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EXPONENTIALLY ACCURATE SEPARATION OF VARIABLES IN DIFFERENTIAL EQUATIONS WITH OPERATOR COEFFICIENTS

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Анотація. Ми пропонуємо спосіб відокремлення змінної t при розв'язуванні задачі Коші $x'(t) = iBx, \ t \in (0, T]; x(0) = x_0$ з необмеженим лінійним оператором B в банаховому просторі. У випадку $B = B_1 + B_2 + \cdots + B_d$ ми пропонуємо метод розділення "просторових" операторів $B_1, B_2, ..., B_d$ на основі тензорного добутку. Результуюча дискретизація має експоненційну точність відносно параметра N і лінійну складність по d, якщо вхідні дані є аналітичними для B

Abstract. We propose a technique to separate the variable t at the solution of the initial value problem (IVP) $x'(t) = iBx, \ t \in (0, T]; x(0) = x_0$ with an unbounded linear operator B in Banach space. In the case $B = B_1 + B_2 + \cdots + B_d$ we propose a method to separate the "spartial" operators $B_1, B_2, ..., B_d$ based on the tensor product. The resulting discretisation possesses an exponentially accuracy with respect to the discretisation parameter N and a linear complexity in d provided the input data is analytical for B.

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1. Introduction

Exponential convergence of approximations to differential equations with unbounded operator coefficients plays a crucial role to obtain algorithms of optimal or near optimal complexity [9, 10, 12]. There are several general ideas leading to exponentially convergent approximations which are inter alia due to V.L.Makarov. One of them is the so called FD-method [25, 26]. The second idea is the representation of the exact solution or of the solution operator through an integral which is then discretized via an exponentially convergent quadrature rule [9–12, 17–19, 21]. The next idea which will be discussed in the present paper is the use of the Cayley transform of unbounded operators. This idea allows one to obtain exponentially convergent approximations in the case of analytical input data but in the same time represents a method to separate the "time variable". In the case when the "spartial" operator involved is of the form $B = B_1 + B_2 + \cdots + B_d$ one can separate the "spatial" operators $B_1, B_2, ..., B_d$ via the tensor product and so arrive at a fully separated approximation [7, 12].

In the theory of operators in Hilbert space the Cayley transform $T^{\alpha,\beta}_{\gamma,\delta} = (\alpha I + \beta A)(\gamma I - \delta A)^{-1}, \alpha, \beta, \gamma, \delta \in \mathbb{C}$ is frequently used to switch from the study of closed but in general unbounded linear operator A with dense domain $D(A)(\overline{D}(A) = H)$ to that of the bounded operators $T^{\alpha,\beta}_{\gamma,\delta}$ (see e.g. [1] where the transform $T_\gamma = (\gamma I + A)(\gamma I - A)^{-1}, \gamma = -i$ converts ta self adjoint (symmetric, dissipative) operator A into unitary (respectively, isometric, contractive) operator T_γ).

In [2, 3, 6, 13–16, 20, 22] the Cayley transform has been used to obtain explicit and constructive representations of the solutions of various evolution differential equations with operator coefficients where, in fact, the solution with continuous time parameter

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where represented through the ones with discrete time. A furher important feature of these representation is the fact that they serve as the basis for algorithms without accuracy saturation, i.e. their accuracy increases automatically and unboundedly together with the smoothness of the solution. In the case of analytical input data the convergence rate becomes exponential.

A similar idea to turn processes with continuous time parameter into such with discrete time via the Cayley transform was used in prediction theory of stationary stochastic processes [24].

Besides, these representations can be considered as a method to separate the time and the "spatial" variables.

In [8] this idea was applied to the Schrödinger differential equation in abstract setting with a self-adjoint "spatial" operator coefficient B in some Hilbert space. On the basis of an exact representation of the solution with use of the Cayley transform, an approximation was proposed with the accuracy depending on the smoothness of this solution. It was shown that for the analytical initial vectors this approximation possesses a super exponential convergence rate.

In the present paper we extend the results from [8] to the case of a strongly positive operator coefficient in a Banach space.

The paper is organized as follows. In Section 2 we derive an explicit representation of the solution operator as a series where the time variable is separated in the Laguerre polynomials and the "spatial" operator is isolated at the powers of a Cayley transform. The convergence properties of the series are studied. Section 3 is devoted to the truncated series as an approximation to the exact solution. We show that this approximation does not possess the accuracy saturation, i.e. its accuracy depends on the smoothness of the exact solution. In the case of an analytical initial vector we prove a super exponential convergence of our approximation. The error of the new approximation tends to zero as $N \to \infty$ as well as $t \to 0$. The second property allows us to develope a preconditioning technique which is the topic of Section 4. The idea of the tensor-product representation of the solution operator is discussed in Section 5.

2. An exact representation of the solution

Let X be a Banacht space and B be a closed unbounded operator in this space with the domain $D(B)$ and the spectrum $\Sigma(B)$. We suppose the operator B to be strongly positive, i.e. there exists a closed path Γ in the complex plane which consists of two rays

$$\mathcal{R}(\pm\varphi) = \{\rho e^{\pm\varphi} : 0 < \underline{\gamma} \le \rho \le \overline{\gamma} \le \infty\} \tag{2.1}$$

and of the circular arc

$$\mathcal{A}(\underline{\gamma}, \varphi) = \mathcal{A}(\underline{\gamma}, \varphi) = \{\rho e^{z} : |z| = \underline{\gamma}, |\mathrm{arg} z| \le \varphi\} \tag{2.2}$$

for an unbounded B or of two arcs $\mathcal{A}(\underline{\gamma}, \varphi)$ and $\mathcal{A}(\overline{\gamma}, \varphi)$ for a bounded one surrounding a domain Ω_Γ such that $\varphi \in (0, \pi/2), \Sigma(A) \in \Omega_\Gamma$ and the estimate for the resolvent

$$\|(zI - B)^{-1}\| \le M/(1 + |z|) \tag{2.3}$$

with a positive constant M holds on Γ and outside of Ω_Γ. The angle φ is called the spectral angle of the operator B. As an example of a strongly positive operator can serve a strongly elliptic linear partial differential operator (see e.g. [27]).

Let us consider the following initial value problem for an equation of the Schrödinger type

$$u'(t) = iBu(t), \quad u(0) = u_0. \tag{2.4}$$

The solution operator of this problem is the operator family $S(t) = S(t; B)$ such that the solution of the IVP is given by $u(t) = S(t)u_0$. This operator is the so called Schrödinger operator exponential $S(t) = e^{iBt}$ which can be represented by the Dunford-Cauchy integral

$$S(t) = S(t; B) = \frac{1}{2\pi i}\int_\Gamma e^{izt}(zI - B)^{-1}dz, \tag{2.5}$$

provided that the path Γ envelopes the spectrum of B and the improper integral converges. We say that this operator is generated by B as well as by the function $F(t, z) = e^{itz}$. One can separate the variable t from the operator B by the separation of the variables t and z in the function $F(t, z) = e^{itz}$.

The general idea is the following: if B is an unbounded operator then the variable z belongs to an unbounded domain in the complex plane. If we will to switsh to the study of bounded operators then an option can be to use some rational transform $z = \frac{\lambda\alpha-\beta w}{-\lambda+w}, w = \lambda\frac{\alpha+z}{\beta+z} = \frac{a+bz}{c+dz}, a/c \neq b/d, \lambda = b/d, \alpha = a/b, \beta = c/d, \alpha \neq \beta$ where the variable w can remain in some bounded domain. The function $F(t, z) = F(t, \frac{\lambda\alpha-\beta w}{-\lambda+w})$ can be represented as a power series in w or approximated by a polynomial of w and we obtain a function of the bounded variable w.

If $F(t, z)$ is analytical with respect to z in the unit disc $|z| < 1$ then the Taylor expansion

$$F(t, z) = \sum_{n=0}^{\infty} c_n(t)z^n, \ |z| < 1 \tag{2.6}$$

separates the both variables. The Taylor coefficients are given by

$$c_n(t) = \frac{1}{n!}F_z^{(n)}(t, z)\big|_{t=0} = \frac{1}{2\pi i}\int_{|\xi|=\rho} \frac{F(t, z)}{\xi^{n+1}}d\xi. \tag{2.7}$$

Example 2.1 Let $F(t, z) = F(t, z; \alpha) = (1-z)^{-\alpha-1}e^{\frac{tz}{z-1}}$, where for not integer parameter α we mean the principal value [28]. This is the generating function for the Laguerre polynomials $L_n^{(\alpha)}(t)$. For each fixed t the function $F(t, z)$ is analytic in the disc $|z| < 1$. We have for it's Taylor coefficients

$$c_n(t) = c_n(t; \alpha) = \frac{1}{2\pi i}\int_{|\xi|=\rho} (1 - \xi)^{-\alpha-1}e^{\frac{t\xi}{\xi-1}}\xi^{-n-1}d\xi, \ 0 < \rho < 1. \tag{2.8}$$

We change the variables by

$$u = \frac{t}{1 - \xi}, \xi = 1 - \frac{t}{u}, d\xi = \frac{t}{u^2}du, \tag{2.9}$$

where the linear-fractional mapping $u = \frac{t}{1-\xi}$ translates the circle $|\xi| = \rho < 1$ into a circle Γ which includesthe point $t > 0$ but does not include the point 0. Then we get

$$\begin{aligned} c_n(t) = c_n(t; \alpha) &= \frac{1}{2\pi i}\int_\Gamma \frac{u}{t}e^{t-u}(\frac{u - t}{u})^{-n-1}\frac{t}{u^2}du \\ = t^{-\alpha}e^t\frac{1}{2\pi i}\int_\Gamma \frac{u^{\alpha+n}e^{-u}}{(u - t)^{n+1}}du &= t^{-\alpha}e^t\frac{1}{n!}(t^{\alpha+n}e^{-t})^{(n)}. \end{aligned} \tag{2.10}$$

Comparing this equality with the Rodrigue's formula for the Laguerre polynomials we see that $c_n(t) = c_n(t; \alpha) = L_n^{(\alpha)}(t)$, i.e. we have the expansion (see also [4, v.2, Ch. 10.12])

$$(1 - z)^{-\alpha-1}e^{\frac{tz}{z-1}} = \sum_{n=0}^{\infty} L_n^{(\alpha)}(t)z^n \tag{2.11}$$

Using the last expansion after the substitution $z \to iB(iB-I)^{-1}$ we obtain formally

$$e^{iBt} = -(iB-I)^{-1}\sum_{n=0}^{\infty} L_n^{(0)}(t)T^n, \tag{2.12}$$

where $T = T(B) = iB(iB-I)^{-1}$ is the Cayley transform of the operator B.

Now, the solution of the IVP can be formally represented by

$$x(t) = \sum_{n=0}^{\infty} L_n^{(0)}(t)u_n, \tag{2.13}$$

where the elements u_n can be found from the recursion

$$\begin{aligned} u_0 &= -(iB-I)^{-1}x_0, \\ u_{p+1} &= iB(iB-I)^{-1}u_p, \quad p = 0, 1, ... \end{aligned} \tag{2.14}$$

In other words, the elements u_p of the sequence are solutions of the operator equations

$$\begin{aligned} (iB-I)u_0 &= -x_0, \\ (iB-I)u_{p+1} &= iBu_p, \quad p = 0, 1, ... \end{aligned} \tag{2.15}$$

From (2.15) it follows that

$$u_p = iB(u_p - u_{p-1}), p = 0, 1, 2, ...; u_{-1} = iB^{-1}x_0 \tag{2.16}$$

or

$$u_p = T^p x_0, p = 0, 1, 2, ... \tag{2.17}$$

Substituting this representation into (2.13) and using the summation by parts

$$\sum_{n=1}^{N-1} u_n v_n = u_N v_{N-1} - u_0 v_0 - \sum_{n=0}^{N-1} u_n v_n \tag{2.18}$$

we obtain

$$\begin{aligned} x(t) &= u_0 + iB\sum_{n=1}^{\infty} L_n^{(0)}(t)(u_n - u_{n-1}) = \\ &= u_0 - iBu_0 L_0^{(0)}(t) - iB\sum_{n=0}^{\infty}(L_{n+1}^{(0)}(t) - L_n^{(0)}(t))u_n = \\ &= x_0 - iB\sum_{n=0}^{\infty}(L_{n+1}^{(0)}(t) - L_n^{(0)}(t))u_n. \end{aligned} \tag{2.19}$$

Here we have used the following properties of the Laguerre polynomials (see e.g. [28, p. 243, 248], [4, vol. 2, Ch. 10.18]:)

$$L_n^{(\alpha)}(t) = \frac{1}{\sqrt{\pi}} e^{t/2} t^{-\alpha/2-1/4} n^{\alpha/2-1/4} \cos(2\sqrt{nt} - \frac{\alpha\pi}{2} - \frac{\pi}{4}) + \mathcal{O}\left(n^{\alpha/2-3/4}\right), \tag{2.20}$$

$$\begin{aligned} &L_n^{(\alpha)}(t) = \pi^{-1/2} e^{t/2} t^{-\alpha/2-1/4} n^{\alpha/2-1/4}[\cos(2\sqrt{nt} - \beta\pi) + \mathcal{O}(n^{-1/2})],\ t \in [a, b], \\ &0 < a < b < \infty,\ \beta = (2\alpha+1)/4, \\ &|L_n^{(\alpha)}(t)| \le c n^{\frac{\alpha}{2}-\frac{1}{4}} t^{-\frac{\alpha}{2}-\frac{1}{4}} e^{\frac{t}{2}}(1 + n^{-\frac{1}{4}} t^{\frac{5}{4}}), \alpha + \frac{1}{2} \ge 0, t \ge 0, \\ &e^{-t/2}|L_n^{(0)}| \le 1, t \ge 0, \end{aligned} \tag{2.21}$$

where c is a constant independent of k and $a, b (0 < a < b)$ are arbitrary fixed numbers. The first property yields $\lim_{n\to\infty} L_n^{(0)}(t) = 0$ (for each fixed t).

Remark 2.2 One can prove that representation (2.19) satisfies equation (2.4) also using the well known relation [4, 28]

$$\frac{d}{dt}\left[L_{n+1}^{(0)}(t) - L_n^{(0)}(t)\right] = -L_n^{(0)}(t), \tag{2.22}$$

which together with (2.13) yields

$$\dot{x}(t) = iB\sum_{n=0}^{\infty} L_n^{(0)}(t)u_n = iBx(t). \tag{2.23}$$

To investigate the convergence of the series (2.19) we will need the next lemmas.

Lemma 2.3 *Let B be a strongly positive operator in Banach space X and*

$$T = iB(iB - I)^{-1}$$

be its Cayley transform , then

$$\|T^n B^{-\sigma}\| = \|[iB(iB - I)^{-1}]^n B^{-\sigma}\| \le cn^{-\sigma+\varepsilon} \tag{2.24}$$

where c is a constant independent of n, σ is a fixed positive number and $\varepsilon \in (0, \sigma)$ is an arbitrarily small number.

Proof. We have for n large enough

$$\begin{aligned}
\|T^n B^{-\sigma}\| = \| & \frac{1}{2\pi i}\int_{\mathcal{R}(+\varphi)} \left(\frac{i\rho e^{i\varphi}}{i\rho e^{i\varphi} - 1}\right)^n (\rho e^{i\varphi})^{-\sigma}(\rho e^{i\varphi} - B)^{-1} e^{i\varphi} d\rho \\
&+ \frac{1}{2\pi i}\int_{\mathcal{A}(\underline{\gamma},\varphi)} (\frac{i\underline{\gamma} e^{i\theta}}{i\underline{\gamma} e^{i\theta} - 1})^n (\underline{\gamma} e^{i\theta})^{-\sigma}(\underline{\gamma} e^{i\theta} - B)^{-1}\underline{\gamma} d\theta \\
&+ \frac{1}{2\pi i}\int_{\mathcal{R}(-\varphi)} (\frac{i\rho e^{-i\varphi}}{i\rho e^{-i\varphi} - 1})^n (\rho e^{-i\varphi})^{-\sigma}(\rho e^{-i\varphi} - B)^{-1} e^{-i\varphi} d\rho\| \\
&\le c\Big\{\int_{\underline{\gamma}}^{\infty} \left(\frac{\rho^2}{1 + 2\rho\sin\varphi + \rho^2}\right)^{n/2} \frac{\rho^{-\sigma+\varepsilon}}{\rho^{\varepsilon}(1+\rho)} d\rho \\
&+ \int_{\underline{\gamma}}^{\infty} \left(\frac{\rho^2}{1 - 2\rho\sin\varphi + \rho^2}\right)^{n/2} \frac{\rho^{-\sigma+\varepsilon}}{\rho^{\varepsilon}(1+\rho)} d\rho \\
&+ \int_{-\varphi}^{\varphi} \left(\frac{\underline{\gamma}^2}{1 + 2\underline{\gamma}\sin\theta + \underline{\gamma}^2}\right)^{n/2} \frac{1}{\underline{\gamma}^{\varepsilon}(1+\underline{\gamma})} d\theta\Big\}
\end{aligned} \tag{2.25}$$

where c denotes a constant independent of n and

$$\psi_\pm(\rho) = \psi_\pm(\rho, n, \varepsilon) = \left(\frac{\rho^2}{1 \pm 2\rho\sin\varphi + \rho^2}\right)^{n/2} \frac{1}{\rho^{\sigma+\varepsilon}}. \tag{2.26}$$

It is easy to see that the functions $\psi_\pm(\rho)$ take their maximums

$$\psi_{\pm,max} = \psi(\rho_\pm, n, \varepsilon) = \mathcal{O}(n^{-\sigma+\varepsilon}) \tag{2.27}$$

at some points

$$\rho_\pm = \mathcal{O}(n) \tag{2.28}$$

and the last summand in(2.25) decays exponentially in n which proves the lemma. □

Lemma 2.4 *Let B be a densely defined, strongly positive operator and $T = iB(iB-I)^{-1}$ its Cayley transform, then*

$$\|T^n e^{-sB} B^{\varepsilon}\| \le c \exp\Big(- c_1 s \sqrt[3]{\frac{n}{s}}\Big) \Big(\sqrt[3]{\frac{n}{s}}\Big)^{\varepsilon} \tag{2.29}$$

where s is a fixed positive number, ε is an arbitrarily small positive number and the positive constants c, c_1 do not depend on n.

Proof. Analogously as above we have

$$\begin{aligned}
\|T^n e^{-sB} B^{\varepsilon}\| &= \|\frac{1}{2\pi i}\int_{\mathcal{R}(+\varphi)} \Big(\frac{i\rho e^{i\varphi}}{i\rho e^{i\varphi}-1}\Big)^n e^{-s\rho e^{i\varphi}} (\rho e^{i\varphi} - B)^{-1} (\rho e^{i\varphi})^{\varepsilon} e^{i\varphi} d\rho \\
&+ \frac{1}{2\pi i}\int_{\mathcal{A}(\underline{\gamma},\varphi)} \Big(\frac{i\underline{\gamma} e^{i\theta}}{i\underline{\gamma} e^{i\theta}-1}\Big)^n e^{-s\underline{\gamma} e^{i\theta}} (\underline{\gamma} e^{i\theta} - B)^{-1} (\underline{\gamma} e^{i\theta})^{\varepsilon} \underline{\gamma} d\theta \\
&+ \frac{1}{2\pi i}\int_{\mathcal{R}(-\varphi)} \Big(\frac{i\rho e^{-i\varphi}}{i\rho e^{-i\varphi}-1}\Big)^n e^{-\rho e^{-i\varphi}} (\rho e^{-i\varphi} - B)^{-1} (\rho e^{-i\varphi})^{\varepsilon} e^{-i\varphi} d\rho\| \\
&\le c[\max_{\rho\in[\underline{\gamma},\infty)} \psi_+(\rho) + \max_{\rho\in[\underline{\gamma},\infty)} \psi_-(\rho) + \max_{\theta\in(-\varphi,\varphi)} \psi_0(\theta)]
\end{aligned} \tag{2.30}$$

where

$$\begin{aligned}
\psi_{\pm}(\rho) &= \psi_{\pm}(\rho, n, \varphi, \varepsilon) = \Big(\frac{\rho^2}{1 \pm 2\rho \sin\varphi + \rho^2}\Big)^{n/2} e^{-s\rho\cos\varphi} \rho^{\varepsilon} \\
\psi_0(\theta) &= \psi_0(\theta, n, \varphi, \varepsilon) = \Big|\frac{i\underline{\gamma} e^{i\theta}}{i\underline{\gamma} e^{i\theta} - 1}\Big|^n \cdot \Big|e^{-s\underline{\gamma} e^{i\theta}} (\underline{\gamma} e^{i\theta})^{\varepsilon}\Big|.
\end{aligned} \tag{2.31}$$

Since

$$\begin{aligned}
\psi'_{\pm}(\rho) &= \frac{n}{2}\Big(\frac{\rho^2}{1 \pm 2\rho\sin\varphi + \rho^2}\Big)^{n/2-1} \frac{2\rho \pm 4\rho^2 \sin\varphi \mp 2\rho^2 \sin\varphi}{(1 \pm 2\rho\sin\varphi + \rho^2)^2} e^{-s\rho\cos\varphi}\rho^{\varepsilon} \\
&- s\cos\varphi\Big(\frac{\rho^2}{1 \pm 2\rho\sin\varphi + \rho^2}\Big)^{n/2} e^{-s\rho\cos\varphi}\rho^{\varepsilon} + \varepsilon\Big(\frac{\rho^2}{1 \pm 2\rho\sin\varphi + \rho^2}\Big)^{n/2} e^{-s\rho\cos\varphi}\rho^{\varepsilon-1} \\
&= \Big(\frac{\rho^2}{1 \pm 2\rho\sin\varphi + \rho^2}\Big)^{n/2-1} e^{-s\rho\cos\varphi}\rho^{\varepsilon}\Big[\frac{n}{2} \cdot \frac{2\rho \pm 2\rho^2\sin\varphi}{(1 \pm 2\rho\sin\varphi + \rho^2)^2} \\
&- s\cos\varphi \frac{\rho^2}{1 \pm 2\rho\sin\varphi + \rho^2} + \varepsilon\frac{\rho}{1 \pm 2\rho\sin\varphi + \rho^2}\Big]
\end{aligned} \tag{2.32}$$

then the equations $\psi'_{\pm}(\rho) = 0$ imply the cubic equations

$$a\rho^3 + b\rho^2 + c\rho + d = 0 \tag{2.33}$$

with

$$a = s\cos\varphi, b = [-\varepsilon \pm s\sin(2\varphi)], c = [\mp n\sin\varphi + s\cos\varphi \mp 2\varepsilon\sin\varphi], d = -(n+\varepsilon). \tag{2.34}$$

Below we will refere to these equations with various signs at the coefficients (2.34) as the (+)-equation (coefficients with the upper signs) and the (-)-equation (coefficients with the lower signs). The behavior of their solutions is determined by the discriminants [5]

$$D_{\pm} = q^2 + p^3, \tag{2.35}$$

where

$$q = \frac{b^3}{27a^3} - \frac{bc}{6a^2} + \frac{d}{2a}, p = \frac{3ac - b^2}{9a^2}. \tag{2.36}$$

It is easy to see that for n large enough we have

$$D_+ \asymp \begin{cases} -\left(\frac{n\tan\varphi}{s}\right)^3, & \text{if } \varphi \neq 0, \\ \left(\frac{n}{s}\right)^2, & \text{if } \varphi = 0 \end{cases} \qquad D_- \asymp \begin{cases} \left(\frac{n\tan\varphi}{s}\right)^3, & \text{if } \varphi \neq 0, \\ \left(\frac{n}{s}\right)^2, & \text{if } \varphi = 0. \end{cases} \tag{2.37}$$

If $\varphi \neq 0$ then we have at the $(+)$-equation (see [5]) $p \asymp -\frac{n\tan\varphi}{s} < 0, q \asymp -\frac{n}{s\cos\varphi} < 0, r = \operatorname{sign}(q)\sqrt{|p|} \asymp -\sqrt{\frac{n\tan\varphi}{s}} < 0$, therefore the equation possesses a positive real root $\rho_1 = -2r\cos\frac{\phi}{3} - b/a \asymp \sqrt{\frac{n\tan\varphi}{s}}$, where $\cos\phi = q/r^3$. In the case $\varphi = 0$ we obtain $p = \frac{3s^2-\varepsilon^2}{3s^2} > 0, q \asymp -\frac{n}{s\cos\varphi} < 0$, $r = \operatorname{sign}(q)\sqrt{|p|} < 0$ and $\rho_1 = -2r\sinh(\phi/3) \asymp -2r\sqrt[3]{q/r^3}$ with $\sinh\phi = q/r^3$ is a positive root with the asymptotics $\rho_1 \asymp \sqrt[3]{n/(s\cos\varphi)}$. If $\varphi \neq 0$ then we have at the (-)-equation $p \asymp \frac{n\tan\varphi}{s} > 0, q \asymp -\frac{n}{s\cos\varphi} < 0, p^3 + q^2 > 0, r = \operatorname{sign}(q)\sqrt{|p|} \asymp -\sqrt{\frac{n\tan\varphi}{s}} < 0$, therefore the equation possesses a positive real root $\rho_2 = -2r\cosh\frac{\phi}{3} - b/a \asymp -r\sqrt[3]{q/r^3} \asymp \sqrt[3]{\frac{n}{s\cos\varphi}}$, where $\cosh\phi = q/r^3 > 0$. In the case $\varphi = 0$ we obtain $p = \frac{3s^2-\varepsilon^2}{3s^2} > 0, q \asymp -\frac{n}{s} < 0$, and $\rho_2 = -2r\sinh(\phi/3) \asymp -2r\sqrt[3]{q/r^3} > 0$ with $\sinh\phi = q/r^3 > 0$ is a positive root with the asymptotics $\rho_2 \asymp \sqrt[3]{n/s}$. Thus, the both values $\max_{\rho\in[\underline{\gamma},\infty)} \psi_\pm(\rho)$ are bounded by $c\,\exp\big(-c_2\sqrt[3]{\frac{n}{s}}\big)\left(\sqrt[3]{\frac{n}{s}}\right)^{\varepsilon}$ with constants c, c_1 independent of n. The third summand in (2.31) can be estimated as follows

$$\max_{\theta\in(-\varphi,\varphi)} |\psi_0(\theta)| \leq \left|\frac{\underline{\gamma}^2}{\underline{\gamma}^2 + 1 - 2\underline{\gamma}\sin\theta}\right|^{n/2} \cdot \left|e^{-s\underline{\gamma}\cos\theta}(\underline{\gamma}^{\varepsilon}\right| \leq cq^{n/2} \tag{2.38}$$

where $\underline{\gamma}$ is choosen so that $0 < q = \frac{\underline{\gamma}^2}{\underline{\gamma}^2+1-2\underline{\gamma}\sin\theta} < 1$ and c does not depend on n.

Thus, we arrive at the estimates (2.29) and the lemma is proved. □

It holds [28]

$$|L_n^{(0)}(t) - L_{n-1}^{(0)}(t)| = \frac{t}{n}|L_{n-1}^{(1)}(t)| \leq ct^{1/4}e^{t/2}n^{-3/4}(1 + n^{-1/4}t^{5/4}) \tag{2.39}$$

uniformly in $t \in [0, T]$. Combining this estimate with (2.24) we obtain the following majorant for the series (2.19)

$$\sum_{p=1}^{\infty} p^{-\sigma+\varepsilon-3/4} \tag{2.40}$$

which proves the following assertion.

Theorem 2.5 *Series (2.19) converges for all* $t \geq 0$ *provided that* $x_0 \in D(B^\sigma), \sigma > 1/4$.

3. Approximation of the exact solution

The truncated series

$$x_N^{(1)}(t) \equiv x_N^{(1)}(t;x_0) \equiv T_N(t;B)x_0 = x_0 - iB\sum_{n=0}^{N}(L_{n+1}^{(0)}(t) - L_n^{(0)}(t))u_n \tag{3.1}$$

represents an algorithm for the approximate solution of problem (2.4). The accuracy of this approximation is given by the next theorem.

Theorem 3.1 *Let $x_0 \in D(B^\sigma), \sigma > 1/4$, then trancated series (3.1) approximates the exact solution of (2.4) given by (2.19) with the error estimate*

$$\|x(t) - x_N^{(1)}(t)\| \le ct^{1/4}e^{t/2}N^{-\sigma+1/4+\varepsilon}\|B^\sigma x_0\|. \tag{3.2}$$

Proof. The proof follows immediately from (2.39), (2.40). □

The convergence rate of the approximation (3.1) becomes exponetial if we assume that the initial vector x_0 is analytical with respect to B. Let us recall that vectors from $\bigcap_{n=1}^{\infty} D(B^n)$ are called $C^\infty-$vectors for the operator B [23, Ch. 1, §9.20]. For example, for the operator $B = \frac{d}{dx}$ the functions of the class C^∞ are $C^\infty-$vectors. A vector f is called analytical for B if $f \in C^\infty$ and the power series

$$\sum_{p=0}^{\infty}\frac{z^n}{n!}\|B^n f\| \tag{3.3}$$

possesses a positive convergence radius r. For an analytical vector f we introduce the following notation

$$|||f|||_{r,B} = \sum_{k=0}^{\infty}\frac{r^k}{k!}\|B^k f\|.$$

Example 3.2 Let B be an operator in $\mathbb{R}^n$ represented by a matrix B. Each vector $f \in \mathbb{R}^n$ is analytical because the power series

$$\sum_{p=0}^{\infty}\frac{z^n}{n!}\|B^n f\| \le e^{z\|B\|}\|f\| \tag{3.4}$$

converges for all $z \in \mathbb{R}$ due to $\|B^n f\| \le \|B\|^n\|f\|$.

Example 3.3 Let us consider the operator B in $L_2(0,\pi)$ defined by

$$\begin{aligned} &D(B) = \{u \in H^2(0,\pi) : u(0) = u(\pi) = 0\},\\ &B = -\frac{d^2u}{dt^2} \quad \forall \quad u \in D(B). \end{aligned} \tag{3.5}$$

The vector $f = f(t) = \sin t$ is analytical for B since $B^n f = \sin t$,

$$\|B^n f\| = \left(\int_0^\pi \sin^2 tdt\right)^{1/2} = \sqrt{\pi/2}\,\forall\, n$$

and the power series

$$\sum_{p=0}^{\infty}\frac{t^n}{n!}\|B^n f\| = \sqrt{\pi/2}\sum_{p=0}^{\infty}\frac{t^n}{n!} = \sqrt{\pi/2}e^t$$

converges for all $t \in \mathbb{R}$.

Example 3.4 Let us consider the operator B in $L_2(-\infty,\infty)$ defined by

$$\begin{aligned} &D(B) = \{u \in H^2(-\infty,\infty) : \psi(-\infty) = \psi(\infty) = 0\}, \\ &B = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{m\omega^2}{2}x^2\psi \quad \forall \quad \psi \in D(B). \end{aligned} \tag{3.6}$$

In the quantum mechanics the operator B is the Hamiltonian of an oscillated particle of mass m subject to a potential V(x) given by $V(x) = \frac{1}{2}m\omega^2x^2$, where ω is the angular frequency of the oscillator and $\hbar$ is the Planck's constant devided by 2π. It can be shown that the normalized eigenfunctions (subject to $\int_{-\infty}^{\infty}\psi^2(x)dx = 1$) are

$$\psi_n(x) = C_n e^{-\xi^2/2}H_n(\xi), \tag{3.7}$$

where $x = \alpha\xi, \alpha = \sqrt{\frac{\hbar}{m\omega}}, C_n = \frac{1}{\sqrt{\alpha}} \cdot \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}$ and $H_n(\xi)$ are the Hermit polynomials [28,29]. These eigenfunctions correspond to the eigenvalues $E_n = (n+1/2)\hbar\omega$. The vector $u_0(x) = \sum_{k=0}^{m}\gamma_k\psi_k(x)$ is analytical for B since

$$\begin{aligned} &B^n u_0 = (\hbar\omega)^n\sum_{k=0}^{m}\gamma_k(k+1/2)^n\psi_k(x), \\ &\|B^n u_0\| = (\hbar\omega)^n\left(\int_{-\infty}^{\infty}\left[\sum_{k=0}^{m}\gamma_k(k+1/2)^n\psi_k(x)\right]^2\right)^{1/2} \\ &\leq (\hbar\omega)^n\sqrt{m}\left(\sum_{k=0}^{m}\gamma_k^2(k+1/2)^{2n}\right)^{1/2}\left(\sum_{k=0}^{m}\int_{-\infty}^{\infty}\psi_k^2(x)dx\right)^{1/2} \\ &\leq (\hbar\omega)^n m(m+1/2)^n\left(\sum_{k=0}^{m}\gamma_k^2\right)^{1/2}. \end{aligned} \tag{3.8}$$

and the power series

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}\|B^n f\| \leq \left(\sum_{k=0}^{m}\gamma_k^2\right)^{1/2}\sum_{n=0}^{\infty}\frac{(\hbar\omega)^n m(m+1/2)^n}{n!}z^n$$

converges for all $z \in \mathbb{R}$.

If B_h is a grid approximation of an unbounded operator B on a grid with n nodes and with the mesh size h, then $n, \|B_h\|$ tend to infinity as the characteristic mesh value h tends to zero. We call such operators *quasibounded*. In this case the convergence radius of the series

$$\sum_{n=0}^{\infty}\frac{\|B_h^n x_0\|z^n}{n!} \tag{3.9}$$

for an initial vector x_0 can tend to zero together with $h \to 0$ and the definition of the analyticity above has no sense more. This fact motivates the following definition of the analytical vectors for B_h.

Definition 3.5 *A vector $x_0 \in \mathbb{R}^n$ is called analytical for $B = B_h$ uniformly in h if the series (3.9) possesses a positive convergence radius independent of h.*

Example 3.6 Let $\omega = \omega_h = \{\xi_i = ih : i = 1, 2, ..., n; h = l/(n+1)\}$ be a grid on the interval $(0, l)$ and $\overline{\omega} = \omega \cup \{\xi_0 = 0, \xi_{n+1} = l\}$ be a grid covering the closed interval $[0, l]$. Let $\mathbb{R}^n$ be the space of grid functions defined on $\overline{\omega}$ and vanishing at $\xi_0 = 0, \xi_{n+1} = l$. The matrix $B = B_h$ corresponding to the difference operator

$$\begin{aligned} &D(B_h) = \{y(x), x \in \overline{\omega} : y(0) = 0, y(l) = 0\}, \\ &By = -y_{\overline{x}x} \ \forall\, y \in \mathbb{R}^n \end{aligned} \tag{3.10}$$

possesses the eigenvalues $\lambda_{h,k} = \frac{4}{h^2}\sin^2\frac{k\pi h}{2l}$ corresponding to the eigenfunctions $y_k(x) = \sin\frac{k\pi x}{l}$, $x \in \overline{\omega}, k = 1, 2, ..., n$ with the norm

$$\|y_k(x)\| = \left(\sum_{j=1}^{n} h y_k^2(x_j)\right)^{1/2} = \sqrt{l/2}. \tag{3.11}$$

The vector $x_0 = \sin\frac{m\pi x}{l}$ for each fixed m is analytical for B uniformly in h, since due to

$$\begin{aligned} &B^j x_0 = \lambda_{h,m}^j \sin\frac{m\pi x}{l}, \\ &\|B^j x_0\| = \lambda_{h,m}^j \sqrt{l/2}, \\ &|\lambda_{h,m}| \le \frac{4}{h^2}\cdot\left[\frac{2}{\pi}\frac{m\pi h}{2l}\right]^2 = \left(\frac{2m}{l}\right)^2 \end{aligned} \tag{3.12}$$

we have that the series

$$\sum_{j=0}^{\infty}\frac{\|B_h^j x_0\| z^j}{j!} \le \sqrt{l/2}\sum_{j=0}^{\infty}\left(\frac{2m}{l}\right)^{2j}\frac{z^j}{j!} \tag{3.13}$$

converges for all $z \in (-\infty, \infty)$ uniformly with respect to h.

The next theorem shows a super exponential convergence rate of approximation (3.1) provided that the initial vector is analytical.

Theorem 3.7 *If x_0 is analytical for B, then approximation (3.1) converges super-exponentially with the error estimate*

$$\|x(t) - x_N^{(1)}(t)\| \le ct^{1/4}e^{t/2}N^{-1/12+\varepsilon/3}r^{-(2+\varepsilon)/3}e^{-c_1 r^{2/3}N^{1/3}}|||x_0|||_{r,B}, \tag{3.14}$$

where φ is the spectral angle of the operator B, c, c_1 are positive constants independent of N, $\varepsilon > 0$ is an arbitrarily small positive number.

Proof. First of all we note that estimate (2.39) yields

$$\|x(t) - x_N(t)\| \le ct^{1/4}e^{t/2}\sum_{p=N+1}^{\infty} p^{-3/4}\|u_p\|, \tag{3.15}$$

where

$$\begin{aligned} u_p = T^p x_0 &= \frac{1}{2\pi i}\int_\Gamma \left(\frac{iz}{iz-1}\right)^p (zI - B)^{-1} dz x_0 \\ &= \frac{1}{2\pi i}\int_\Gamma e^{-zs}\left(\frac{iz}{iz-1}\right)^p \left(\sum_{k=0}^{\infty}\frac{z^k s^k}{k!}\right)(zI - B)^{-1} dz x_0 \\ &= T^p e^{-sB} B^{\varepsilon}\left(\frac{1}{2\pi i}\int_\Gamma z^{-\varepsilon}(zI - B)^{-1} dz\right)\left(\sum_{k=0}^{\infty}\frac{s^k B^k x_0}{k!}\right). \end{aligned} \tag{3.16}$$

This representation together with Lemma 2.4 yields the estimate

$$\begin{aligned}\|u_p\| &\le \|T^p e^{-rB} B^{\varepsilon}\| \left(\frac{1}{2\pi i} \int_{\Gamma} \frac{1}{z^{\varepsilon}(1+|z|)} |dz| \right) \left(\sum_{k=0}^{\infty} \frac{r^k \|B^k x_0\|}{k!} \right) \\ &\le c e^{-c_1 r \sqrt[3]{p/r}} \left(\sqrt[3]{p/r}\right)^{\varepsilon} |||x_0|||_{r,B}.\end{aligned} \tag{3.17}$$

This inequality together with (3.15) implies

$$\begin{aligned}\|x(t) - x_N(t)\| &\le c t^{1/4} e^{t/2} r^{-\varepsilon/3} |||x_0|||_{r,B} \sum_{p=N+1}^{\infty} e^{-c_1 r^{2/3} p^{1/3}} p^{-3/4+\varepsilon/3} \\ &\le c t^{1/4} e^{t/2} N^{-1/12+\varepsilon/3} r^{-\varepsilon/3} |||x_0|||_{r,B} \sum_{p=N+1}^{\infty} e^{-c_1 r^{2/3} p^{1/3}} p^{-2/3} \\ &\le c t^{1/4} e^{t/2} N^{-1/12+\varepsilon/3} r^{-\varepsilon/3} |||x_0|||_{r,B} \int_N^{\infty} e^{-c_1 r^{2/3} x^{1/3}} x^{-2/3} dx \\ &\le c t^{1/4} e^{t/2} N^{-1/12+\varepsilon/3} r^{-(2+\varepsilon)/3} e^{-c_1 r^{2/3} N^{1/3}} |||x_0|||_{r,B}\end{aligned} \tag{3.18}$$

with some constants c, c_1 independent of N. The proof is complete. □

Estimate (3.2) indicates that the error of the approximation $x_N^{(1)}(t)$ tends to zero exponentially in N as $N \to \infty$. The error also tends to zero polynomially in t with the order $\mathcal{O}(t^{1/4})$ as $t \to 0$. The last fact can be used to arrive computational stability.

Note, that we can rewrite representation (2.19) and algorithm (3.1) in the form

$$\begin{aligned}x(t) &= u_0 + iB \sum_{n=1}^{\infty} L_n^{(0)}(t)(u_n - u_{n-1}) = u_0 + iB \sum_{n=1}^{\infty} L_n^{(0)}(t)(u_n - u_{n-1}) \\ &= x_0 - \sum_{n=0}^{\infty} (L_{n+1}^{(0)}(t) - L_n^{(0)}(t)) \tilde{u}_n, \\ x_N^{(1)}(t) &= x_0 - \sum_{n=0}^{N} (L_{n+1}^{(0)}(t) - L_n^{(0)}(t)) \tilde{u}_n,\end{aligned} \tag{3.19}$$

where (compare with (2.15))

$$\begin{aligned}(iB - I)\tilde{u}_0 &= -iBx_0, \\ (iB - I)\tilde{u}_{p+1} &= iB\tilde{u}_p, \quad p = 0, 1, \ldots\end{aligned} \tag{3.20}$$

From (3.20) we have

$$\begin{aligned}\tilde{u}_0 &= -Tx_0, \\ \tilde{u}_{p+1} &= T\tilde{u}_p = T^{p+2}x_0, \quad p = 0, 1, \ldots\end{aligned} \tag{3.21}$$

The equivalent form of (3.19)) is

$$\begin{aligned}x(t) &= e^{iB}x_0, e^{iB} = I - \sum_{n=0}^{\infty} (L_{n+1}^{(0)}(t) - L_n^{(0)}(t)) T^{n+1}, \\ x_N^{(1)}(t) &= e_N^{iB} x_0, e_N^{iB} = I - \sum_{n=0}^{N} (L_{n+1}^{(0)}(t) - L_n^{(0)}(t)) T^{n+1}.\end{aligned} \tag{3.22}$$

Given the solution operator $S(t) = S(t; B) = e^{iBt}$ for the homogeneous Schrödinger equation (2.4) the solution of the inhomogeneous Schrödinger equation

$$\dot{x}(t) = iBx(t) + f(t), \quad x(0) = x_0. \tag{3.23}$$

with a given function $f(t)$ can be represented by

$$x(t) = e^{iBt}x_0 + \int_0^t e^{iB(t-\tau)} f(\tau)d\tau. \tag{3.24}$$

4. Scaling and Squaring

Estimate (3.2) indicates that approximation (3.1) can become worse for large t. In this case one can combine (3.1) with the idea of the Scaling and Squaring Method [30, P.241] which exploits the relation $e^{iB2t_0} = (e^{iBt_0})^2$. This algorithm can be considered as a form of preconditioning to stabilize the computations.

Suppose we can compute $x(t_0)$ efficiently by (3.1) for some (small enough) t_0 but should compute $x(t)$ for $t = 2^M t_0$ with a natural M. This can be done via following algorithm.

Algorithm ES1 .

This algorithm computes e^{iBt} for $t = 2^M t_0$ using (3.1).

Input: x_0, t_0, M, N

Output: $x_N^{(1)}(t)$

1. Set $X_0 = x_0$ and compute $X = x_N^{(1)}(t_0; X_0)$ by (3.1).
2. For $k = 1 : M$ do

begin

2.1. $X_0 = X$;

2.2. Compute $X = x_N^{(1)}(t_0; X_0)$ by (3.1)

end

Let $C(N)$ denotes the computational costs for one evaluation of (3.1). Then the computational costs of the Algorithm ES1 are of the order $\mathcal{O}(C(N) \log M)$.

4.1. Tensor-product representation of $\exp(iBt)$

Let $B = \sum_{j=1}^d B_j$ be a strongly positive operator, where B_j are mutually commutative, strongly positive operators with the respective spectral sectors S_j and positive spectral angles. Then we introduce the tensor-product approximant

$$T(t; B) = \prod_{j=1}^d T(t; B_j) = \prod_{j=1}^d e^{itB_j} \approx T_{\mathbf{m}}(t) = T_{\mathbf{m}}(t; B) = \prod_{j=1}^d T_{m_j}(t; B_j), \tag{4.1}$$

where each of the Schrödinger operator exponentials $T_{m_j}(t; B_j)$ can be computed by the algorithm (3.1). Here we use the notation $\mathbf{m} = (m_1, \ldots, m_d)$. We denote by m_j the discretization parameter in the algorithm (3.1) with the operator B_j. For simplicity, we consider only the case $\mathbf{m} = (m, \ldots, m)$ with fixed $m_j = m$.

Lemma 4.1 *For any fixed $t > 0$ and analytical x_0 for all $B_j, j = 1, ..., d$, the approximation error by (4.1) satisfies*

$$\|[e^{itB} - T_{\mathbf{m}}(t; B)]x_0\| \le Cde^{-c\sqrt[3]{m}} |||x_0|||, \tag{4.2}$$

where C, c depend neither on d nor on $\mathbf{m}$ and $|||x_0||| = \max_{j=1,\ldots,d} |||x_0|||_{r_j, B_j}$.

Proof. Representing the error by a chain sum, we arrive at the estimate

$$\begin{aligned}\|[e^{itB} - T_{\mathbf{m}}(t;B)]x_0\| &= \|\{[e^{itB_1} - T_m]e^{itB_2}\cdots e^{itB_d} + T_m(t)[e^{itB_2} - T_m(t)]e^{itB_3}\cdots e^{itB_d} \\ &\quad + \ldots + T_m(t)\cdots T_m(t)[e^{itB_d} - T_m(t)]\}x_0\| \\ &\le Cde^{-c\sqrt[3]{m}}|||x_0|||\end{aligned}$$

providing an error bound (4.2) with C, c being independent of d, $\mathbf{m}$. □

Example 4.2 Consider the elliptic operator of divergent type,

$$B = \mathcal{A} := -\sum_{j=1}^{d} \partial_j a_j(x_j)\partial_j, \qquad x \in \Omega := (0,1)^d,$$

with $\partial_j v = \frac{\partial v}{\partial xj}$ defined on the Sobolev space $H_0^1(\Omega)$ and the Schrödinger equation (2.4) with this spartial operator, where we denote the unknown function by $u(t) \equiv u(t, x_1, ..., x_d)$. We assume that $a_j \ge a_0 > 0$. Introduce a uniform grid with step size h and $N = n^d$ interior nodes. Using the $(2d+1)$-point stencil, we obtain the finite difference discretisation

$$A_h z := -\sum_{j=1}^{d} \frac{2a^j_{i_j} z_{i_1\ldots i_d} - b^j_{i_j-1} z_{i_1\ldots(i_j-1)\ldots i_d} - c^j_{i_j+1} z_{i_1\ldots(i_j+1)\ldots i_d}}{h^2}, \qquad 1 \le i_j \le n, \tag{4.3}$$

where z denotes the vector corresponding to $[z_{i_1\ldots i_d}]_{i_j=1}^{n} \in \mathbb{R}^N$ given in the tensor-product numbering. In fact, we can regard d-dimensional $n \times \ldots \times n$ arrays (tensors) also as one-dimensional ones (vectors) with n^d components. Then the matrix A_h in (4.3) takes the form

$$A_h = \sum_{j=1}^{d} A_j,$$

where

$$A_1 = V^1 \times I \times \ldots \times I, \quad A_2 = I \times V^2 \times \ldots \times I, \ \ldots, \ A_d = I \times \ldots \times I \times V^d$$

with

$$V^j = \frac{1}{h^2}\begin{bmatrix} 2a_1^j & -c_1^j & & & \\ -b_2^j & 2a_2^j & -c_2^j & & \\ & \ddots & \ddots & \ddots & \\ & & -b_{n-1}^j & 2a_{n-1}^j & -c_{n-1}^j \\ & & & -b_n^j & 2a_n^j \end{bmatrix}_{n\times n}, \qquad I = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}_{n\times n}.$$

It is easy to see that $A_j > 0$ for all $j = 1, \ldots, d$, and that they commute pairwise, i.e., $A_j A_m = A_m A_j$. Finally, (4.1) implies the following tensor-product representation

$$e^{itA} \approx \bigotimes_{j=1}^{d} T_{m_j}(t; A_j).$$

Assume that our input data can be represented in the tensor-product form

$$x_0 \equiv u_0 \approx \sum_{k=1}^{r} u_1^k(x_1) \otimes \ldots \otimes u_d^k(x_d)$$

with $u_i^k \in \mathbb{R}^n$, $i = 1, ..., d$ and with some rank r independent of $h, \mathbf{m}$. Then we obtain the tensor-product approximation for the solution of the Schrödinger equation

$$\widetilde{u}(t) = \sum_{k=1}^{r} \bigotimes_{j=1}^{d} T_{n_j}(t; A_j) u_j^k(x_j) \approx u(t)$$

which can be implemented with the linear with respect to d complexity $\mathcal{O}(rdn)$.

Bibliography

1. Achieser N. I. Theorie der linearen operatoren im Hilbert-Raum / N. I. Achieser, I. M. Glasmann.– Berlin: Akademie-Verlag, 1975.
2. Arov D. Z. A method for solving initial value problems for linear differential equations in Hilbert space based on the Cayley transform / D. Z. Arov, I. P. Gavrilyuk // Numer. Funct. Anal. Optimization.– 1993.– Vol. 14.– No 5-6.– P. 456-473.
3. Arov D. Z. Representation and approximation of the solution of an initial value problem for a first order differential equation with an unbounded operator coefficient in Hilbert space based on the Cayley transform / D. Z. Arov, I. P. Gavrilyuk, V. L. Makarov // Progress in partial differential equations.– Pitman Res. Notes Math. Sci.– 1994.– Vol. 1.– P. 40-50.
4. Bateman H. Higher Transcendental Functions. Vol. 2 / H. Bateman, A. Erdelyi New York, Toronto, London: MC Graw-Hill Book Comp., Inc., 1988.
5. Bronstein I.N. Taschenbuch der Mathematik / I. N. Bronstein, K. A. Semendjajew, G. Musiol, H. Mühlig.– Frankfurt am Main, Verlag Harri Deutsch, 1999.
6. Gavrilyuk I. P. Strongly P-positive operators and explicit representation of the solutions of initial value problems for second order differential equations in Banach space / I. P. Gavrilyuk // Journ.of Math. Analysis and Appl.– 1999.– Vol. 236.– P. 327-349.
7. Gavrilyuk I. P. Approximation of the operator exponential and applications / I. P. Gavrilyuk // CMAM.– 2007.– Vol. 7.– No 4.– P. 294-320.
8. Gavrilyuk I. P. Super exponentially convergent approximation to the solution of the Schrödinger equation in abstract settings / I. P. Gavrilyuk // CMAM.– 2010.– Vol. 10.– No 4.– P. 345-358.
9. Gavrilyuk I. P. $\mathcal{H}$-matrix Approximation for the Operator Exponential with Applications / Gavrilyuk, I.P. Hackbusch, W. Khoromskij, B.N. // Numer. Math.– 2002.– Vol. 92.– P. 83-111.
10. Gavrilyuk I. P. Data- Sparse Approximation to the Operator-Valued Functions of Elliptic Operator / I. P. Gavrilyuk, W. Hackbusch, B. N. Khoromskij // Math. Comp.– 2004.– Vol. 73.– P. 1297-1324.
11. Gavrilyuk I. P. Data-sparse approximation of a class of operator-valued functions / I. P. Gavrilyuk, W. Hackbusch, B. N. Khoromskij // Math. Comp.– 2005.– Vol. 74.– P. 681-708.
12. Gavrilyuk I. P. Tensor-Product Approximation to Elliptic and Parabolic Solution Operators in Higher Dimensions / I. P. Gavrilyuk, W. Hackbusch, B. N. Khoromskij // Computing.– 2005.– Vol. 74.– P. 131-157.
13. Gavrilyuk I. P. The Cayley transform and the solution of an initial value problem for a first order differential equation with an unbounded operator coefficient in Hilbert space / I. P. Gavrilyuk, V. L. Makarov // Numer. Func. Anal. Optimiz.– 1994.– Vol. 15.– P. 583-598.
14. Gavrilyuk I. P. Representation and approximation of the solution of an initial value problem for a first order differential eqation in Banach space / I. P. Gavrilyuk, V. L. Makarov // Z. Anal. Anwend.(ZAA).– 1996.– Vol. 15.– P. 495-527.
15. Gavrilyuk I. P. Explicit and approximate solutions of second order elliptic differential equations in Hilbert- and Banach spaces / I. P. Gavrilyuk, V. L. Makarov // Numer. Func. Anal. Optimiz.– 1999.– Vol. 20.– No 7&8.– P. 695-717.
16. Gavrilyuk I. P. Explicit and approximate solutions of second order evolution differential equations in Hilbert space / I. P. Gavrilyuk, V. L. Makarov // Numerical Methods for Partial Differential Equations.– 1999.– Vol. 15.– P. 111-131.

17. Gavrilyuk I. P. Exponentially convergent parallel discretization methods for the first order evolution equations / I. P. Gavrilyuk, V. L. Makarov // Computational Methods in Applied Mathematics (CMAM).– 2001.– Vol. 1.– No 4.– P. 333-355.
18. Gavrilyuk I. P. Exponentially convergent parallel discretization methods for the first order differential equations / I. P. Gavrilyuk, V. L. Makarov // Doklady of the Ukrainian Academy of Scienses.– 2002.– No 3.– P. 1-6.
19. Gavrilyuk I. P. Strongly positive operators and computational algorithms without accuracy saturation / I. P. Gavrilyuk, V. L. Makarov.– Kiev: Publishing House of the Institute of Mathematics of NASU, 2004. (in Russian).
20. Gavrilyuk I. P. Algorithms without accuracy saturation for evolution equations in Hilbert and Banach spaces / I. P. Gavrilyuk, V. L. Makarov // Math. Comp.– 2005.– Vol. 74.– P. 555-583.
21. Gavrilyuk I. P. Exponentially convergent algorithms for the operator exponential with applications to inhomogeneous problems in Banach spaces / I. P. Gavrilyuk, V. L. Makarov // SIAM Journal on Numerical Analysis.– 2005.– Vol. 43.– No 5.– P. 2144-2171.
22. Gavrilyuk I. P. An explicit boundary integral representation of the solution of the two-dimensional heat equation and its discretization / I. P. Gavrilyuk, V. L. Makarov // J. Integral Equations Appl.– 2000.– Vol. 12.– No 1.– P. 63-83.
23. Goldstein J. A. Semigroups of Linear Operators and Applications / J. A. Goldstein.– New York, Oxford: Oxford University Press and Clarendon Press, 1985.
24. Krein M. G. On fundamental approximating problem in the extrapolation theory and filtration of stationary stochastic processes / M. G. Krein // Dokl. Akad. Nauk SSSR.– 1954.– Vol. 94.– No 1.– P. 13-16. (in Russian).
25. Makarov V. L. About functional-discrete method of arbitrary accuracy order for solving Sturm-Liouville problem with piecewise smooth coefficients / V. L. Makarov // DAN SSSR.– 1991.– Vol. 320.– No 1.– P. 34-39.
26. Makarov V. L. FD-method: the exponential rate of convergence / V. L. Makarov // J. Math. Sciens.– 1997.– Vol. 104.– No 6.– P. 1648-1653.
27. Pazy A. Semigroups of linear operator and applications to partial differential equations / A. Pazy.– New York, Berlin, Heidelber: Springer Verlag, 1983.
28. Suetin P. K. Classical Orthogonal Polynomials / P. K. Suetin.– Moscow: Naukova Dumka, 1979. (in Russian)
29. Szegö G. Orthogonal Polynomials / G. Szegö.– New York: American Mathematical Society, 1959.
30. Higham I. Functions of Matrices. Theory and Computation / I. Higham.– Manchester: SIAM, 2008.

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