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VARIATIONAL APPROACH TO SOLVING A CLASS OF NONLINEAR MULTIPARAMETER SPECTRAL PROBLEMS

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V. V. KHLOBYSTOV AND B. M. PODLEVSKYI

Анотація. Нелінійній багатопараметричній спектральній задачі у дійсному абстрактному гільбертовому просторі ставиться у відповідність варіаційна задача на мінімум деякого функціоналу. Доведена еквівалентність спектральної та варіаційної задач. На базі модифікованого методу Ньютона запропоновано чисельний алгоритм знаходження її власних значень та власних векторів.

Abstract. In the real abstract Hilbert space the nonlinear multiparameter spectral problem is assigned to the variation problem on a minimum of some functional. The equivalence of spectral and variation problems is proved. On the base of modified Newton method a numerical algorithm of finding its eigenvalues and eigenvectors is sproposed.

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1. Introduction

When studying the solvability of operator equations of the form

$$T(\lambda)x \;=\; f$$

with the operator-valued function $T(\lambda) : \; E^m \to X(H)$ ($X(H)$ is a set of the linear operators in the real Hilbert space), which linearly or nonlinearly depends on several spectral parameters λ_1, λ_2, ... , λ_m, the problems of finding such parameters λ_i, $i = 1, 2,$ ... , m, for which exists a nontrivial solution of the corresponding homogeneous equation $T(\lambda)x \;=\; 0$ exists, arise. Such problems arise in many areas of analysis and mathematical physics and have a variety of formulations. These differences in formulation are determined, in particular, by the following: the formulation of the matrix or operator problem , the number of equations and the scalar parameters, which are considered as spectral and also by dependence on them, for example, linear or nonlinear.

Various statements of such problems, the corresponding spectral theory, application and some numerical methods for solving them is the subject of research, for example, in [1] - [17].

This paper deals with nonlinear, with respect to spectral parameters, multiparameter eigenvalue problem of the form

$$T(\lambda)x \;=\; 0, \quad x \in H, \quad x \neq 0. \tag{1.1}$$

in the abstract Hilbert space H, all scalar parameters of which $\lambda = (\lambda_1, \lambda_2, ..., \lambda_{m-1}, \lambda_m)$, $i = 1, 2,$ ... , m, are spectral.

Such problems are little investigated both from a theoretical point of view (unlike the linear weakly coupled multparameter eigenvalue problems, for which the spectral theory is developed see., eg, [3], [16], [17], and some numerical methods , see., eg, [1] - [5]), and from the point of view of numerical methods for solving them.

Key words. Multiparameter eigenvalue problem, numerical algorithm, modified Newton method.

This explains to a considerable extentthe interest in such problems, in particular, in developing numerical methods for solving them since they arise in many application problems.

In this paper a variational approach to solving such problems, in which the multiparameter eigenvalue problem is replaced by the equivalent variational problem on a minimum of some functional, is proposed. In the basis of the numerical algorithm of minimization of the functional, the modified Newton method to the problem in an extended space, which is a direct sum of the abstract Hilbert space H and the real space E^m, is proposed. As a result we obtain the numerical algorithm of finding the eigenvector and a set of eigenvalues.

2. Eigenvectors and eigenvalues as the points of minimum

Let H be the real Hilbert space with the scalar product $(\cdot,\cdot)$ and the norm $||\ \cdot\ ||$.

The nonlinear multiparameter eigenvalue problems consist in finding such set of spectral parameters $\lambda^* = (\lambda_1^*,\ \dots\ ,\lambda_m^*)$, for which the nontrivial solution $x \neq 0$ of equation (1.1) exists. The set of spectral parameters $\lambda^* = \{\lambda_1^*,\ \dots\ ,\lambda_m^*\}$ we will name the generalized eigenvalue or eigenvalue set, and the corresponding vector $x^* \in H$ we will name the generalized eigenvector of the problem (1.1).

Along with the problem (1.1) we consider the problem of finding such a set of parameters $\lambda = \{\lambda_1,\ \dots\ ,\ \lambda_m\}$ and such vectors x on which functional

$$F(u) = \frac{1}{2}\|\, T(\lambda)x\,\|^2, \quad \forall\, u = \{x,\lambda\}\ \in \tilde{H} = H\, \oplus E^m,\ \ x \neq 0 \tag{2.1}$$

reaches its minimum value, i.e.

$$F(u) \to \min_u, \quad u \in \tilde{U} \subset \tilde{H}, (x \neq 0), \tag{2.2}$$

where $\tilde{U}$ is a set containing points $u^* = \{x^*, \lambda^*\}$ satisfying equation (1.1), $\tilde{H}$ is the Hilbert space in which the scalar product and norm are defined as follows:

$$(u,v)_{\tilde{H}} = (u_1,u_2) + (v_1,v_2)_{E^m}, \quad \|u\|_{\tilde{H}} = \sqrt{\|u_1\|^2 + \|v_1\|^2_{E^m}},$$

$$u = \{u_1,v_1\}, \quad v = \{u_2,v_2\}, \quad u_1,u_2 \in H, \quad v_1,v_2 \in E^m.$$

Further, we consider that the operator-valued function $T(\lambda)$ is differentiable by Frechet, i.e. for any $\lambda_k \in R$, $k = 1,\, 2,\, \dots,\, m$, partial derivatives $\frac{\partial T(\lambda)}{\partial \lambda_k}$, $k = 1,\, 2,\, \dots,\, m$, exist and we will prove, that the problems (1.1) and (3) are equivalent.

Theorem 2.1 *Each eigenvector x^* that corresponds to its eigenvalue set $\lambda^* = (\lambda_1^*, \dots, \lambda_m^*)$ of problem (1.1) is the stationary point $u^* = \{x^*,\lambda^*\}$ of functional (2.1) and, conversely, every stationary point $u^* = \{x^*,\lambda^*\}$ of functional (2.1) corresponds to eigenvector x^* and its eigenvalue set $\lambda^* = (\lambda_1^*,\ \dots\ ,\lambda_m^*)$ of problem (1.1).*

Proof. Consider the increment of functional $F(u+\Delta u) - F(u) = F(x+h,\lambda+q) - F(x,\lambda)$ for arbitrary $u,\ u+\Delta u\ \in \tilde{U}$, where $\Delta u = \{h,q\} \in \tilde{U}$. After simple transformations we obtain

$$\begin{aligned} F(u+\Delta u) - F(u) = F(x+h,\lambda+q) - F(x,\lambda) = \\ = (\ T(\lambda)x\,,\ T(\lambda)h\,) + (T(\lambda)x, \sum_{i=1}^{m} \frac{\partial T(\lambda)}{\partial \lambda_i} x q_i) \ +\ o(\|\Delta u\|) \end{aligned} \tag{2.3}$$

Consequently, the first differential of $F(x)$ will be written as

$$d\{F(x,\lambda);(h,q)\} = F'(u)\Delta u =$$
$$= (\,T(\lambda)x\,,\,T(\lambda)h\,) + \sum_{i=1}^{m}(T(\lambda)x\,,\,B_i(\lambda)x\,)q_i =$$
$$= (\,T^*(\lambda)\,T(\lambda)x\,,\;h\,) + (f(\lambda,x)\,,\;q\,)_{E^m} = (\tilde{u},\Delta u)_{\tilde{H}}, \quad (2.4)$$

where $f\,(\lambda,x) = (f_1(\lambda,x), f_2(\lambda,x),...,f_m(\lambda,x)$, $f_i(\lambda,x) = (T(\lambda)x, B_i(\lambda)x)$, $B_i = \frac{\partial T(\lambda)}{\partial \lambda_i}$, $i = 1,\,2,\,...\,,m$, $\tilde{u} = \{T^*(\lambda)T(\lambda)x, f_1\{\lambda,x), f_2(\lambda,x),...f_m(\lambda,x)\}$ is an ordered system.

Hence, for the first derivative of the functional (2.1) we have

$$F'(u)\;(\cdot) = (\tilde{u},\cdot)_{\tilde{H}}. \quad (2.5)$$

Let $T(\lambda)x = 0$, $x \neq 0$. Then from (2.4) immediately implies that $\tilde{u} = 0$. Let $\tilde{u} = 0$. Then from (2.4) we also have

$$T^*(\lambda)T(\lambda)x = 0 \Rightarrow (T^*(\lambda)T(\lambda)x,x) = 0 \Rightarrow (T(\lambda)x, T(\lambda)x) = 0 \Rightarrow T(\lambda)x = 0,$$

that proves the theorem statement. □

Thus, the problem (1.1) and the problem of finding the stationary points of the functional $F(u)$ are equivalent.

Remark 2.2 Since $F(u) \geq 0$, $F(u^*) = 0$, $u, u^* \in \tilde{U}$, then each stationary point u^* of functional $F(u)$ is a point of it local (and global) minimum.

3. Numerical algorithm

We consider that the operator-valued function $T(\lambda)$ is twice differentiable by Frechet, i.e. for any $\lambda_k \in R$, $k = 1,\,2,\,...\,,\,m$, the partial derivatives $\frac{\partial^2 T(\lambda)}{\partial\lambda_k\partial\lambda_l}$, $k,l = 1,\,2,\,...\,,\,m$, exist.

To simplify the calculations we consider two-parameter problem, that is $\lambda = \{\lambda,\mu\}$. Consequently that $u = \{x,\lambda,\mu\} \in H \oplus R^1 \oplus R^1$, i.e. for $F'(u)$ we obtain

$$F'(u) \equiv \Psi(u) =$$
$$= \begin{bmatrix} T^*(\lambda)T(\lambda)x \\ f_1(\lambda,x) \\ f_2(\lambda,x) \end{bmatrix} = \begin{bmatrix} T^*(\lambda,\mu)T(\lambda,\mu)x \\ (T(\lambda,\mu)x, T'_\lambda(\lambda,\mu)x) \\ (T(\lambda,\mu)x, T'_\mu(\lambda,\mu)x) \end{bmatrix} = \begin{bmatrix} T^*(\lambda,\mu)T(\lambda,\mu)x \\ (T(\lambda,\mu)x, B_1(\lambda,\mu)x) \\ (T(\lambda,\mu)x, B_2(\lambda,\mu)x) \end{bmatrix}.$$

Modified Newton method (about the conditions of convergence of this method see, for example, [18]), applied to the equation $\Phi(u) = 0$ (see remark 3.1), where

$$\Phi(u) = \begin{bmatrix} T(\lambda)x \\ f_1(\lambda,x) \\ f_2(\lambda,x) \end{bmatrix} = \begin{bmatrix} T(\lambda,\mu)x \\ (T(\lambda,\mu)x, B_1(\lambda,\mu)x) \\ (T(\lambda,\mu)x, B_2(\lambda,\mu)x) \end{bmatrix}, \quad (3.1)$$

which is equivalent to the equation $\Psi(u) = 0$, yields

$$\Phi'(u_0)(u_{k+1} - u_k) = -\Phi(u_k), k = 0, 1, 2, ..., \quad (3.2)$$

where $\Phi'(u_0)$ is calculated by the formula

$$\Phi'(u_0) =$$

$$\left[\begin{array}{ccc} T & B_1x & B_2x \\ T^*B_1x+B_1^*Tx & (B_1x,B_1x)+(Tx,B'_{1\lambda}x) & (B_1x,B_2x)+(Tx,B'_{1\mu}x) \\ T^*B_2x+B_2^*Tx & (B_2x,B_1x)+(Tx,B'_{2\lambda}x) & (B_2x,B_2x)+(Tx,B'_{2\mu}x) \end{array}\right]\Bigg|_{\substack{\lambda=\lambda_0 \\ x=x_0}} \tag{3.3}$$

To make one step by the method (3.2) it is necessary to solve a linear system

$$\left[\begin{array}{ccc} T & B_1x_0 & B_2x_0 \\ T^*B_1x_0+B_1^*Tx_0 & (B_1x_0,B_1x_0)+(Tx_0,B'_{1\lambda}x_0) & (B_1x_0,B_2x_0)+(Tx_0,B'_{1\mu}x_0) \\ T^*B_2x_0+B_2^*Tx_0 & (B_2x_0,B_1x_0)+(Tx_0,B'_{2\lambda}x_0) & (B_2x_0,B_2x_0)+(Tx_0,B'_{2\mu}x_0) \end{array}\right]\times$$

$$\times\left[\begin{array}{c} \Delta x_k \\ \Delta\lambda_k \\ \Delta\mu_k \end{array}\right] = -\left[\begin{array}{c} T(\lambda_k)x_k \\ (T(\lambda_k)x_k, B_1(\lambda_k)x_k) \\ (T(\lambda_k)x_k, B_2(\lambda_k)x_k) \end{array}\right] \tag{3.4}$$

and compute a new approximation

$$\begin{aligned} x_{k+1} &= x_k + \Delta x_k, \\ \lambda_{k+1} &= \lambda_k + \Delta\lambda_k, \\ \mu_{k+1} &= \mu_k + \Delta\mu_k. \end{aligned} \tag{3.5}$$

The following notations $T = T(\lambda_0)$, $B_1 = \frac{\partial T(\lambda_0)}{\partial\lambda}$, $B_2 = \frac{\partial T(\lambda_0)}{\partial\mu}$, $B'_{1\lambda} = \frac{\partial^2 T(\lambda_0)}{\partial\lambda^2}$, $B'_{1\mu} = \frac{\partial^2 T(\lambda_0)}{\partial\lambda\partial\mu}$, $B'_{2\lambda} = \frac{\partial^2 T(\lambda_0)}{\partial\mu\partial\lambda}$ are introduced to reduce the above formulae.

From (3.4) formally we obtain

$$T\Delta x_k + B_1x_0\Delta\lambda_k + B_2x_0\Delta\mu_k = -T(\lambda_k)x_k,$$

$$\begin{aligned} &[T^*B_1x_0 + B_1^*Tx_0]\Delta x_k + [(B_1x_0,B_1x_0)+(Tx_0,B'_{1\lambda}x_0)]\Delta\lambda_k + \\ &\quad +\left[(B_1x_0,B_2x_0)+(Tx_0,B'_{1\mu}x_0)\right]\Delta\mu_k = -(T(\lambda_k)x_k, B_1(\lambda_k)x_k), \end{aligned} \tag{3.6}$$

$$\begin{aligned} &[T^*B_2x_0 + B_2^*Tx_0]\Delta x_k + [(B_2x_0,B_1x_0)+(Tx_0,B'_{2\lambda}x_0)]\Delta\lambda_k + \\ &\quad +\left[(B_2x_0,B_2x_0)+(Tx_0,B'_{2\mu}x_0)\right]\Delta\mu_k = -(T(\lambda_k)x_k, B_2(\lambda_k)x_k). \end{aligned}$$

From the first equation (3.6) we have

$$\Delta x_k = -T^{-1}\left(B_1x_0\Delta\lambda_k + B_2x_0\Delta\mu_k\right) - T^{-1}T(\lambda_k)x_k. \tag{3.7}$$

To calculate $\Delta\lambda_k$ and $\Delta\mu_k$ we will use the second and third equation of (3.5) "substituting" the expression (3.7) instead of Δx_k (actually it is the scalar product). Then we obtain a linear system of equations

$$\left[\begin{array}{cc} (Tx_0,B'_{1\lambda}x_0)-(B_1^*Tx_0,T^{-1}B_1x_0) & (Tx_0,B'_{1\lambda}x_0)-(B_1^*Tx_0,T^{-1}B_2x_0) \\ (Tx_0,B'_{2\lambda}x_0)-(B_2^*Tx_0,T^{-1}B_1x_0) & (Tx_0,B'_{2\lambda}x_0)-(B_2^*Tx_0,T^{-1}B_2x_0) \end{array}\right]\left[\begin{array}{c} \Delta\lambda_k \\ \Delta\mu_k \end{array}\right] =$$

$$= -\left[\begin{array}{c} (T(\lambda_k)x_k,B_1(\lambda_k)x_k)+(B_1x_0,T(\lambda_k)x_k)+(B_1^*Tx_0,T^{-1}T(\lambda_k)x_k) \\ (T(\lambda_k)x_k,B_2(\lambda_k)x_k)+(B_2x_0,T(\lambda_k)x_k)+(B_2^*Tx_0,T^{-1}T(\lambda_k)x_k) \end{array}\right] \tag{3.8}$$

Further, we consider that the determinant of a matrix system (3.8) is different from zero.

Thus, the algorithm scheme is as follows. After solving a system (3.8) with respect to $\Delta\lambda_k$ and $\Delta\mu_k$ we compute Δx_k using the formula (3.7).

For m-parametric eigenvalue problem similar to (3.8) linear system of m equations to calculate $\Delta\lambda_k^i = \lambda_{k+1}^i - \lambda_k^i$, $i = 1, ..., m$ is obtained. For calculation Δx_k we obtain equation of the form

$$\Delta x_k = -T^{-1}\left(\sum_{i=1}^{m} B_ix_0\Delta\lambda_k^i\right) - T^{-1}T(\lambda_k)x_k$$

similar to (3.7).

Remark 3.1 For Newton method when $\lambda_k \underset{k\to\infty}{\longrightarrow} \lambda_*$, $\lim\limits_{k\to\infty} T^{-1}(\lambda_k)$ does not exist, that's why for equation $\Phi(u) = 0$ was have chosen the modified Newton method.

Note that in [6], [10] and [11] for $x \in E^n$, as in [9] for $x \in L^2[-1,1]$ another formulation of variational problem, which, in particular, for matrix formulation of the problem leads, for example, to a gradient procedure for calculating eigenvectors, and for calculating its eigenvalues a system of linear equations obtained similar to (3.8) was proposed. In [15] a similar statement of the problem of finding eigenvalues and eigenvectors of linear homogeneous equations $T(\lambda)x = x$ for the case when $T(\lambda) : L^2(\Omega) \to L^2(\Omega)$ is a linear selfadjoint positive define operator, which depends nonlinearly on the spectral parameter $\lambda = \{\lambda_1, \ldots, \lambda_m\} \in R^m$, and in the basis of numerical algorithm to minimize the functional is a method of coordinate descent, was proposed.

4. Numerical example

Consider the application of the proposed algorithm for finding eigenvalues and eigenfunctions of two-parameter eigenvalue problem

$$u(\xi_1, \xi_2) = T(\lambda_1, \lambda_2) u(\xi_1, \xi_2) \tag{4.1}$$

with integral operators form

$$T(\lambda_1, \lambda_2)\, u(\xi_1, \xi_2) = \iint_{\Omega} F(\xi_1', \xi_2') K(\xi_1, \xi_2, \xi_1', \xi_2', \lambda_1, \lambda_2) \frac{u(\xi_1', \xi_2')}{f_0(\xi_1, \xi_2)}\, d\xi_1' d\xi_2',$$

whose kernel nonlinearly depends on two spectral parameters $\lambda_1, \lambda_2 \in R^1$. Here $F(\xi_1, \xi_2)$ is continuous in $\Omega = \{\, |\xi_1| \le 1, |\xi_2| \le 1 \,\}$ really and positive function, and

$$f_0(\xi_1, \xi_2) = \iint_{\Omega} F(\xi_1', \xi_2') K(\xi_1, \xi_2, \xi_1', \xi_2', \lambda_1, \lambda_2) d\xi_1' d\xi_2'.$$

Equation (4.1) arises in the theory of antennas synthesis of finding points of possible branching of solutions of nonlinear integral equation [19]

$$f(\xi_1, \xi_2) = \iint_{\Omega} F(\xi_1', \xi_2') K(\xi_1, \xi_2, \xi_1', \xi_2', \lambda_1, \lambda_2) e^{i \arg f(\xi_1', \xi_2')} d\xi_1 d\xi_2, \tag{4.2}$$

It is easy to verify that for arbitrary finite values λ_1, λ_2, the function $f_0(\xi_1, \xi_2)$ is the eigenfunction of equation (4.2). It follows that the operator $T(\lambda_1, \lambda_2)$ has a spectrum, which coincides with the first quadrant of the plane R^2.

The problem consists in finding such a range of real parameters λ_1 and λ_2 of the problem (4.1), for which there are solutions different from $f_0(\xi_1, \xi_2)$ that we call trivial.

We have brought the problem (4.1) to self-adjoint form and its spectrum we have excluded from continual set of eigenvalues, which coincides with the first quadrant of the plane. We obtain the self-adjoint eigenvalue problem

$$L(\lambda_1, \lambda_2)\varphi \equiv (T(\lambda_1, \lambda_2) - I)\, \varphi = 0, \tag{4.3}$$

with continuously differentiable, with respect to parameters λ_1 and λ_2, operator

$$T(\lambda_1, \lambda_2)\varphi(\xi_1, \xi_2) \equiv \int_{-1}^{1}\int_{-1}^{1} E(\xi_1, \xi_2, \xi_1', \xi_2', \lambda_1, \lambda_2)\varphi(\xi_1', \xi_2') d\xi_1' d\xi_2'.$$

The existence of partial Fréchet derivatives of operator follows from the continuity of the kernel $E(\xi_1, \xi_2, \xi_1', \xi_2', \lambda_1, \lambda_2)$ on the set of its variables in the $\Omega \times \Omega$. Due to the awkwardness of the formulas, expressions for the derivatives are not quoted here.

Numerous experiments on the use of the described algorithm for finding the eigenvalues of the problem (4.2) for different functions $F(\xi_1, \xi_2)$ were carried out.

The following table shows the eigenvalues obtained for two given functions $F(\xi_1, \xi_2) \equiv F_1(\xi_1, \xi_2) = \sqrt{1 - (\xi_1^2 + \xi_2^2)/2}$ and $F(\xi_1, \xi_2) \equiv F_2(\xi_1, \xi_2) = 1 - (\xi_1^2 + \xi_2^2)/2$. When selecting the initial approximations for the eigenvectors the properties of invariance of the integral operator relative to the parity functions $\varphi(\xi_1, \xi_2)$ (see [19]) were taken into account. Namely, in both cases as the initial approximation of eigenvector we take $\varphi^{(0)}(\xi_1, \xi_2) = \xi_1 \cdot \xi_2$ and that of eigenvalue we take $\lambda_1 = \lambda_2 = \pi$.

Tabl. 4.1. Eigenvalues of the problem (4.3)

$F(\xi_1, \xi_2)$	λ_1	λ_2	*Number of iteration*	$F(u)$
$\sqrt{1-(\xi_1^2+\xi_2^2)/2}$	3.464427	3.464427)	23	0.161080E-13
$1-(\xi_1^2+\xi_2^2)/2$	3.797144	3.797144	32	0.100711E-14

The figure shows the eigenvalue curves of the problem (4.3) for two given functions $F(\xi_1, \xi_2)$. The curve 1 corresponds to the function $F_1(\xi_1, \xi_2)$, and the curve 2 corresponds to the function $F_2(\xi_1, \xi_2)$. The same figure indicate the eigenvalues which are obtained by the proposed algorithm. Namely, the point A_1 and the point A_2, that correspond to the functions $F_1(\xi_1, \xi_2)$ and $F_2(\xi_1, \xi_2)$, respectively.

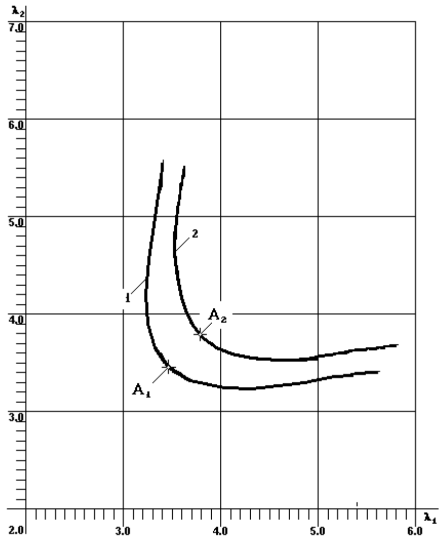

Fig. 1. Eigenvalue curves of the problem (4.3)

The figure shows that the obtained eigenvalues (point A_1 and point A_2) with given accuracy belong to the eigenvalue curves of problem (4.3), which has been otherwise proposed in [12].

By providing different initial approximations for the parameters λ and μ ($\lambda \neq \mu$), we obtain the curve of eigenvalues, for example, the curve 1, which was first obtained for the problem (4.3) in [12].

So for the parameter values λ and μ belonging, for example, to the curve 1 (respectively, the curve 2), from the trivial solution other solutions of equation (4.2) branch off.

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Institute of Mathematics,
National Academy of Sciences of Ukraine,
3 Tereshchenkivska st., Kyiv, 01601, Ukraine.
E-mail address: khl@i.com.ua

Pidstrygach Institute for Applied Problems of Mechanics and Mathematics,
National Academy of Sciences of Ukraine,
3-b Naukova st., Lviv, 79060, Ukraine.
E-mail address: podlev@iapmm.lviv.ua

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