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OPERATOR INTERPOLATION HERMITE-BIRKHOFF FORMULAS IN SPACES OF SMOOTH FUNCTIONS

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Анотація. Для скалярних функцій побудовано новий клас алгебраїчних інтерполяційних многочленів Ерміта-Бірхофа довільного фіксованого порядку. Спираючись на побудований інтерполяційний оператор Ерміта-Бірхофа отримані різні структури формул в просторах гладких функцій та матриць. Тип поліноміального оператора показує інваріантність цих формул.

Abstract. New class of algebraic interpolation Hermite-Birkhoff arbitrary fixed degree polynomials for scalar functions is constructed. On this base interpolation operator Hermite-Birkhoff difference structure formulas in space of smooth functions and matrices are obtained. The type of operator polynomials, for which these formulas are invariant, is indicated.

1. Introduction

At the construction of interpolation formulas for operators, given on functional spaces, the known interpolation polynomials for scalar functions can be applied. On the base of such approach a series of operator interpolation formulas has been obtained [1–3]. The given paper is devoted to further development of this direction.

Let $X = C^p([a, b])$ be the space p-times differentiable on $[a, b] \subseteq \mathbb{R}$ functions. Elements $x_k \in X$, $k = 0, 1, ..., n$, are some given nodes of interpolation. We will define an operator $F : X \to Y$, where Y is some function space. Through $\delta^\nu F[x; h_1, h_2, ..., h_\nu]$ we will designate Gateaux differential of order ν for operator F at a point $x = x(t) \in X$ in directions $h_i = h_i(t) \in X$ $(i = 1, 2, ..., \nu;\ t \in [a, b])$.

Interpolation Hermite problem for $F(x)$ $(x \in X)$ consists in the construction of operator polynomial $H_m(F; x) \equiv H_m(x) : X \to Y$ of degree not above m, satisfying the conditions

$$\delta^{\beta_j} H_m[x_j; h_1, h_2, ..., h_{\beta_j}] = \delta^{\beta_j} F[x_j; h_1, h_2, ..., h_{\beta_j}], \tag{1.1}$$

$$\beta_j = 0, 1, ..., \alpha_j - 1;\ j = 0, 1, ..., n;\ \alpha_0 + \alpha_1 + \cdots + \alpha_n = m + 1.$$

The problem of Hermite-Birkhoff arises, when in the interpolation condition (1.1) some from orders of differentials β_j are absent.

Even for scalar functions the construction of interpolation Hermite-Birkhoff formulas is usually connected with considerable difficulties. In series of cases interpolation Hermite-Birkhoff problem has no solution at all [4–6]. Complexities substantially increase at operator analogue research of this problem.

In this paper we will consider only those operators $F(x)$, for which differentials $\delta^\nu F[x; h_1, h_2, ..., h_\nu]$ contain the product of directions $h_1, h_2, ..., h_\nu$. In particular, if $F(x) = f(s, x(t))$, where $f(s, u)$ is some scalar function of arguments $s \in \mathbb{C}^m$ $(m \in \mathbb{N})$ and $u \in \mathbb{R}$, differentiable with respect to variable u not less, than ν times, then differential $\delta^\nu F[x; h_1, h_2, ..., h_\nu]$ equals $\dfrac{\partial^\nu}{\partial x^\nu} f(s, x)\, h_1 h_2 \cdots h_\nu$, where $x = x(t)$ and $h_i = h_i(t)$

Key words. Algebraic Hermite-Birkhoff interpolation, operator interpolation polynomial, interpolation error estimate.

$(i = 1,\ 2,\ ...,\ \nu)$. Many operators of integral type, for example, Hammerstein, Uryson and others operators, possess this property.

By means of $\delta^\nu F[x;\ h]$ we will designate differential of order ν, where first $\nu - 1$ directions $h_1,\ h_2,\ ...,\ h_{\nu-1}$ are identically equal to unit, and last direction $h_\nu = h$.

2. Interpolation formulas for functions of scalar argument

At first we will consider interpolation Hermite-Birkhoff problem of special kind for scalar functions. Let interpolation nodes $t_0, t_1,\ ..., t_n$ be different points of numerical axis, and values $f(t_0), f(t_1),\ ..., f(t_n)$ of function $f(t)$ and values of its derivatives $f^{(n+j)}(t_0)$, $f^{(n+j)}(t_1), ..., f^{(n+j)}(t_n)$ of the fixed order $n+j$, $j \in \mathbb{N}$, $t \in \mathbb{R}$, be also known. It is required to construct an algebraic polynomial $H_{2n+j}(f,\ t) \equiv H_{2n+j}(t)$ of degree not above $2n + j$, satisfying the conditions

$$H_{2n+j}(t_k) = f(t_k),\ \ H^{(n+j)}_{2n+j}(t_k) = f^{(n+j)}(t_k),\ \ k = 0,\ 1,\ ...,\ n, \tag{2.1}$$

where j is some fixed natural number.

Now we introduce the notations

$$l_{nk}(t) = \frac{\omega_n(t)}{(t - t_k)\omega'_n(t_k)},\ \omega_n(t) = (t - t_0)(t - t_1)\cdots(t - t_n); \tag{2.2}$$

$$\widetilde{l}_{nkj}(t) = y_{nkj}(t) - \sum_{\nu=0}^{n} l_{n\nu}(t)y_{nkj}(t_\nu),\ \ y_{nkj}(t) = \frac{1}{(n+j-1)!}\int_0^t (t-s)^{n+j-1} l_{nk}(s)ds, \tag{2.3}$$

where $k = 0,\ 1,\ \ldots,\ n$; $j \in \mathbb{N}$.

We notice that $y_{nkj}(t)$ is algebraic polynomial of degree $2n + j$ and it is possible to represent $\widetilde{l}_{nkj}(t)$ in the form $\widetilde{l}_{nkj}(t) = \omega_n(t)p_{nkj}(t)$, where $p_{nkj}(t)$ is a polynomial of degree $n + j - 1$.

In that case, when $j = 1$, the solution of investigated problem is known [2] and the required Hermite-Birkhoff polynomial of degree not above $2n + 1$ looks like $H_{2n+1}(t) = L_n(t) + \sum\limits_{k=0}^{n} \widetilde{l}_{nk1}(t)f^{(n+1)}(t_k)$, where $L_n(t) = \sum\limits_{k=0}^{n} l_{nk}(t)f(t_k)$ is the algebraic interpolation Lagrange polynomial for function $f(t)$, constructed on the nodes $t_0, t_1,\ ..., t_n$.

Lemma 2.1 *For the algebraic polynomial*

$$H_{2n+j}(t) = \sum_{k=0}^{n} \left[l_{nk}(t)f(t_k) + \widetilde{l}_{nkj}(t)f^{(n+j)}(t_k) \right], \tag{2.4}$$

where j is some fixed natural number, the conditions (2.1) *are satisfied. In the case $j > 1$ the interpolation formula* (2.4) *is invariant with respect to algebraic polynomials of degree less or equal n, and for $j = 1$ it is exact for polynomials of degree not above $2n + 1$.*

Proof. The first group equalities (2.1) hold, because $l_{nk}(t_\nu) = \delta_{k\nu}$, $\widetilde{l}_{nkj}(t_\nu) = 0$, where $\delta_{k\nu}$ is the Kronecker symbol; $k,\ \nu = 0,\ 1,\ ...,\ n$; $j \in \mathbb{N}$.

Taking into account, further, that $l_{nk}(t)$ is a polynomial of degree n, we obtain $l^{(n+j)}_{nk}(t) \equiv 0$ for $j \in \mathbb{N}$. Since

$$\widetilde{l}^{(n+j)}_{nkj}(t) = \frac{d^{n+j}}{dt^{n+j}} \left(\frac{1}{(n+j-1)!}\int_0^t (t-s)^{n+j-1} l_{nk}(s)ds \right) = l_{nk}(t),$$

where $k=0,\ 1,\ ...,\ n;\ j\in\mathbb{N}$, then $H_{2n+j}^{(n+j)}(t)=\sum\limits_{k=0}^{n} l_{nk}(t)f^{(n+j)}(t_k)$. Hence $H_{2n+j}^{(n+j)}(t_\nu)=$ $=f^{(n+j)}(t_\nu),\ \nu=0,\ 1,\ ...,\ n$, i.e. the second group of conditions (2.1) is also satisfied.

If $f(t)=Q_n(t)$ is an algebraic polynomial of degree not above n, then $H_{2n+j}(t)$ coincides with the Lagrange polynomial for the same function $f(t)$ with the same interpolation nodes and consequently $H_{2n+j}(t)\equiv Q_n(t)$. Further it is enough to prove invariance of the formula (2.4) for the polynomials $f(t)=\omega_n(t)t^\nu$ $(\nu=0,\ 1,\ ...,\ n+j-1)$. In this case $f(t_k)=0,\ k=0,\ 1,\ ...,\ n$, and $H_{2n+j}(t)=\sum\limits_{k=0}^{n}\widetilde{l}_{nkj}(t)f^{(n+j)}(t_k)$. From here $H_{2n+j}^{(n+j)}(t)=$ $=\sum\limits_{k=0}^{n} l_{nk}(t)f^{(n+j)}(t_k)\equiv f^{(n+j)}(t)$, because for considered functions $f(t)=\omega_n(t)t^\nu$ the derivative $f^{(n+j)}(t)$ is a polynomial of degree not above n. Therefore $r_{2n+j}^{(n+j)}(t)\equiv 0$ for error $r_{2n+j}(t)=f(t)-H_{2n+j}(t)$, i.e. $r_{2n+j}(t)$ is an algebraic polynomial of degree less or equal $n+j-1$. According to the interpolation conditions, $r_{2n+j}(t_k)=0,\ k=0,\ 1,\ ...,\ n$. Thus, the polynomial $r_{2n+j}(t)$ of degree not above $n+j-1$ is equal to zero in $n+1$ points. Hence, only in that case, when $j=1$, for the polynomials $f(t)=\omega_n(t)t^\nu$ $(\nu=0,\ 1,\ \ldots,\ n)$ of degree not above $2n+1$ the error $r_{2n+1}(t)\equiv 0$.

So, the lemma 2.1 is proved. □

Note that the interpolation error $R_{2n+j}(f;\ t)=f(t)-H_{2n+j}(t)$ can be written as $R_{2n+j}(f;\ t)=r_n(f;\ t)-\sum\limits_{k=0}^{n} r_n(y_{nkj};\ t)f^{(n+j)}(t_k)$, where $r_n(g;\ t)$ is interpolation remainder for the function g in a point t by means of Lagrange polynomial of degree not above n with respect to nodes $t_0,\ t_1,\ ...,\ t_n$. It is known that for functions $g\in C^{(n+1)}([a,\ b])$, where $[a,\ b]$ is minimal segment, containing nodes $t_0,\ t_1,\ ...,\ t_n$ and a point t, interpolation error $r_n(g;\ t)$ can be presented [7] in Lagrange form as $r_n(g;\ t)=\dfrac{g^{(n+1)}(\xi)}{(n+1)!}\omega_n(t),\ \xi\in(a,\ b)$. Therefore for the interpolation error $R_{2n+j}(f;\ t)$ of the constructed interpolation Hermite-Birkhoff formula $H_{2n+j}(t)$ one may write

$$R_{2n+j}(f;\ t)=\left[f^{(n+1)}(\xi)-\sum_{k=0}^{n} y_{nkj}^{(n+1)}(\xi_k)f^{(n+j)}(t_k)\right]\frac{\omega_n(t)}{(n+1)!},$$

where $\xi,\ \xi_0,\ \xi_1,\ ...,\ \xi_n\in(a,\ b)$, $a=\min\{t_0,\ t_1,\ \ldots,\ t_n,\ t\}$, $b=\max\{t_0,\ t_1,\ \ldots,\ t_n,\ t\}$, $j\in\mathbb{N}$.

3. Interpolation polynomials containing differentials of interpolated operator

Further we will consider operator polynomials $P_m(x):X\to Y$ of degree not above m in the form

$$P_m(x)=c_0(s)+\sum_{q=1}^{m}\int\limits_c^d c_q(s,\ t)\frac{d^\nu}{dt^\nu}x^q(t)dt\ \ (\nu=0,\ 1,\ ...,\ p), \tag{3.1}$$

where $c_0(s),\ c_q(s,\ t)$ are some fixed functions $(s\in\mathbb{C}^\alpha,\ \alpha\in\mathbb{N};\ t\in[c;\ d]\,;\ q=0,\ 1,\ ...,\ m)$.

Note that if $F(x)=\int\limits_c^d c_q(s,\ t)\dfrac{d^\nu}{dt^\nu}x^q(t)dt,\ \nu=0,\ 1,\ ...,\ p;\ q=0,\ 1,\ ...,\ m$, then

$$\delta^j F[x;\ h_1,\ h_2,\ ...,\ h_j]= \tag{3.2}$$

$$= \begin{cases} \dfrac{q!}{(q-j)!} \int\limits_c^d c_q(s,\ t) \dfrac{d^\nu}{dt^\nu} \left[x^{q-j}(t) h_1(t) h_2(t) \cdots h_j(t) \right] dt, & j \le q; \\ 0, & j > q. \end{cases}$$

We assume that functions $\{x_k(t)\}_{k=0}^n \in X$, used further as interpolation nodes, are such that $x_i(t) \neq x_j(t)$ at $i \neq j$ for any $t \in [a,\ b]$.

Let $g(\tau, t; x)$ be a linear on X operator, satisfying the conditions

$$g(a, t; x) \equiv 0,\ g(b, t; x) \equiv x(t);\quad \tau, t \in [a,\ b],\ x \in X, \tag{3.3}$$

and for interpolated operator $F(x)$ the equality

$$\delta F[x_0(\cdot) + g(\tau, \cdot; h);\ g'_\tau(\tau, \cdot; h)] = F'_\tau(x_0(\cdot) + g(\tau, \cdot; h)),\ h \in X, \tag{3.4}$$

takes place.

Notice that one of the types of linear operators $g(\tau, t; x)$ can be defined directly through invertible integral transformations in the space $X = C^p([a,\ b])$. In particular, it can be based on integral Fourier, Abel or others transformations and be presented as

$$g(\tau, s; x) = \int\limits_a^\tau \rho(s, t) \psi(t, x) dt = \begin{cases} 0, & \tau = a; \\ x(s), & \tau = b. \end{cases}$$

For each of such transformations function $\rho(s, t)$ and operator $\psi(t, x)$ are defined by corresponding formulas. So, in the case Abel transformation in the space $X = C^1([a,\ b])$, operator $g(\tau, s; x)$ has the form

$$g(\tau, s; x) = \frac{\sin(\alpha\pi)}{\pi} \int\limits_a^\tau \frac{\chi(s, t)}{(s-t)^{1-\alpha}} \psi(t, x) dt,\quad s > a,\ 0 < \alpha < 1,$$

where

$$\psi(t, x) = \frac{d}{dt} \int\limits_a^t \frac{x(s)}{(t-s)^\alpha} ds,\quad \chi(s, t) = \begin{cases} 1, & s > t; \\ 0, & s < t. \end{cases}$$

Theorem 3.1 *For the operator polynomial*

$$H_{2n+j}(F;\ x) = F(x_0) + \sum_{k=0}^n \delta^{n+j} F[x_k;\ \widetilde{l}_{nkj}(x)] + \tag{3.5}$$

$$+ \sum_{k=1}^n \int\limits_a^b \delta F[x_0(\cdot) + g(\tau, \cdot; x_k - x_0);\ l_{nk}(x(\cdot)) g'_\tau(\tau, \cdot; x_k - x_0)] d\tau,$$

where j is some fixed natural number, $l_{nk}(x) \equiv l_{nk}(x(t))$ and $\widetilde{l}_{nkj}(x) \equiv \widetilde{l}_{nkj}(x(t))$ are defined by equalities (2.2) and (2.3), in which points t and t_k are replaced with the functions $x = x(t)$ and $x_k = x_k(t)$ respectively, and for operator $g(\tau,\ t; x)$ relations (3.3) take place, the following conditions

$$H_{2n+j}(F;\ x_k) = F(x_k), \tag{3.6}$$

$$\delta^{n+j} H_{2n+j}[x_k;\ h_1,\ h_2,\ \ldots,\ h_{n+j}] = \delta^{n+j} F[x_k;\ h_1,\ h_2,\ \ldots,\ h_{n+j}],\ k = 0,\ 1,\ \ldots,\ n, \tag{3.7}$$

are satisfied. In the case $j > 1$ interpolation formula (3.5) is invariant with respect to operator polynomials $F(x) = P_n(x)$ of the form (3.1) of degree less or equal n. If $j = 1$ and $F(x) = P_{2n+1}(x)$ is operator polynomial of degree not above $2n+1$, then $H_{2n+1}(F;\ x) \equiv \equiv P_{2n+1}(x)$.

Proof. Since $\widetilde{l}_{nkj}(x_\nu) = 0$, and $l_{nk}(x_\nu) = \delta_{k\nu}$ for $j \in \mathbb{N}$ and $k,\ \nu = 0,\ 1,\ ...,\ n$, then, using the identities (3.3) end relation (3.4), we obtain

$$H_{2n+j}(F;\ x_\nu) = F(x_0) + \int\limits_a^b \delta F[x_0(\cdot) + g(\tau,\cdot;x_\nu - x_0);\ g'_\tau(\tau,\cdot;x_\nu - x_0)]d\tau =$$

$$= F(x_0) + \int\limits_a^b F'_\tau(x_0(\cdot) + g(\tau,\cdot;x_\nu - x_0))d\tau = F(x_0) + F(x_\nu) - F(x_0) = F(x_\nu)$$

for $\nu = 0,\ 1,\ ...,\ n$. Taking into account, further, the following equalities and identities: $\delta^{n+j}\widetilde{l}_{nkj}[x_i;\ h_1,\ h_2,\ ...,\ h_{n+j}] = \widetilde{l}^{(n+j)}_{nkj}(x_i(t))h_1(t)h_2(t)\cdots h_{n+j}(t) = \delta_{ki}h_1(t)\cdots h_{n+j}(t)$, $\delta^{n+j}l_{nk}[x;\ h_1,\ h_2,\ ...,\ h_{n+j}] \equiv 0$ for $j \in \mathbb{N}$ and $k,\ i = 0,\ 1,\ ...,\ n$, we come to (3.7).

We will prove invariance of the interpolation formula (3.5) with respect to operator polynomials of the form (3.1). Let

$$F(x) = \int\limits_c^d c_q(s,\ t)\frac{d^\nu}{dt^\nu}x^q(t)dt, \tag{3.8}$$

where $\nu = 0,\ 1,\ ...,\ p;\ q = 0,\ 1,\ ...,\ 2n+j$. For operators (3.8) in the case $q = 0,\ 1,\ ...,\ n$ the first sum in (3.5) is equal to zero, as long as the order of differentials $n + j > n$ and the equality (3.2) holds.

Since

$$\int\limits_a^b \delta F[x_0(\cdot) + g(\tau,\cdot;x_k - x_0);\ l_{nk}(x(\cdot))g'_\tau(\tau,\cdot;x_k - x_0)]d\tau =$$

$$= \int\limits_a^b q\int\limits_c^d c_q(s,\ t)\frac{d^\nu}{dt^\nu}\left\{[x_0(t) + g(\tau,t;x_k - x_0)]^{q-1}l_{nk}(x(t))g'_\tau(\tau,t;x_k - x_0)\right\}dt\ d\tau =$$

$$= \int\limits_c^d c_q(s,\ t)\frac{d^\nu}{dt^\nu}\left\{\int\limits_a^b q[x_0(t) + g(\tau,t;x_k - x_0)]^{q-1}g'_\tau(\tau,t;x_k - x_0)d\tau l_{nk}(x(t))\right\}dt =$$

$$= \int\limits_c^d c_q(s,\ t)\frac{d^\nu}{dt^\nu}\left\{\int\limits_a^b \frac{d}{d\tau}[x_0(t) + g(\tau,t;x_k - x_0)]^q d\tau l_{nk}(x(t))\right\}dt =$$

$$= \int\limits_c^d c_q(s,\ t)\frac{d^\nu}{dt^\nu}\left[l_{nk}(x(t))(x_k^q(t) - x_0^q(t))\right]dt,$$

then, using the identity $\sum\limits_{k=0}^{n} l_{nk}(x(t))x_k^q(t) \equiv x^q(t),\ q = 0,\ 1,\ \ldots,\ n$, we come to equality

$$H_{2n+j}(F;\ x) = \int\limits_c^d c_q(s,\ t)\frac{d^\nu}{dt^\nu}\left[x_0^q(t) + \sum_{k=1}^{n} l_{nk}(x(t))(x_k^q(t) - x_0^q(t))\right]dt =$$

$$= \int\limits_c^d c_q(s,\ t)\frac{d^\nu}{dt^\nu}\left[\sum_{k=0}^{n} l_{nk}(x(t))x_k^q(t)\right]dt = \int\limits_c^d c_q(s,\ t)\frac{d^\nu}{dt^\nu}x^q(t)dt = F(x).$$

For further proof of the theorem it is more convenient to consider operators of the form

$$\tilde{F}(x) = \int_c^d c_q(s,\ t)\frac{d^\nu}{dt^\nu}\left[\omega_n(x(t))x^q(t)\right]dt, \tag{3.9}$$

where $\nu = 0,\ 1,\ \dots,\ p;\ q = 0,\ 1,\ \dots,\ n+j-1$.

Notice that if the formula (3.5) is exact for operators (3.9), then it will be exact and for operators $F(x)$ of the form (3.8) for $q = n+1,\ n+2,\ \dots,\ 2n+j$.

Since first order differential $\delta\tilde{F}[x;\ h]$ for the operators (3.9) is evaluated by the rule $\delta\tilde{F}[x;\ h] = \int_c^d c_q(s,\ t)\frac{d^\nu}{dt^\nu}\left\{\frac{d}{dx}[\omega_n(x(t))x^q(t)]h(t)\right\}dt$, then second sum in the right-hand side of formula (3.5) will be transformed to the form

$$\sum_{k=1}^n \int_a^b \delta\tilde{F}[x_0(\cdot) + g(\tau,\cdot;x_k - x_0);\ l_{nk}(x(\cdot))g'_\tau(\tau,\cdot;x_k - x_0)]d\tau =$$

$$= \sum_{k=1}^n \int_a^b \int_c^d c_q(s,\ t)\frac{d^\nu}{dt^\nu}\left\{\frac{d}{dx}\Big\{\omega_n(x_0(t) + g(\tau,t;x_k - x_0))[x_0(t) + g(\tau,t;x_k - x_0)]^q\Big\}\times\right.$$

$$\left.\times l_{nk}(x(t))g'_\tau(\tau,t;x_k - x_0)\right\}dt\ d\tau = \sum_{k=1}^n \int_c^d c_q(s,\ t)\times$$

$$\times\frac{d^\nu}{dt^\nu}\left\{\int_a^b \frac{d}{d\tau}\Big\{\omega_n(x_0(t) + g(\tau,t;x_k - x_0))[x_0(t) + g(\tau,t;x_k - x_0)]^q\Big\}d\tau l_{nk}(x(t))\right\}dt =$$

$$= \sum_{k=0}^n \int_c^d c_q(s,\ t)\frac{d^\nu}{dt^\nu}\Big\{[\omega_n(x_k(t))x_k^q(t) - \omega_n(x_0(t))x_0^q(t)]l_{nk}(x(t))\Big\}dt =$$

$$= \int_c^d c_q(s,\ t)\frac{d^\nu}{dt^\nu}\Big\{\sum_{k=0}^n l_{nk}(x(t))[\omega_n(x_k(t))x_k^q(t)]\Big\}dt - \tilde{F}(x_0).$$

For operators (3.9) the differential

$$\delta^{n+j}\tilde{F}[x;\ h_1,\ h_2,\ \dots,\ h_{n+j}] = \int_a^b c_q(s,\ t)\frac{d^\nu}{dt^\nu}[\psi_p(x(t))h_1(t)h_2(t)\cdots h_{n+j}(t)]dt,$$

where $\psi_p(x(t))$ is algebraic polynomial of degree p $(0 \le p = n+1+q-n-j = 1+q-j \le n)$ in regard to $x(t)$. Therefore for first sum in (3.5) the equality

$$\sum_{k=0}^n \delta^{n+j}\tilde{F}[x_k;\ \tilde{l}_{nkj}(x)] = \sum_{k=0}^n \int_c^d c_q(s,\ t)\frac{d^\nu}{dt^\nu}\Big\{\psi_p(x_k(t))\tilde{l}_{nkj}(x(t))\Big\}dt =$$

$$= \int_c^d c_q(s,\ t)\frac{d^\nu}{dt^\nu}\Big\{\sum_{k=0}^n \tilde{l}_{nkj}(x(t))\frac{d^{n+j}}{dx^{n+j}}[\omega_n(x_k(t))x_k^q(t)]\Big\}dt$$

takes place. Hence, for the operators (3.9) we finally obtain $H_{2n+j}(\tilde{F};\ x) =$

$$= \int\limits_c^d c_q(s,\ t)\frac{d^\nu}{dt^\nu}\Big\{\sum_{k=0}^{n} l_{nk}(x(t))[\omega_n(x_k(t))x_k^q(t)] + \sum_{k=0}^{n} \widetilde{l}_{nkj}(x(t))\frac{d^{n+j}}{dx^{n+j}}[\omega_n(x_k(t))x_k^q(t)]\Big\}dt.$$

>From here for $j = 1$ on the basis of proved lemma 2.1 the identity

$$H_{2n+1}(\tilde{F};\ x) = \int\limits_c^d c_q(s,\ t)\frac{d^\nu}{dt^\nu}\left[\omega_n(x(t))x^q(t)\right]dt \equiv \tilde{F}(x)$$

takes place. The theorem 3.1 is proved. □

Notice that special case of the formula (3.5), when the operator $g(\tau, t; x) = \tau x(t)$, and $\tau,\ t \in [a,\ b] = [0;\ 1]$ and $j = 1$, is obtained in [2].

Let's designate, further,

$$Q_{nk}(x) = l_{nk}(x) - l_{n-1k}(x),\ G_{nkj}(x) = \widetilde{l}_{nkj}(x) - \widetilde{l}_{n-1kj}(x)\ (k = 0,\ 1,\ ...,\ n+1;\ j \in \mathbb{N}),$$

where $l_{nn+1}(x) = \widetilde{l}_{nn+1j}(x) \equiv 0$.

For the error $R_{2n+j}(F;\ x) = F(x) - H_{2n+j}(F;\ x)$, where $H_{2n+j}(F;\ x)$ is interpolation polynomial (3.5), the representation

$$R_{2n+j}(F;\ x) = \sum_{k=0}^{n+1}\delta^{n+j}F\left[x_k; G_{n+1kj}(x)\right] + \tag{3.10}$$

$$+\sum_{k=1}^{n+1}\int\limits_a^b \delta F[x_0(\cdot) + g(\tau, \cdot; x_k - x_0);\ Q_{n+1k}(x(\cdot))g'_\tau(\tau, \cdot; x_k - x_0)]d\tau,$$

where $x_{n+1} = x$, takes place. Indeed, $R_{2n+j}(F;\ x_\nu) = 0$ for $\nu = 0,\ 1,\ ...,\ n$. If $\nu = n+1$, we obtain

$$R_{2n+j}(F;\ x_{n+1}) = -\sum_{k=1}^{n}\int\limits_a^b \delta F\left[x_0(\cdot) + g(\tau, \cdot; x_k - x_0);\ l_{nk}(x_{n+1}(\cdot))g'_\tau(\tau, \cdot; x_k - x_0)\right]d\tau +$$

$$+\int\limits_a^b \delta F\left[x_0(\cdot) + g(\tau, \cdot; x_{n+1} - x_0);\ g'_\tau(\tau, \cdot; x_{n+1} - x_0)\right]d\tau - \sum_{k=0}^{n}\delta^{n+j}F\left[x_k;\ \widetilde{l}_{nkj}(x_{n+1})\right] =$$

$$= \left[-H_{2n+j}(F;\ x_{n+1}) + F(x_0)\right] + \left[F(x_{n+1}) - F(x_0)\right].$$

Thus, the equality $R_{2n+j}(F;\ x_{n+1}) = F(x_{n+1}) - H_{2n+j}(F;\ x_{n+1})$ is true, i.e. expression (3.10) really represents interpolation error of operator $F(x)$ by polynomial $H_{2n+j}(F;\ x)$ of the form (3.5).

Notice that the interpolation formula (3.5) belongs to Lagrange's class , i.e. at the adding a new note in such formula all fundamental interpolation polynomials vary. This fact is certain shortage of Lagrange's formulas in contrast to polynomials of Newton's type, in which with increasing number of nodes new items are simplify added to previous ones.

Now we obtain Newton's variant of the formula (3.5), using identity

$$H_{2n+j}(F;\ x) = F(x_0) + \sum_{k=1}^{n}\triangle H_k(x), \tag{3.11}$$

where $\triangle H_\nu(x) = H_{2\nu+j}(x) - H_{2(\nu-1)+j}(x)$ for $\nu = 1,\ 2,\ ...,\ n;\ j \in \mathbb{N}$. Let polynomials $H_{2k+j}(x)$ $(k = 0,\ 1,\ ...,\ n;\ j \in \mathbb{N})$ are given by the equality (3.5), then for $\triangle H_k(x)$ we obtain the representation

$$\triangle H_k(x) = \sum_{\nu=1}^{k} \int_a^b \delta F\left[x_0(\cdot) + g(\tau, \cdot; x_\nu - x_0);\ Q_{k\nu}(x(\cdot))g'_\tau(\tau, \cdot; x_\nu - x_0)\right] d\tau +$$

$$+ \sum_{\nu=0}^{k} \delta^{n+j} F\left[x_\nu;\ G_{k\nu j}(x)\right].$$

Hereinafter we will suppose that for polynomials $l_{k-1k}(x)$ and $\widetilde{l}_{k-1kj}(x)$, entering the expressions $Q_{kk}(x)$ and $G_{kkj}(x)$, the equalities

$$l_{k-1k}(x) = \widetilde{l}_{k-1kj}(x) = 0 \quad (k = 1,\ 2,\ ...,\ n;\ j \in \mathbb{N}) \tag{3.12}$$

take place. It allows to formulate the following statement.

Corollary 3.2 *The operator*

$$H_{2n+j}(x) = F(x_0) + \sum_{k=1}^{n} \sum_{\nu=0}^{k} \delta^{n+j} F\left[x_\nu;\ G_{k\nu j}(x)\right] + \tag{3.13}$$

$$+ \sum_{k=1}^{n} \sum_{\nu=1}^{k} \int_a^b \delta F\left[x_0(\cdot) + g(\tau, \cdot; x_\nu - x_0);\ Q_{k\nu}(x(\cdot))g'_\tau(\tau, \cdot; x_\nu - x_0)\right] d\tau,$$

where j is some fixed natural number, is Newton interpolation polynomial for operator $F(x)$, satisfying the conditions (3.6) and (3.7). In the case $j > 1$ formula (3.13) is invariant with respect to operator polynomials $F(x) = P_n(x)$ of the form (3.1) of degree less or equal n. If $j = 1$ and $F(x) = P_{2n+1}(x)$ is operator polynomial of degree not above $2n+1$, then $H_{2n+1}(F;\ x) \equiv P_{2n+1}(x)$.

Note that error representation formula $R_{2n+j}(F;\ x) = F(x) - H_{2n+1}(F;\ x)$, where $H_{2n+1}(F;\ x)$ is interpolation polynomial (3.13), coincides with the equality (3.10).

4. Formulas containing differentials and Stieltjes integrals of interpolated operator

We construct, further, interpolation operator formulas of other structure, containing differentials and Stieltjes integrals of interpolated operator.

Let's introduce the scalar function

$$\chi(\tau,\ t) = \begin{cases} b, & \tau \geq t; \\ a, & \tau < t, \end{cases}$$

where $a < \tau < b$, $\chi(a,\ t) \equiv a$ and $\chi(b,\ t) \equiv b$.

By $Q_{2n+j}(x)$ we define operator polynomial of the form

$$Q_{2n+j}(x) = \sum_{k=0}^{2n+j} \int_a^b a_k(s,\ t)x^k(t)dt, \tag{4.1}$$

where $a_k(s,\ t)$ are some given functions $(s \in \mathbb{R}^\alpha,\ \alpha \in \mathbb{N};\ t \in [a,\ b];\ k = 0,\ 1,\ ...,\ 2n+j;\ j \in \mathbb{N})$.

Theorem 4.1 *For the operator polynomial*

$$H_{2n+j}(F;\ x) = F(x_0) + \sum_{k=0}^{n} \delta^{n+j} F[x_k;\ \widetilde{l}_{nkj}(x)] + \tag{4.2}$$

$$+\sum_{k=1}^{n} \int_a^b l_{nk}[x(\tau)] d_\tau F[x_0(\cdot) + g(\chi(\tau,\ \cdot), \cdot; x_k - x_0)],$$

where $l_{nk}(x) \equiv l_{nk}(x(t))$ *and* $\widetilde{l}_{nkj}(x) \equiv \widetilde{l}_{nkj}(x(t))$ *are the same as in theorem 3.1, and for operator* $g(\tau,\ t; x)$ *the relations (3.3) take place, interpolation conditions (3.6) and (3.7) are satisfied. In the case* $j > 1$ *interpolation formula (4.2) is invariant with respect to operator polynomials* $F(x) = Q_n(x)$ *of the form (4.1) of degree less or equal* n*. If* $j = 1$ *and* $F(x) = Q_{2n+1}(x)$ *is operator polynomial of degree not above* $2n + 1$*, then* $H_{2n+1}(F;\ x) \equiv Q_{2n+1}(x)$*.*

Proof. At $x = x_\nu$ the equalities $\widetilde{l}_{nkj}(x_\nu) = 0$, $l_{nk}(x_\nu) = \delta_{k\nu}$ take place for integer $j \geq 1$ and $k,\ \nu = 0,\ 1,\ ...,\ n$. Hence

$$H_{2n+j}(F;\ x_\nu) = F(x_0) + \int_a^b d_\tau F[x_0(\cdot) + g(\chi(\tau,\ \cdot), \cdot; x_\nu - x_0)] = F(x_\nu),$$

where $\nu = 0,\ 1,\ ...,\ n$.

Using, as well as earlier, the identities $\delta^{n+j} l_{nk}[x;\ h_1,\ h_2,\ ...,\ h_{n+j}] \equiv 0$ and equalities $\delta^{n+j}\widetilde{l}_{nkj}[x_i;\ h_1,\ h_2,\ ...,\ h_{n+j}] = \delta_{ki} h_1(t) h_2(t) \cdots h_{n+j}(t)$ for $k,\ i = 0,\ 1,\ ...,\ n$ we come to the relations (3.7).

Let operator $F(x)$ has the form

$$F(x) = \int_a^b a_q(s,\ t) x^q(t) dt, \tag{4.3}$$

where degree $q = 0,\ 1,\ ...,\ n$, then first sum in (4.2) will turn into zero, because the order of the differential is equal to $n + j > q$.

For k-th item of second sum in (4.2) and this type of operators, taking into account properties of the function $\chi(\tau,\ t)$, we get the equality

$$\int_a^b l_{nk}[x(\tau)] d_\tau F[x_0(\cdot) + g(\chi(\tau,\ \cdot), \cdot; x_k - x_0)] = \int_a^b l_{nk}[x(\tau)] \times$$

$$\times d_\tau \left\{ \int_a^\tau a_q(s,\ t) x_k^q(t) dt + \int_\tau^b a_q(s,\ t) x_0^q(t) dt \right\} = \int_a^b a_q(s,\ \tau) l_{nk}[x(\tau)] \left\{ x_k^q(\tau) - x_0^q(\tau) \right\} d\tau.$$

From here

$$H_{2n+j}(F;\ x) = \int_a^b a_q(s,\ t) \left[x_0^q(t) + \sum_{k=1}^{n} l_{nk}[x(t)] \left\{ x_k^q(t) - x_0^q(t) \right\} \right] dt =$$

$$= \int_a^b a_q(s,\ t) \sum_{k=0}^{n} l_{nk}[x(t)] x_k^q(t) dt = \int_a^b a_q(s,\ t) x^q(t) dt = F(x).$$

For further proof of the theorem, as before, it is more conveniently to consider operators of the form

$$\tilde{F}(x) = \int_a^b a_q(s,\ t)\omega_n(x(t))x^q(t)dt \ (\ q = 0,\ 1,\ ...,\ n+j-1), \tag{4.4}$$

since if the formula (4.2) is exact for operators (4.4), then it will be exact and for operators $F(x)$ of the form (4.3) for $q = n+1,\ n+2,\ ...,\ 2n+j$. The theorem 4.1 is proved. □

For interpolation error $R_{2n+j}(F;\ x) = F(x) - H_{2n+j}(F;\ x)$, where $H_{2n+j}(F;\ x)$ is a polynomial of the form (4.2), following representation takes place:

$$R_{2n+j}(F;\ x) = \sum_{k=1}^{n+1} \int_a^b Q_{n+1k}[x(\tau)]d_\tau F[x_0(\cdot) + g(\chi(\tau,\ \cdot), \cdot; x_k - x_0)] + \tag{4.5}$$

$$+ \sum_{k=0}^{n+1} \delta^{n+j} F\left[x_k; G_{n+1kj}(x)\right].$$

Here $x_{n+1} = x$. Indeed, at nodes x_ν for $\nu = 0,\ 1,\ ...,\ n$ we have $R_{2n+j}(F;\ x_\nu) = 0$. If $\nu = n+1$, then we obtain

$$R_{2n+j}(F;\ x_{n+1}) = -\sum_{k=1}^{n} \int_a^b l_{nk}[x_{n+1}(\tau)]d_\tau F[x_0(\cdot) + g(\chi(\tau,\ \cdot), \cdot; x_k - x_0)] +$$

$$+ \int_a^b d_\tau F[x_0(\cdot) + g(\chi(\tau,\ \cdot), \cdot; x_{n+1} - x_0)] - \sum_{k=0}^{n} \delta^{n+j} F\left[x_k; \widetilde{l}_{nkj}(x_{n+1})\right] =$$

$$= F(x_{n+1}) - H_{2n+j}(F;\ x_{n+1}).$$

Thus, the formula (4.5) really defines interpolation error of operator $F(x)$ by polynomial $H_{2n+j}(F;\ x)$ of the form (4.2).

Now we obtain Newton's variant of Lagrange formula (4.2). From the equality (4.2), taking into account the relation (3.12), we have

$$\triangle H_k(x) = \sum_{\nu=1}^{k} \int_a^b Q_{k\nu}[x(\tau)]d_\tau F[x_0(\cdot) + g(\chi(\tau,\ \cdot), \cdot; x_\nu - x_0)] + \sum_{\nu=0}^{k} \delta^{n+j} F\left[x_\nu; G_{k\nu j}(x)\right].$$

After substitution of this expression into the formula (3.11) we obtain Newton type formula.

Corollary 4.2 *The operator*

$$H_{2n+j}(F;\ x) = F(x_0) + \tag{4.6}$$

$$+ \sum_{k=1}^{n} \sum_{\nu=1}^{k} \int_a^b Q_{k\nu}[x(\tau)]d_\tau F[x_0(\cdot) + g(\chi(\tau,\ \cdot), \cdot; x_\nu - x_0)] + \sum_{k=1}^{n} \sum_{\nu=0}^{k} \delta^{n+j} F\left[x_\nu; G_{k\nu j}(x)\right],$$

where j is some fixed natural number, is Newton interpolation polynomial for operator $F(x)$, satisfying the conditions (3.6) and (3.7). In the case $j > 1$ formula (4.6) is invariant with respect to operator polynomials $F(x) = Q_n(x)$ of the form (4.1) of degree less or equal n. If $j = 1$ and $F(x) = Q_{2n+1}(x)$ is operator polynomial of degree not above $2n+1$, then $H_{2n+1}(F;\ x) \equiv Q_{2n+1}(x)$.

Interpolation error $R_{2n+j}(F;\ x) = F(x) - H_{2n+j}(F;\ x)$, where $H_{2n+j}(F;\ x)$ is interpolation polynomial (4.6), coincides with the representation (4.5).

Let, further, X and Y be any normed spaces. We assume, that for differentials $\delta^{\nu}F[x;\ h]$ $(\nu = 1,\ 2,\ ...,\ n+j;\ \ x,\ \ h \in X)$ the following inequalities are fulfilled: $\| \ \delta^{\nu}F[x;\ h] \ \| \leq M_{\nu} \ \| \ h \ \|$ $(0 \leq M_{\nu} < \infty)$. Let $M = \max\{M_1,\ \ M_{n+j}\}$, where j is some fixed natural number, then for error (3.10) the estimate

$$\| \ R_{2n+j}(F;\ x) \ \| \leq M \left\{ \sum_{k=1}^{n+1} \|Q_{n+1k}(x) g'_{\tau}(\tau, \cdot; x_k - x_0)\| + \sum_{k=0}^{n+1} \|G_{n+1kj}(x)\| \right\}$$

takes place. If function $F[x_0(\cdot) + g(\chi(\tau,\ \cdot), \cdot; x)]$ is of bounded variation on X, then for the interpolation error (4.5) we come to similar estimate

$$\| \ R_{2n+j}(F;\ x) \ \| \leq L \left\{ \sum_{k=1}^{n+1} \|Q_{n+1k}(x)\| + \sum_{k=0}^{n+1} \|G_{n+1kj}(x)\| \right\},\quad 0 \leq L < \infty.$$

One of extremal problems arising here consists in determination of interpolation nodes $x_k(t)$ $(k = 0,\ 1,\ ...,\ n)$ for which the sums of norms in estimates of errors have minimal value.

5. Some interpolation formulas for functions of matrix variables

The interpolation problem of matrix functions was studied earlier [4–6, 8], too. The construction questions of interpolation matrix polynomials of various structure were considered both for stationary, and for functional square and rectangular matrices.

The form and basic properties of interpolation matrix Hermite-Birkhoff polynomials, constructed further for functions, given on the set of stationary matrices, are similar to the interpolation polynomials, constructed earlier for the case of scalar functions.

Let X be a set of stationary square matrices of some fixed size, on which an operator $F : X \to X$, differentiable on X in the Gateaux sense, is defined, and interpolation nodes $A_0,\ A_1,\ ...,\ A_n$ be scalar matrices from X, such that $A_k = \alpha_k I$, where α_k is some pairwise different numbers $(k = 0,\ 1,\ ...,\ n)$, and element $I \in X$ is identity matrix. Let also values $F(A_0),\ F(A_1),\ ...,\ F(A_n)$ of operator $F(A)$ and values of its Gateaux differentials $\delta^{n+j}F[A_k;\ h_1,\ h_2,\ ...,\ h_{n+j}]$ of some fixed order $n+j$, in directions $h_{\nu} \in X$ $(\nu = 1, 2, ..., n+j;\ \ j \in \mathbb{N})$ are known. It is required to construct a matrix algebraic polynomial $H_{2n+j}(F,\ A) \equiv H_{2n+j}(A)$ of degree not above $2n+j$ with numerical coefficients, satisfying the conditions

$$H_{2n+j}(F;\ A_k) = F(A_k), \tag{5.1}$$

$$\delta^{n+j}H_{2n+j}[A_k;\ h_1,\ h_2,\ ...,\ h_{n+j}] = \delta^{n+j}F[A_k;\ h_1,\ h_2,\ ...,\ h_{n+j}], \tag{5.2}$$

where $k = 0,\ 1,\ ...,\ n$, and j is given natural number.

Let's define, as well as earlier, the algebraic polynomial $y_{nkj}(t)$ of degree $2n+j$ with respect to scalar variable t by equality $y_{nkj}(t) = \dfrac{1}{(n+j-1)!}\displaystyle\int_0^t (t-s)^{n+j-1} l_{nk}(s)ds$, where $k = 0,\ 1,\ \ldots,\ n;\ j \in \mathbb{N}$.

Now we introduce the notations

$$l_{nk}(A) = \frac{A - \alpha_0 I}{\alpha_k - \alpha_0} \cdots \frac{A - \alpha_{k-1} I}{\alpha_k - \alpha_{k-1}} \cdot \frac{A - \alpha_{k+1} I}{\alpha_k - \alpha_{k+1}} \cdots \frac{A - \alpha_n I}{\alpha_k - \alpha_n}, \tag{5.3}$$

$$\widetilde{l}_{nkj}(A) = y_{nkj}(A) - \sum_{\nu=0}^{n} l_{n\nu}(A) y_{nkj}(A_\nu), \tag{5.4}$$

where $k = 0, 1, \ldots, n$; $j \in \mathbb{N}$.

Let's assume that the independent variable A and directions $h_1, h_2, ..., h_{n+j}$, entering the interpolation conditions (5.2), are mutually permutable.

Note that in the case $j = 1$ the required Hermite-Birkhoff polynomial of degree not above $2n+1$ is constructed in [4] in the form

$$H_{2n+1}(A) = \sum_{k=0}^{n} \{l_{nk}(A)F(A_k) + \delta^{n+1}F[A_k;\ \widetilde{l}_{nk1}(A)]\}.$$

It is proved that the corresponding interpolation formula is exact for matrix algebraic polynomials of degree less or equal $2n+1$ with numerical coefficients.

Theorem 5.1 *For the matrix polynomial*

$$H_{2n+j}(F;\ A) = \sum_{k=0}^{n} \left[l_{nk}(A)F(A_k) + \delta^{n+j}F[A_k;\ \widetilde{l}_{nkj}(A)] \right], \tag{5.5}$$

where j is some fixed natural number, the conditions (5.1) *and* (5.2) *are satisfied. In the case $j > 1$ interpolation formula* (5.5) *is invariant with respect to matrix algebraic polynomials of degree less or equal n, and for $j = 1$ it is exact for a matrix polynomials of degree not above $2n+1$ with numerical coefficients.*

Proof. The equalities (5.1) follows from the relations $l_{nk}(A_\nu) = \delta_{k\nu}$, $\widetilde{l}_{nkj}(A_\nu) = 0$ for $k, \nu = 0, 1, ..., n$; $j \in \mathbb{N}$.

Taking into account, further, that $l_{nk}(A)$ is a matrix polynomial of degree n, we obtain $l_{nk}^{(n+j)}(A) \equiv 0$ for $k = 0, 1, ..., n$, and any natural value j. Besides, on account of requirement indicated above in regard to the permutability of matrices A and directions $h_1, h_2, ..., h_{n+j}$, we have $\delta^{n+j}\widetilde{l}_{nkj}[A_\nu;\ h_1, h_2, ..., h_{n+j}] = \delta_{k\nu} h_1 h_2 \cdots h_{n+j}$ for $k, \nu = 0, 1, ..., n$; $j \in \mathbb{N}$. It proves that for the formula (5.5) relations (5.2) will be fulfilled.

Let $Q_n(A)$ be a matrix polynomial of degree not above n with numerical coefficients. If $F(A) = Q_n(A)$, then the formula (5.5) coincides with interpolation polynomial Lagrange and, hence, $H_{2n+j}(F;\ A) \equiv Q_n(A)$. Invariance of the formula (5.5) for matrix polynomials of degree not above $2n+1$ in the case $j = 1$, as it was already noted, established in [4]. The theorem 5.1 is proved. □

Example 1. Let $F(x) = \int\limits_a^b K[s,t,x(t)]dt$ is Uryson operator, and interpolation nodes $x_k(t) \in C[a,b]$, $k = 0, 1, ..., n$. Then differential $\delta^\nu \tilde{F}[x;\ h] = \int\limits_a^b K_x^{(\nu)}[s,t,x(t)]h(t)dt$, $\nu = 1, 2, ..., n+m$, and Hermite-Birkhoff interpolation polynomial (3.5), where the operator $g(\tau,t;x) = \tau x(t)$, and $\tau, t \in [0;\ 1]$, takes the form

$$H_{2n+j}(F;\ x) = \sum_{k=0}^{n} \int\limits_a^b \left\{ K[s,t,x_k(t)] l_{nk}(x(t)) + K_x^{(n+j)}[s,t,x_k(t)] \widetilde{l}_{nkj}(x(t)) \right\} dt.$$

It is easy to notice that obtained polynomial satisfies the interpolation conditions (3.6) and (3.7), where $j \in \mathbb{N}$.

In summary we will note that a series of interpolation similar type formulas is obtained in [3, 4], and the theory of operator interpolation is fully enough investigated in monography [5], in which, in particular, special cases of Hermite-Birkhoff interpolation problem are also considered.

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