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# ASYMPTOTIC STABILITY OF NONLOCAL DIFFERENCE SCHEMES

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Анотація. Розглядається сімейство двошарових різницевих схем для рівняння теплопровідності $\partial u/\partial t = \partial^2 u/\partial x^2$, $0 < x < 1$, $u(x,0) = u_0(x)$, $u(0,t) = 0$, $\gamma\,\partial u(0,t)/\partial x = \partial u(1,t)/\partial x$ з нелокальними крайовими умовами та параметром $\gamma > 1$. На деякому інтервалі $\gamma \in (1,\, \gamma_+)$ спектр основного різницевого оператора містить єдине власне значення $\lambda_0$ в лівій комплексній півплощині, тоді як інші власні значення $\lambda_1$, $\lambda_2$, ..., $\lambda_{N-1}$ знаходяться в правій півплощині. Відповідний простір сіткових функцій $H_N$ подається у вигляді прямої суми $H_N = H_0 \oplus H_{N-1}$ одновимірного підпростору $H_0$ та підпростору $H_{N-1}$, який є лінійною оболонкою власних векторів $\mu^{(1)}$, $\mu^{(2)}$, ..., $\mu^{(N-1)}$. Введено означення асимптотичної стійкості в $H_{N-1}$ та сформульовані умови асимптотичної стійкості. Досліджена асимптотична поведінка нульової гармоніки $\mu^{(0)}$ при великих $t$.

Abstract. The family of two-layer difference schemes is considered for the heat conduction equation $\partial u/\partial t = \partial^2 u/\partial x^2$, $0 < x < 1$, $u(x,0) = u_0(x)$, $u(0,t) = 0$, $\gamma\,\partial u(0,t)/\partial x = \partial u(1,t)/\partial x$ with nonlocal boundary conditions and parameter $\gamma > 1$. The spectrum of the main difference operator contains in some interval $\gamma \in (1,\, \gamma_+)$ a single eigenvalue $\lambda_0$ in the left complex half-plane, while the rest eigenvalues $\lambda_1$, $\lambda_2$, ..., $\lambda_{N-1}$ are situated in the right half-plane. The corresponding grid-function space $H_N$ is represented as a direct sum $H_N = H_0 \oplus H_{N-1}$ of one-dimensional subspace $H_0$ and subspace $H_{N-1}$, which is the linear span of eigenvectors $\mu^{(1)}$, $\mu^{(2)}$, ..., $\mu^{(N-1)}$. The definition of asymptotic stability in $H_{N-1}$ is introduced, as well as the conditions of asymptotic stability are formulated. The asymptotic behavior of null harmonic $\mu^{(0)}$ is investigated for large $t$.

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## 1. Introduction

Let us consider the heat conduction equation with nonlocal boundary conditions

$$\begin{aligned} &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad u(x,0) = u_0(x), \\ &u(0,t) = 0, \quad \gamma \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t). \end{aligned} \tag{1.1}$$

Here $\gamma$ is a prescribed real parameter. The eigenvalue problem

$$X''(x) + \lambda X(x) = 0, \quad 0 < x < 1, \quad X(0) = 0, \quad \gamma X'(0) = X'(1). \tag{1.2}$$

arises when we solve the problem (1.1) by means of the method of variables separation. The spectrum of the problem (1.2) depends on parameter $\gamma$ and is defined by formulae

$$\begin{aligned} &\lambda_0 = \psi^2, \quad \psi = \arccos\gamma, \\ &\lambda_{2k-1} = (2\pi k - \psi)^2, \quad \lambda_{2k} = (2\pi k + \psi)^2, \quad k = 1, 2, \ldots \end{aligned}$$

*Key words.* Difference scheme, heat conduction equation, stability criteria, nonlocal boundary conditions.

If $|\gamma| \leqslant 1$ then all the eigenvalues are real and positive. The eigenvalues are complex with $\psi = i \ln\left(\gamma + \sqrt{\gamma^2 - 1}\right)$ when $\gamma > 1$. Moreover for $\gamma > 1$ exist eigenvalues with negative real part. As a consequence the original problem (1.1) became unstable one in view of exponential time increasing of corresponding harmonics. The number of eigenvalues situated in the left half-plane increases with growth of $\gamma$. It can be demonstrated that if $1 < \gamma < \gamma_+ = \mathrm{ch}(2\pi) \approx 267.7$, then the single eigenvalue exists in the left half-plane, namely $\lambda_0 = \psi^2$. In what follows we restrict ourselves to rang $\gamma \in (1,\, \gamma_+)$.

Essential time increasing of unstable harmonics as well as boundedness of a rang representation of real numbers in computers constrained us to carrying out calculations only for bounded time segment. Thereby the primitive evaluation of maximal admissible calculating time $T$ leads us to the condition $\exp\left(|\lambda_0|\, t\right) \leqslant 10^M$, where $10^M$ is a maximal number which is representable in the computer. From here we obtain $T = T(M, \gamma) = M \ln 10\, |\lambda_0(\gamma)|^{-1}$. For example, $T(300, 10) = 77$, $T(300, 100) = 25$ and $T(300, 300) = 17$.

Let us denote

$$y_i^n = y(x_i, t_n), \quad y_i^{(\sigma)} = \sigma y_i^{n+1} + (1-\sigma) y_i^n,$$
$$y_{t,i}^n = \frac{y_i^{n+1} - y_i^n}{\tau}, \quad y_{\bar{x},i}^n = \frac{y_i^n - y_{i-1}^n}{h}, \quad y_{x,i}^n = \frac{y_{i+1}^n - y_i^n}{h}$$

on the uniform mesh $\omega_{h,\tau} = \omega_h \times \omega_\tau$, where $\omega_h = \{x_i = ih\}_{i=0}^N$, $\omega_\tau = \{t_n = n\tau\}_{n=0}^K$.

As usual, we approximate the problem (1.1) by the weighted difference scheme

$$\begin{gathered} y_{t,i}^n - y_{\bar{x}x,i}^{(\sigma)} = 0, \quad i = 1, 2, \ldots, N-1, \quad n = 0, 1, \ldots, \\ y_i^0 = u_0(x_i), \quad y_0^{n+1} = 0, \quad \frac{h}{2} y_{t,N}^n + y_{\bar{x},N}^{(\sigma)} - \gamma y_{x,0}^{(\sigma)} = 0. \end{gathered} \tag{1.3}$$

Difference schemes for the problem (1.1) were discussed primarily by N. I. Ionkin [1]—[3], where the case $\gamma = 1$ (so called the problem Samarskii — Ionkin) is investigated in detail. The case $\gamma \in (-1,\ 1)$ studied in the papers of A. V. Gulin, N. I. Ionkin and V .A. Morozova (see [4]). The fundamental distinction from the case $\gamma = \pm 1$ consists here in the fact that the eigenfunctions system of the difference analogue of (1.2) constitutes the basis in the space of grid functions. As a consequence it is possible to represent the desired solution of unsteady difference problem in the form of expansion in terms of eigenfunctions basis. The certain significant results on the theory of difference schemes for differential equations with nonlocal boundary condition, including quasilinear equations, obtained by V. L. Makarov with co-authors [5] — [8].

## 2. The spectrum of difference operator

The main operator of the difference scheme (1.3) is defined as

$$\begin{gathered} (Ay)_j = -y_{\bar{x}x,j}, \quad j = 1, 2, \ldots, N-1, \quad y_0 = 0, \\ (Ay)_N = \frac{2}{h}\left(y_{\bar{x},N} - \gamma y_{x,0}\right). \end{gathered} \tag{2.1}$$

The eigenvalue problem for operator (2.1)

$$\mu_{\bar{x}x,j} + \lambda \mu_j = 0, \quad j = 1, 2, \ldots, N-1, \quad \mu_0 = 0, \quad \frac{2}{h}\left(\gamma \mu_{x,0} - \mu_{\bar{x},N}\right) + \lambda \mu_N = 0 \tag{2.2}$$

for $\gamma > 1$ has the solution

$$\lambda_{2k-1} = \frac{4}{h^2} \sin^2((\pi k - 0,5\psi)h), \quad \mu^{(2k-1)}(x_j) = \sin((2\pi k - \psi)x_j),$$

$$\lambda_{2k} = \frac{4}{h^2}\sin^2((\pi k + 0,5\psi)h), \quad \mu^{(2k)}(x_j) = \sin((2\pi k + \psi)x_j),$$

where $\psi = i\ln\left(\gamma + \sqrt{\gamma^2 - 1}\right)$.

The following expressions are valid for the real and imaginary parts of eigenvalues. If we denote $a = \mathrm{ch}\left(h\ln\left(\gamma + \sqrt{\gamma^2 - 1}\right)\right)$, then

$$\begin{aligned}
\operatorname{Re}\lambda_{2k-1} &= \frac{2}{h^2}\left(1 - a\cos(2\pi kh)\right), \quad \operatorname{Im}\lambda_{2k-1} = -\frac{2}{h^2}\sqrt{a^2-1}\sin(2\pi kh), \\
\operatorname{Re}\lambda_{2k} &= \frac{2}{h^2}\left(1 - a\cos(2\pi kh)\right), \quad \operatorname{Im}\lambda_{2k-1} = \frac{2}{h^2}\sqrt{a^2-1}\sin(2\pi kh).
\end{aligned} \tag{2.3}$$

For $N$ odd the index $k$ go from 1 to $(N-1)/2$ in formulae for $\lambda_{2k-1}$ and go from 0 to $(N-1)/2$ in formulae for $\lambda_{2k}$. For $N$ even the subscript $k$ go from 1 to $N/2$ in formulae for $\lambda_{2k-1}$ and go from 0 to $N/2-1$ in formulae for $\lambda_{2k}$. For the sake of distinctness we suppose hereinafter that $N$ is even and denote $m = N/2$.

It follows from (2.3) the expressions for modules of eigenvalues:

$$|\lambda_{2k-1}| = |\lambda_{2k}| = \frac{2}{h^2}(a - \cos(2\pi kh)), \quad a = \mathrm{ch}\left(h\ln\left(\gamma + \sqrt{\gamma^2-1}\right)\right).$$

Hereinafter we shall use the following designations relating to the spectrum of a difference operator. As a rule instead of parameter $\gamma \geqslant 1$ we introduce the parameter $a = a(\gamma) = \mathrm{ch}\left(h\ln\left(\gamma + \sqrt{\gamma^2-1}\right)\right)$, from which $\gamma = \mathrm{ch}\left(h^{-1}\ln\left(a + \sqrt{a^2-1}\right)\right)$. At the present paper is supposed that

$$1 < \gamma < \gamma_+^{(h)} = \mathrm{ch}\left(h^{-1}\ln\left(\frac{1+\sin(\pi h)}{\cos(\pi h)}\right)\right),$$

then $1 < a < a_+ = \cos^{-1}(2\pi h)$. Note that under condition $1 < \gamma < \gamma_+^{(h)}$ it exists only one eigenvalue of the difference problem (2.2), situated in the left complex half-plane. Inequalities $1 < a_* < a_+$ are valid, where the number

$$a_* = 0.5\left(1 + \sqrt{1 + 8\sin^2(\pi h)}\right) > 1 \tag{2.4}$$

is the root of quadratic equation $a^2 - a - 2\sin^2(\pi h) = 0$. Here

$$p_2(a) = a^2 - a - 2\sin^2(\pi h) < 0, \text{ if } 1 < a < a_*, \quad p_2(a) > 0, \text{ if } a_* < a < a_+.$$

## 3. Asymptotic stability

### 3.1. Main notions

Let us denote by

$$s_k = \frac{1 - (1-\sigma)\tau\lambda_k}{1 + \sigma\tau\lambda_k}, \quad k = 0, 1, \cdots, N-1$$

the eigenvalues of transition operator of the difference scheme (1.3). The nonlocal difference operator (2.1) contains along with stable harmonics $\mu_k(x)$ c $k = 1, 2, \cdots, N-1$, for which $\operatorname{Re}\lambda_k > 0$, also the unstable harmonic $\mu_0(x)$ infinitely increasing with time. Let $H_{N-1}$ is a linear span of eigenvectors $\mu_k(x)$, where $k = 1, 2, \cdots, N-1$.

We say that the difference scheme is *stable in the subspace* $H_{N-1}$, if inequalities $|s_k| \leqslant 1$ are valid for all eigenvalues of the transition operator except eigenvalue $s_0$. Let us denote

$$a = \mathrm{ch}\left(h\ln\left(\gamma + \sqrt{\gamma^2-1}\right)\right), \quad a_+ = 1/\cos(2\pi h), \quad a_* = \frac{1 + \sqrt{1 + 8\sin^2(\pi h)}}{2}.$$

**Theorem 3.1** *Suppose that* $1 < a < a_+$. *If* $\sigma \geqslant 0.5$, *then the scheme (1.3) is stable in* $H_{N-1}$ *for all* $\kappa > 0$. *If* $\sigma < 0.5$ *and* $1 < a \leqslant a_*$, *then the scheme (1.3) is stable in* $H_{N-1}$ *under condition*

$$0 < \kappa \leqslant \frac{1}{(1-2\sigma)(a+1)}.$$

*If* $\sigma < 0.5$ *and* $a_* \leqslant a < a_+$, *then the scheme (1.3) is stable in* $H_{N-1}$ *under condition*

$$0 < \kappa \leqslant \frac{1 - a\cos(2\pi h)}{(1-2\sigma)(a-\cos(2\pi h))^2}.$$

We omit the proof, because it carrying out similarly to [4, p. 223].

Asymptotic stability requirement imposes more strong restriction on difference scheme's parameters. The conception of asymptotic stability for difference schemes was introduced by A. A. Samarskii [9, p. 201], [10, p. 327] in connection with the difference schemes for heat conduction equation with first kind boundary conditions. Let $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{N-1}$ are all eigenvalues of the main difference operator of such a scheme, and $\mu_k(x)$ are corresponding eigenvectors, $k = 1, 2, \cdots, N-1$. The difference scheme is referred as asymptotically stable, if it stable with respect to initial data and, furthermore, its solution tends for $t \to \infty$ to the asymptotic solution, which is defined by the first eigenvalue $\lambda_1$.

Presence of an unstable harmonic force us to slightly vary the definition of asymptotic stability. The difference scheme (1.3) is said to be *asymptotically stable in the subspace* $H_{N-1}$, if inequalities

$$|s_k| < |s_1| < 1, \quad k = 2, 3, \cdots, N-1$$

are hold for all eigenvalues of the transition operator except the eigenvalue $s_0$.

**Lemma 3.2** *If* $1 < \gamma < \gamma_+^{(h)}$, *then fulfilment of inequalities*

$$f_0(\kappa) = (2\sigma - 1)(a - \cos(2\pi h))^2\kappa + (1 - a\cos(2\pi h)) > 0, \tag{3.1}$$

$$f_k(\kappa) = r_0 + r_1^{(k)}\kappa + r_2^{(k)}\kappa^2 > 0, \quad k = 1, 2, \ldots, m, \tag{3.2}$$

*where* $\kappa = \tau/h^2$, $a = \operatorname{ch}\left(h \ln\left(\gamma + \sqrt{\gamma^2 - 1}\right)\right)$ *and*

$$\begin{aligned} &r_0 = a, \quad r_1^{(k)} = (2\sigma - 1)(2a - \cos(2\pi kh) - \cos(2\pi h)), \\ &r_2^{(k)} = 4\sigma(1-\sigma)[a(a^2 - \cos(2\pi kh)\cos(2\pi h)) - (2a - \cos(2\pi h) - \cos(2\pi kh))] \end{aligned} \tag{3.3}$$

*is necessary and sufficient for asymptotic stability of the scheme (1.3) in* $H_{N-1}$.

*Proof.* It follow from (2.3) that

$$|s_{2k-1}|^2 = |s_{2k}|^2 = \frac{1 + a_1^{(k)}\kappa + a_2^{(k)}\kappa^2}{1 + b_1^{(k)}\kappa + b_2^{(k)}\kappa^2}, \quad k = 1, 2, \ldots, m,$$

where

$$\begin{aligned} &a_1^{(k)} = -4(1-\sigma)(1 - a\cos(2\pi kh)), \quad a_2^{(k)} = 4(1-\sigma)^2(a - \cos(2\pi kh))^2, \\ &b_1^{(k)} = 4\sigma(1 - a\cos(2\pi kh)), \quad b_2^{(k)} = 4\sigma^2(a - \cos(2\pi kh))^2, \end{aligned} \tag{3.4}$$

Let us find from here the asymptotic stability conditions

$$|s_{2k}|^2 < |s_2|^2 = |s_1|^2 < 1, \quad k = 2, 3, \ldots, m. \tag{3.5}$$

We get from (3.4) that

$$|s_2|^2 - |s_{2k}|^2 = \frac{\kappa R}{D}, \quad D = \left(1 + b_1^{(1)}\kappa + b_2^{(1)}\kappa^2\right)\left(1 + b_1^{(k)}\kappa + b_2^{(k)}\kappa^2\right) > 0,$$

where $D = \left(1 + b_1^{(1)}\kappa + b_2^{(1)}\kappa^2\right)\left(1 + b_1^{(k)}\kappa + b_2^{(k)}\kappa^2\right) > 0$ and

$$R = 4(\cos(2\pi h) - \cos(2\pi kh))(r_0 + r_1^{(k)}\kappa + r_2^{(k)}\kappa^2).$$

Here $r_0$ and $r_{1,2}^{(k)}$ are specified in accordance with (3.3). As far as the multiplier $\cos(2\pi h) - \cos(2\pi kh)$ is positive, inequalities (3.5) and (3.2) are true or false simultaneously. So, it remains to find the condition guaranteeing the inequality $|s_1|^2 < 1$. We get

$$1 - |s_1|^2 = \kappa \frac{(b_1^{(1)} - a_1^{(1)}) + (b_2^{(1)} - a_2^{(1)})\kappa}{1 + b_1^{(1)}\kappa + b_2^{(1)}\kappa^2},$$

where $b_1^{(1)} - a_1^{(1)} = 4(1 - a\cos(2\pi h))$, $\quad b_2^{(1)} - a_2^{(1)} = 4(2\sigma - 1)(1 - a\cos(2\pi h))^2$. It follows from here that the inequality $|s_1|^2 < 1$ is equivalent to (3.1). □

Let us consider cases $\sigma = 0$, $\sigma = 1$ and $\sigma = 0,5$ separately.

## 3.2. Asymptotic stability of explicit scheme

**Theorem 3.3** *If $1 < a < a_*$, then the explicit scheme ($\sigma = 0$ )is asymptotically stable in $H_{N-1}$ under condition*

$$\kappa < \frac{a}{2(a + \sin^2(\pi h))}.$$

*If $a_* < a < a_+$, then the explicit scheme asymptotically stable in $H_{N-1}$ under condition*

$$\kappa < \frac{1 - a\cos(2\pi h)}{(a - \cos(2\pi h))^2}.$$

*Proof.* For $\sigma = 0$ we get $r_0 = a$, $\quad r_1^{(k)} = -(2a - \cos(2\pi h) - \cos(2\pi kh))$, $\quad r_2^{(k)} = 0$, therefore the asymptotic stability conditions (3.1), (3.2) assume the form

$$-(a - \cos(2\pi h))^2\kappa + (1 - a\cos(2\pi h)) > 0,$$
$$a - (2a - \cos(2\pi h) - \cos(2\pi kh))\kappa > 0, \quad k = 1, 2, \ldots, m.$$

Let us find

$$\min_{2 \leqslant k \leqslant m} \{a - (2a - \cos(2\pi h))\kappa + \cos(2\pi kh)\kappa\} =$$
$$= \{a - (2a - \cos(2\pi h))\kappa + \cos(2\pi mh)\kappa\} = a - 2(a + \sin^2(\pi h))\kappa.$$

It is seen from here that the asymptotic stability condition reduces to inequality

$$\kappa < \min\left\{\frac{1 - a\cos(2\pi h)}{(a - \cos(2\pi h))^2}, \frac{a}{2(a + \sin^2(\pi h))}\right\}.$$

In order to find the minimum mentioned we rearrange

$$\frac{a}{2(a + \sin^2(\pi h))} = \frac{a}{2a + 1 - \cos(2\pi h)}$$

and solve relative to $a$ the inequality

$$\frac{1 - a\cos(2\pi h)}{(a - \cos(2\pi h))^2} < \frac{a}{2a + 1 - \cos(2\pi h)},$$

which can reduced to inequality $a^3 - 2a - 1 + (1+a)\cos(2\pi h) > 0$ or $a^2 - a - 2\sin^2(\pi h) > 0$. The maximal root of quadratic equation $a^2 - a - 2\sin^2(\pi h) = 0$ is

$$a_* = \frac{1 + \sqrt{1 + 8\sin^2(\pi h)}}{2}.$$

□

Note that according to Theorem 3.1 the explicit scheme is stable (but not asymptotically stable) in $H_{N-1}$ under condition

$$\kappa \leqslant \frac{1}{a+1}, \text{ if } 1 < a < a_* \text{ and } \kappa \leqslant \frac{1 - a\cos(2\pi h)}{(a - \cos(2\pi h))^2}, \text{ if } a_* < a < a_+.$$

Thus, the asymptotic stability condition of explicit scheme coincides with condition of usual stability if $a_* < a < a_+$, and is a more strong requirement in the case $1 < a < a_*$.

## 3.3. The pure implicit scheme

**Theorem 3.4** *If $1 < a < a_+$, then the pure implicit scheme ($\sigma = 1$) is asymptotically stable in $H_{N-1}$ for arbitrary $\kappa > 0$.*

*Proof.* In the case $\sigma = 1$ we get

$$r_0 = a > 1, \quad r_1^{(k)} = 2a - \cos(2\pi h) - \cos(2\pi kh) > 0, \quad r_2^{(k)} = 0,$$

consequently the condition $r_0 + r_1^{(k)}\kappa > 0$ is valid for all $\kappa$. The condition $|s_1|^2 < 1$, which reduces to inequality

$$(a - \cos(2\pi h))^2\kappa + 1 - a\cos(2\pi h) > 0,$$

also is valid for all $\kappa > 0$, if $a < 1/\cos(2\pi h)$. □

In is seen from here and Theorem 3.1 that the pure implicit scheme, being absolutely stable in $H_{N-1}$, is also absolutely asymptotically stable in $H_{N-1}$.

## 3.4. Asymptotic stability of the symmetric scheme

**Theorem 3.5** *If $1 < a < a_*$, then the symmetric scheme ($\sigma = 0.5$) is asymptotically stable in $H_{N-1}$ under condition*

$$\kappa^2 < \frac{a}{(a+1)(-a^2 + a + 2\sin^2(\pi h))}. \tag{3.6}$$

*If $a_* < a < a_+$, then the symmetric scheme is asymptotically stable in $H_{N-1}$ for arbitrary $\kappa > 0$.*

*Proof.* Setting $\sigma = 0.5$, we get from (3.4) that

$$r_0 = a, \quad r_1^{(k)} = 0, \quad r_2^{(k)} = a(a^2 - \cos(2\pi h)\cos(2\pi kh)) - (2a - \cos(2\pi h) - \cos(2\pi kh)).$$

The asymptotic stability conditions (3.1), (3.2) take a form $1 - a\cos(2\pi h) > 0$ and $a + r_2^{(k)}\kappa^2 > 0$. The first inequality is valid by assumption. Further,

$$\begin{aligned} a + r_2^{(k)}\kappa^2 &= a + [a^3 - 2a + \cos(2\pi h)]\kappa^2 + (1 - a\cos(2\pi h))\cos(2\pi kh)\kappa^2 \geqslant \\ &\geqslant a + [a^3 - 2a + \cos(2\pi h)]\kappa^2 + (1 - a\cos(2\pi h))\cos(\pi Nh)\kappa^2 = \\ &= a + [a^3 - 2a + \cos(2\pi h)]\kappa^2 - (1 - a\cos(2\pi h))\kappa^2. \end{aligned}$$

It follows from here that the asymptotic stability condition can be written as inequality $a + (a+1)(a^2 - a - 2\sin^2(\pi h)) > 0$. Consequently the coefficient $r_2^{(k)}$ is positive if and only if $a > a_*$. For $a < a_*$ we get $r_2^{(k)} < 0$ and the inequality $a + r_2^{(k)} > 0$ reduces to the condition (3.6).

□

Note that the symmetric scheme is stable in $H_{N-1}$ in usual sense for all $\kappa > 0$. So, in the case $1 < a < a_*$ requirement of asymptotic stability leads to more essential restriction.

## 4. Increasing of null harmonic

The eigenvalue $s_0$ of the transition operator of difference scheme (1.3) corresponding to null harmonic is defined as

$$s_0 = \frac{1 - (1-\sigma)\tau\lambda_0^{(h)}}{1 + \sigma\tau\lambda_0^{(h)}}, \text{ where } \lambda_0^{(h)} = -\frac{2(a-1)}{h^2}.$$

Let us represent $s_0$ in the form

$$s_0 = e^{\ln s_0} = e^{-\tau\lambda_0} e^{\tau\lambda_0 + \ln s_0} = e^{-\tau\lambda_0}\rho,$$

where $\rho = e^{\varphi}$, $\varphi = \tau\lambda_0 + \ln s_0$ and $\lambda_0 = -\ln^2\left(\gamma + \sqrt{\gamma^2 - 1}\right)$ is an eigenvalue of differential problem (1.2, corresponding to null harmonic. With the notation $\kappa = \tau/h^2$, let us rewrite the exponent $\varphi$ in the form

$$\varphi = \kappa h^2\lambda_0 + \ln\left(1 + \frac{2\kappa(a-1)}{1 - 2\sigma\kappa(a-1)}\right),$$

from where we get

$$\rho = \rho(\kappa) = e^{\varphi} = \left(1 + \frac{2\kappa(a-1)}{1 - 2\sigma\kappa(a-1)}\right) e^{\kappa h^2\lambda_0}. \tag{4.1}$$

The factor $\rho(\kappa)$ characterizes the deviation of solution $u(t) = e^{-\lambda_0 t}$ of the differential problem from corresponding difference solution $y(t,\kappa) = s_0^{t/\tau} = s_0^{t/(\kappa h^2)}$.

If $\rho(\kappa_0) = 1$ for some $\kappa_0 > 0$, then solutions $u(t)$ and $y(t, \kappa_0)$ are coincides. In this case the value $\kappa = \kappa_0$ can be called as optimal one. If for some $\kappa > 0$ the inequality $\rho(\kappa) > 1$ is valid, then $y(t,\kappa) > u(t)$ (approximation from above). If $\rho(\kappa) < 1$, then $y(t,\kappa) < u(t)$ (approximation from below).

Let us consider certain values of $\sigma$.

### 4.1. Explicit scheme

For the explicit scheme ($\sigma = 0$) from (4.1) we get

$$\rho(\kappa) = (1 + 2\kappa(a-1))\, e^{\kappa h^2\lambda_0}.$$

The derivative
$$\frac{d\rho}{d\kappa} = [2(a-1)(1+\kappa h^2\lambda_0) + h^2\lambda_0]$$
vanishes in the point
$$\kappa = \kappa_1 = \frac{2(a-1)+h^2\lambda_0}{2h^2(-\lambda_0)(a-1)}.$$
As far as
$$2(a-1)+h^2\lambda_0 = \frac{h^4}{12}\ln^4\left(\gamma+\sqrt{\gamma^2-1}\right) + O(h^6) > 0$$
we get $\kappa_1 > 0$. The derivative $\rho'(\kappa)$ is positive right up to the point $\kappa_1$ and is negative for $\kappa > \kappa_1$. Consequently, $\kappa_1$ is a maximum point of $\rho(\kappa)$, and $\rho(\kappa_1) > 1$.

The graph of function $\rho(\kappa)$ is represented on the Figure 1 (left picture). Here $N = 20$ and $\gamma = 100$, that corresponds $a = 1.035$, $a_* = 1.047$ and $a_+ = 1.051$. The scheme is stable in $H_{N-1}$ under condition
$$\kappa \leqslant \frac{1}{a+1} = 0.491,$$
and it asymptotically stable in $H_{N-1}$ under condition
$$\kappa < \frac{a}{2(a+\sin^2(\pi h))} = 0.488.$$
For the example concerned the derivative $\rho'(\kappa)$ vanishes in the point $\kappa_1 = 0.083$, and $\rho(\kappa_1) = 1.000017$. The equality $\rho(\kappa) = 1$ is valid in the point $\kappa = \kappa_0 = 0.171$.

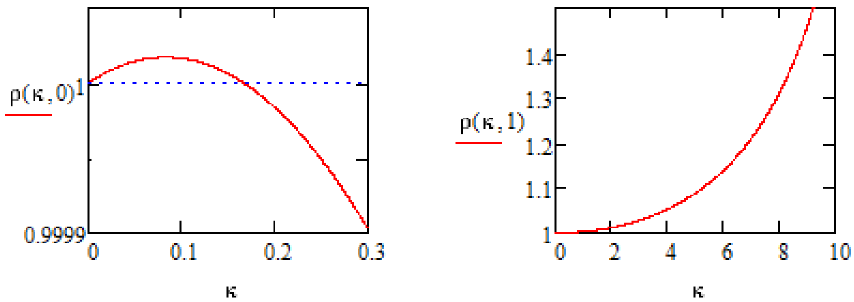


Fig. 1. The multiplier $\rho(\kappa)$ of explicit scheme (left) and pure implicit scheme (right)

The small deviation of the factor $\rho$ from 1 result nevertheless in considerable distinction between difference solution $y(t,\kappa)$ and differential solution $u(t)$ for large time. So, for $t = 10$ we get $u(t) = 8.217\cdot 10^{121}$, whereas
$$y(t,\,0.171) = 7.855\cdot 10^{121}, \quad y(t,\,0.083) = 1.868\cdot 10^{122}, \quad y(t,\,0.488) = 3.653\cdot 10^{120}.$$

## 4.2. Pure implicit scheme

In the case $\sigma = 1$ we obtain
$$\rho(\kappa) = \frac{1}{1-2\kappa(a-1)} e^{\kappa h^2\lambda_0}.$$

The derivative

$$\frac{d\rho}{d\kappa} = \frac{2(a-1)+h^2[1-2\kappa(a-1)]\lambda_0}{(1-2\kappa(a-1))^2} e^{\kappa h^2\lambda_0}$$

vanishes in the point $\kappa_1 = \left(h^2\lambda_0\right)^{-1} < 0$ and is positive for $\kappa \geqslant 0$. As far as $\rho(0) = 1$ and the function $\rho(\kappa)$ is monotone increasing for $\kappa \geqslant 0$, the inequality $\rho(\kappa) > 1$ is valid for $\kappa > 0$. As opposed to explicit scheme, the solution $y(t,\kappa)$ of the difference problem exceeds the solution $u(t)$ of the differential problem for all $\kappa > 0$.

The graph of function $\rho(\kappa)$ is represented on the Figure 1 (right picture). Here, alike previous example, $N = 20$ and $\gamma = 100$, what corresponds $a = 1.035$, $a_* = 1.047$ and $a_+ = 1.051$. The scheme is stable and asymptotically stable in в $H_{N-1}$ for any $\kappa > 0$. The distinction between difference solution and differential solution for $t = 10$ is characterized by following data: $u(10) = 8.217 \cdot 10^{121}$,

$$y(10, 0.171) = 2.374 \cdot 10^{123}, \quad y(10, 0.083) = 9.773 \cdot 10^{122}, \quad y(10, 0.488) = 6.186 \cdot 10^{124}$$
$$y(10, 0.5) = 7.012 \cdot 10^{124}, \quad y(10, 1) = 1.489 \cdot 10^{127}, \quad y(10, 2) = 1.578 \cdot 10^{132}.$$

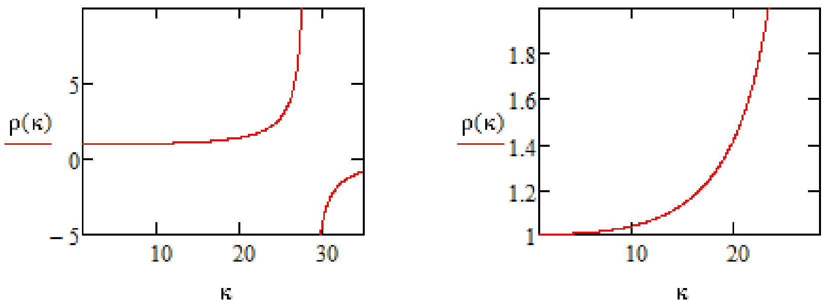


Fig. 2. Factor $\rho(\kappa)$ in the case $\sigma = 0.5$

## 4.3. Symmetric scheme

For $\sigma = 0.5$ from (4.1) we get

$$\rho(\kappa) = \left(\frac{1+\kappa(a-1)}{1-\kappa(a-1)}\right) e^{\kappa h^2\lambda_0}, \quad \rho'(\kappa) = \frac{-(a-1)^2h^2\lambda_0\kappa^2 + 2(a-1) + h^2\lambda_0}{(1-\kappa(a-1))^2} e^{\kappa h^2\lambda_0}.$$

The graph of the multiplier

$$v(\kappa) = \frac{1+\kappa(a-1)}{1-\kappa(a-1)}$$

is the hyperbola with vertical asymptote $\kappa = \kappa_2 = (a-1)^{-1} = O(h^{-2})$ and horizontal asymptote $v = -1$.

For $\kappa > \kappa_2$ the function $\rho(\kappa)$ is negative, so that it is sufficient not go beyond $\kappa = \kappa_2$. It was demonstrated in Theorem 3.5 that for $1 < a < a_*$ the symmetric scheme is asymptotically stable in $H_{N-1}$ under condition

$$\kappa = \kappa_1 = \left(\frac{a}{(a+1)(-a^2+a+2\sin^2(\pi h))}\right)^{1/2}.$$

The graph of function $\rho(\kappa)$ is represented on the Figure 2. For input data $N = 20$, $\gamma = 100$, $a = 1.035$, $a_* = 1.047$ and $a_+ = 1.051$ the scheme is stable in $H_{N-1}$ for any $\kappa > 0$ and asymptotically stable in $H_{N-1}$ for $\kappa < \kappa_1 = 6.404$. The vertical asymptote is $\kappa = \kappa_2 = 28.332$.

Distinctions between difference and differential solutions for $t = 10$ are

$$u(10) = 8.217 \cdot 10^{121}, \quad y(10,1) = 4.79 \cdot 10^{122},$$
$$y(10,\kappa_1) = 6.088 \cdot 10^{124}, \quad y(10,40) = 5.801 \cdot 10^{76}, \quad y(10,80) = 1.194 \cdot 10^{16}.$$

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