\xi_1,\xi_2,\lambda,\mu)$,we obtain the equation

$$\begin{aligned}\varphi(\xi_1,\xi_2,\lambda,\mu)=T(\lambda,\mu)\varphi(\xi_1,\xi_2,\lambda,\mu)\equiv\\ \equiv\int_{-1}^{1}\int_{-1}^{1}E(\xi_1,\xi_2,\xi_1',\xi_2',\lambda,\mu)\varphi(\xi_1',\xi_2',\lambda,\mu)d\xi_1'd\xi_2'\end{aligned} \tag{4.2}$$

with a symmetric kernel

$$\begin{aligned}E(\xi_1,\xi_2,\xi_1',\xi_2',\lambda,\mu)=\sqrt{w(\xi_1,\xi_2,\lambda,\mu)w(\xi_1',\xi_2',\lambda,\mu)}\,\times\\ \times\left[K(\xi_1,\,\xi_2,\,\xi_1',\,\xi_2',\,\lambda,\,\mu)-\frac{f_0(\xi_1,\,\xi_2,\,\lambda,\,\mu)f_0(\xi_1',\,\xi_2',\,\lambda,\,\mu)}{||\;\varphi_0(\xi_1,\,\xi_2,\,\lambda,\,\mu)\;||^2}\right],\end{aligned}$$

where $\varphi_0(\xi_1,\xi_2,\lambda,\mu)=\sqrt{F(\xi_1,\xi_2)f_0(\xi_1,\xi_2,\lambda,\mu)}$.

From Schmidt's lemma [18] follows, that $\varphi_0(\xi_1,\xi_2,\lambda,\mu)$ will not be any more the eigenfunction of the equation (4.2). Thus, from a spectrum of the operator (4.2) we have eliminated a continual set of eigenvalues, which coincides with the first quadrant of the plane R^2 that responds of the function $\varphi_0(\xi_1,\xi_2,\lambda,\mu)$.

Now, using the property of degeneracy of the kernel $K(\xi_1, \xi_2, \xi_1', \xi_2', \lambda, \mu)$, we will reduce the equation (4.2) to the equivalent system of the algebraic equations

$$\mathbf{T}_N(\lambda, \mu)\mathbf{u}_N \equiv (\mathbf{A}_N(\lambda, \mu) - \mathbf{I}_N)\mathbf{u}_N = 0.$$

Here $\mathbf{A}_N(\lambda, \mu)$ is the matrix of dimension $N = (2M_1 + 1)(2M_2 + 1)$, $\mathbf{I}_N$ is the unit matrix, $\mathbf{u}_N \in R^N$. Because of a bulky size of the formulas, the expressions for elements of the matrix $\mathbf{A}_N(\lambda, \mu)$ here are not given.

Using algorithm 1 in which at the Step 2 the next values μ were defined as $\mu = \alpha_k \lambda$, and α_k varied in an interval $[0.7 \div 1.6]$ with the step $\Delta\alpha = 0.05$, the eigenvalue curves for two problems are constructed, for which the function $F(\xi_1, \xi_2)$ was assigned by the formulas $F(\xi_1, \xi_2) = \cos\frac{\pi}{2}\xi_1 \cdot |\sin\pi\xi_2|$ and $F(\xi_1, \xi_2) = \sqrt{1 - \frac{1}{2}(\xi_1^2 + \xi_2^2)}$ and which are presented on Fig.1 and Fig. 2. For the same problems the biffurcation points of eigenvalue curves calculated on algorithm 2, are indicated in Tab.1.

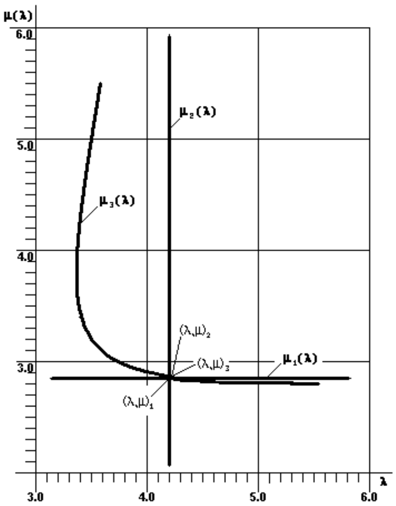

Fig. 1. Eigenvalue curves of the problem (4.1) for the formula $F(\xi_1, \xi_2) = \cos\frac{\pi}{2}\xi_1 \, |\sin\pi\xi_2|$.

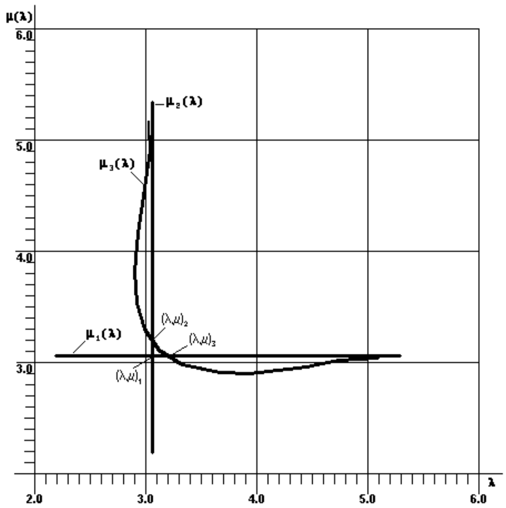

Fig. 2. Eigenvalue curves of the problem (4.1) for the formula $F(\xi_1, \xi_2) = \sqrt{1 - \frac{1}{2}(\xi_1^2 + \xi_2^2)}$.

Tabl. 4.1. Biffurcation points of eigenvalue curves

$F(\xi_1, \xi_2)$	*Biffurcation points* $(\lambda_b, \mu_b)_i$	*Biffurcation points obtained by other method*
$\cos\frac{\pi\xi_1}{2} \cdot \|\sin\pi\xi_2\|$	$(\lambda_b, \mu_b)_1 = (\lambda_b, \mu_b)_2 =$ $(\lambda_b, \mu_b)_3 = (4.207065,\ 2.855434)$	$(\lambda_b, \mu_b)_1 = (4.207065,\ 2.855434)$
$\sqrt{1 - \frac{1}{2}(\xi_1^2 + \xi_2^2)}$	$(\lambda_b, \mu_b)_1 = (3.064250,\ 3.064250)$ $(\lambda_b, \mu_b)_2 = (3.064250,\ 3.186696)$ $(\lambda_b, \mu_b)_3 = (3.186696,\ 3.064250)$	-

5. Conclusions and remarks

The application of algorithm to calculate the derivatives of matrix determinant allows to build the accurate and effective (in the sense of construction of all eigenvalue curves which belong to the region of change of parameters λ and μ) algorithms for finding the eigenvalue curves and calculation of bifurcation points of linear and nonlinear spectral problems. So, for example, by means of the above algorithm the curves of eigenvalues which were not to obtained by existing methods, for example in [1] and [19], were found (that is spectral lines $\mu_3(\lambda)$ in Fig. 1 and Fig. 2). Moreover, the eigenvalue curves (that is spectral lines $\mu_1(\lambda)$ and $\mu_2(\lambda)$ in Fig.2), which, as was considered earlier, do not exist for the problems, for which the function $F(\xi_1, \xi_2)$ does not suppose a separation of variables, are found.

In summary we shall mark, that the offered algorithm of calculation of derivatives of matrix determinant can be used and in the mentioned above approach in which basis the implicit function theorem is. At such approach it is necessary to solve the Cauchy problem

$$\frac{d\lambda}{d\mu} = -\frac{[\det T_n(\lambda,\mu)]'_\mu}{[\det T_n(\lambda,\mu)]'_\lambda}, \qquad \lambda(\mu_1) = \lambda_1, \tag{5.1}$$

for which the right part of equation (5.1) can be calculated by the algorithm of calculation of derivatives of matrix determinant. Besides by the algorithm, given in this paper, it is possible numerically to define a quantity of eigenvalues, and, therefore, the eigenvalue curves, which are in the given range of spectral parameters and to calculate initial value for Cauchy problem for each curve.

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