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EXACT THREE-POINT DIFFERENCE SCHEME FOR NONLINEAR STATIONARY DIFFERENTIAL EQUATIONS IN CYLINDRICAL COORDINATES

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Анотація. Для чисельного розв'язування крайових задач для нелінійних стаціонарних рівнянь в циліндричній системі координат побудовано та обгрунтовано точну триточкову різницеву схему на нерівномірній сітці. Доведено існування та єдиність розв'язку цієї схеми, збіжність ітераційного методу послідовних наближень для її розв'язування.

Abstract. Exact three-point difference scheme on a irregular grid for the numerical solution boundary value problems for nonlinear stationary equations in cylindrical coordinate system is constructed and justified. The existence and uniqueness of the solution of this scheme, the convergence of iterative method of successive approximation for its solution are proved.

1. Introduction

An approach for the construction of exact three-point difference schemes (ETDS) and the corresponding three-point difference schemes (TDS) of an arbitrary given order of accuracy m for nonlinear boundary-value problems (BVP) of the form

$$\frac{d}{dx}\left[k(x)\frac{du}{dx}\right]=-f(x,u(x)),\quad x\in[0,1],\quad u(0)=\mu_1,\quad u(1)=\mu_2$$

was proposed for the first time in [1]. Further development of the ideas of [1] was obtained in [2,3] and in the case of nonlinear monotone ordinary differential equations in [4, 5].

In this paper, the exact three-point difference scheme for BVP

$$\frac{1}{r}\frac{d}{dr}\left[rk(r)\frac{du}{dr}\right]=-f(r,u),\quad r\in[0,R],\quad \lim_{r\to 0} rk(r)\frac{du}{dr}=0,\quad u(R)=\mu_2 \tag{1.1}$$

on the irregular grid $\widehat{\overline{\omega}}_h=\{r_j\in[0,R], j=0,1,\dots,N, r_0=0, r_N=R\}$ is constructed. Such difference scheme requires for each point $r_j, j=1,2,\dots,N-1$ of the grid $\widehat{\omega}_h$ the four auxiliary initial value problems: the two nonlinear ordinary differential equations and the two linear ordinary differential equations on the intervals $[r_{j-1},r_j]$ (forward) and $[r_j,r_{j+1}]$ (backward) to be solved.

2. Existence and uniqueness of the solution to the problem

The following theorem gives sufficient conditions for the existence and uniqueness of a solution of BVP (1.1), that is based on the linearization method and the principle of contraction mapping (see, e.g., [6, 7]).

Theorem 2.1 *Suppose that*

$$0<c_1\le k(r)\le c_2\quad \forall r\in[0,R],\ k(r)\in Q^1[0,R], \tag{2.1}$$

$$f_u(r)\equiv f(r,u)\in Q^0[0,R],\quad |f(r,u)|\le K\quad \forall r\in[0,R],\ u\in\Omega([0,R],\rho), \tag{2.2}$$

Key words. Nonlinear ordinary differential equation, boundary value problem, three-point exact difference scheme, linearization method, principle of contraction mapping.

$$|f(r,u)-f(r,v)| \le L|u-v| \quad \forall r \in [0,R],\ u,v \in \Omega([0,R],\rho), \tag{2.3}$$

$$q = \frac{LR^2}{4c_1}\max(1,2c_1) < 1. \tag{2.4}$$

Then the BVP (1.1) has a unique solution $u(r) \in \Omega([0,R],\rho)$, that can be found by the method of successive approximations

$$\begin{aligned} &\frac{1}{r}\frac{d}{dr}\left[rk(r)\frac{du^{(n)}}{dr}\right] = -f(r,u^{(n-1)}(r)), \quad r \in (0,R), \\ &\lim_{r\to 0} rk(r)\frac{du^{(n)}(0)}{dr} = 0, \quad u^{(n)}(R) = \mu_2, \quad n = 1,2,\dots, \quad u^{(0)}(r) = \mu_2 \end{aligned} \tag{2.5}$$

with the error estimate

$$\left\|u^{(n)} - u\right\|_{1,\infty,[0,R]} \le \frac{q^n}{1-q}\rho. \tag{2.6}$$

Here $Q^p[0,R]$ is the class of function with piecewise continuous derivatives up to the pth order, including the finite number of discontinuity points of the first kind; $\Omega([0,R],\rho)$ is a set of functions of the form

$$\Omega([0,R],\rho) = \Big\{u(r) : u(r) \in W^1_\infty[0,R],\ u(r), rk(r)\frac{du}{dr} \in C[0,R],$$

$$\left\|u - u^{(0)}\right\|^*_{1,\infty,[0,R]} \le \rho\Big\}, \quad \rho = \frac{KR^2}{4c_1}\max(1,2c_1),$$

$$\|u\|_{0,\infty,[0,R]} = \max_{r\in[0,R]}|u(r)|, \quad \|u\|^*_{1,\infty,[0,R]} = \max\left\{\|u\|_{0,\infty,[0,R]}, \left\|rk(r)\frac{du}{dr}\right\|_{0,\infty,[0,R]}\right\}.$$

Proof. We write problem (1.1) in an equivalent integral form

$$u(r) = \mathrm{Re}\ (r,u(\cdot)) = \int_0^R G(r,\xi)f(\xi,u(\xi))d\xi + \mu_2, \quad 0 \le r \le R, \tag{2.7}$$

where

$$G(r,\xi) = \begin{cases} \xi V(\xi), & 0 \le r \le \xi, \\ \xi V(r), & \xi \le r \le R, \end{cases} \qquad V(r) = \int_r^R \frac{dt}{tk(t)}.$$

Denote, the boundary condition at $r \to 0$ is satisfied, since

$$\frac{du}{dr} = -\frac{1}{rk(r)}\int_0^r \xi f(\xi,u(\xi))d\xi.$$

From the equality (2.7) follows, that the solution of problem (1.1)

$$u(r) = \int_r^R \xi V(\xi)f(\xi,u(\xi))d\xi + V(r)\int_0^r \xi f(\xi,u(\xi))d\xi + \mu_2$$

has a logarithmic singularity at the point $r = 0$.

Let us show that operator (2.7) maps the set $\Omega([0,R],\rho)$ onto itself. Taking into account condition (2.2), we have

$$\begin{aligned} \|\mathrm{Re}\ (r,v(\cdot)) - u^{(0)}\|^*_{1,\infty,[0,R]} &\le \int_0^R \|G(r,\xi)\|^*_{1,\infty,[0,R]}\,|f(\xi,v(\xi))|\,d\xi \le \\ &\le K\int_0^R \|G(r,\xi)\|^*_{1,\infty,[0,R]}d\xi \quad \forall v \in \Omega([0,R],\rho). \end{aligned} \tag{2.8}$$

Due to (2.1)

$$\int_0^R |G(r,\xi)|d\xi = \int_r^R \xi V(\xi)d\xi + V(r)\int_0^r \xi d\xi = V(\xi)\frac{\xi^2}{2}\bigg|_r^R + \frac{1}{2}\int_r^R \frac{\xi}{k(\xi)}d\xi +$$

$$+ V(r)\frac{\xi^2}{2}\bigg|_0^r = \frac{1}{2}\int_r^R \frac{\xi d\xi}{k(\xi)} \le \frac{1}{2c_1}\int_r^R \xi d\xi = \frac{R^2-r^2}{4c_1} \le \frac{R^2}{4c_1},$$

$$\int_0^R \left| rk(r)\frac{\partial G(r,\xi)}{\partial r}\right| d\xi = \int_0^r \xi d\xi = \frac{r^2}{2} \le \frac{R^2}{2},$$

we obtain

$$\int_0^R \|G(r,\xi)\|^*_{1,\infty,[0,R]}d\xi \le \frac{R^2}{4c_1}\max(1,2c_1). \tag{2.9}$$

From the inequalities (2.8), (2.9) follows

$$\left\|\text{Re }(r,v(\cdot)) - u^{(0)}\right\|^*_{1,\infty,[0,R]} \le \frac{KR^2}{4c_1}\max(1,2c_1) = \rho, \quad \forall v\in\Omega([0,R],\rho).$$

Moreover, Re $(r,u(\cdot))$ is a contraction operator on the $\Omega([0,R],\rho)$. Taking into account condition (2.3) and inequality (2.9), we have

$$\|\text{Re }(r,u(\cdot)) - \text{ Re }(r,v(\cdot))\|^*_{1,\infty,[0,R]} \le \int_0^R \|G(r,\xi)\|^*_{1,\infty,[0,R]}\,|f(\xi,u(\xi)) - f(\xi,v(\xi))|\,d\xi \le$$

$$\le L\int_0^R \|G(r,\xi)\|^*_{1,\infty,[0,R]}\,|u(\xi)-v(\xi)|\,d\xi \le L\|u-v\|^*_{1,\infty,[0,R]}\int_0^R \|G(r,\xi)\|^*_{1,\infty,[0,R]}d\xi \le$$

$$\le \frac{LR^2}{4c_1}\max(1,2c_1)\,\|u-v\|^*_{1,\infty,[0,R]} = q\|u-v\|^*_{1,\infty,[0,R]} \quad \forall u,v\in\Omega([0,R],\rho).$$

Thus, for the operator Re $(r,u(\cdot))$ all conditions of the principle of contraction mapping are fulfilled and, hence, problem (1.1) has a unique solution, which can be obtained by the method of successive approximations (2.5) with the error estimate (2.6). The way of obtaining of the estimate (2.6) is standard (see, e.g., [6]), and, therefore, we do not describe it. □

3. Existence of an exact three-point difference scheme

On the closed interval $[0,R]$ we introduce an irregular grid

$$\widehat{\overline{\omega}}_h = \{r_j\in[0,R], \quad j=0,1,\dots,N, \quad r_0=0,\ r_N=R\},$$
$$h_j = r_j - r_{j-1} > 0, \quad j=1,2,\dots,N, \quad |h| = \max_{1\le j\le N} h_j$$

such that the discontinuity points of functions $k(r), f(r,u)$ coincide with the nodes of the grid $\widehat{\omega}_h = \{r_j,\ j=1,2,\dots,N-1\}$. Denote by θ the set of all discontinuity points and assume that N is such that $\theta\subseteq\widehat{\omega}_h$. At discontinuity points the usual consistency conditions

$$u(r_i-0) = u(r_i+0), \quad rk(r)\frac{du}{dr}\bigg|_{r=r_i-0} = rk(r)\frac{du}{dr}\bigg|_{r=r_i+0} \quad \forall r_i\in\theta$$

must be satisfied.

Let us introduce the set

$$\Omega(\widehat{\omega}_h, r) = \left\{(v_j)_{j=0}^N : \|v-u^{(0)}\|^*_{1,\infty,\widehat{\omega}_h} \le \rho\right\}.$$

We will use the following notation:

$$\|y\|_{0,\infty,\widehat{\omega}_h}=\max_{\xi\in\widehat{\omega}_h}\|y(\xi)\|,\quad \|y\|^*_{1,\infty,\widehat{\omega}_h}=\max\left\{\|y\|_{0,\infty,\widehat{\omega}_h},\left\|rk(r)\frac{dy}{dr}\right\|_{0,\infty,\widehat{\omega}_h}\right\}.$$

Consider the boundary value problems

$$\begin{aligned}&\frac{1}{r}\frac{d}{dr}\left[rk(r)\frac{dY_1^1(r,u)}{dr}\right]=-f(r,Y_1^1(r,u)),\quad 0<r<r_1,\\&\lim_{r\to 0}rk(r)\frac{dY_1^1(r,u)}{dr}=0,\quad Y_1^1(r_1,u)=u_1,\end{aligned}\tag{3.1}$$

$$\begin{aligned}&\frac{1}{r}\frac{d}{dr}\left[rk(r)\frac{dY_\alpha^j(r,u)}{dr}\right]=-f(r,Y_\alpha^j(r,u)),\quad r_{j-2+\alpha}<r<r_{j-1+\alpha},\\&Y_\alpha^j(r_{j-2+\alpha},u)=u(r_{j-2+\alpha}),\quad Y_\alpha^j(r_{j-1+\alpha},u)=u(r_{j-1+\alpha}),\\&j=3-\alpha,4-\alpha,\dots,N+1-\alpha,\quad \alpha=1,2.\end{aligned}\tag{3.2}$$

The following lemma is valid.

Lemma 3.1 *Suppose that assumptions (2.1) – (2.4) are satisfied. Then problems (3.1), (3.2) have a unique solution $Y_1^1(r,u),Y_\alpha^j(r,u),j=3-\alpha,4-\alpha,\dots,N+1-\alpha,\alpha=1,2$. Moreover, the solution of problem (1.1) can be represented in the form*

$$\begin{aligned}&u(r)=Y_\alpha^j(r,u),\quad r\in[r_{j-2+\alpha},r_{j-1+\alpha}],\\&j=3-\alpha,4-\alpha,\dots,N+1-\alpha,\quad \alpha=1,2,\\&u(r)=Y_1^1(r,u),\quad r\in[0,r_1].\end{aligned}\tag{3.3}$$

Proof. We write problems (3.1), (3.2) in an equivalent integral form

$$\begin{aligned}&Y_1^1(r,u)=\int_0^{r_1}\widetilde{G}^1(r,\xi)f(\xi,Y_1^1(\xi,u))d\xi+u_1,\quad r\in[0,r_1],\\&Y_\alpha^j(r,u)=\int_{r_{j-2+\alpha}}^{r_{j-1+\alpha}}\widetilde{G}^{j-1+\alpha}(r,\xi)f(\xi,Y_\alpha^j(\xi,u))d\xi+\hat{u}(r),\\&r\in[r_{j-2+\alpha},r_{j-1+\alpha}],\quad j=3-\alpha,4-\alpha,\dots,N+1-\alpha,\quad \alpha=1,2,\\&\hat{u}(r)=\frac{u(r_j)V_1^j(r)+u(r_{j-1})V_2^{j-1}(r)}{V_1^j(r_j)},\quad r\in[r_{j-1},r_j],\end{aligned}\tag{3.4}$$

where

$$V_1^j(r)=\int_{r_{j-1}}^{r}\frac{dt}{tk(t)},\quad V_2^j(r)=\int_r^{r_{j+1}}\frac{dt}{tk(t)},$$

$$\widetilde{G}^1(r,\xi)=\begin{cases}\xi V_2^0(\xi),&0\le r\le\xi,\\ \xi V_2^0(r),&\xi\le r\le r_1,\end{cases}\tag{3.5}$$

$$\widetilde{G}^{j-1+\alpha}(r,\xi)=\begin{cases}\dfrac{V_1^{j-1+\alpha}(r)V_2^{j-2+\alpha}(\xi)\xi}{V_1^{j-1+\alpha}(r_{j-1+\alpha})},&r_{j-2+\alpha}\le r\le\xi,\\[2ex] \dfrac{V_1^{j-1+\alpha}(\xi)V_2^{j-2+\alpha}(r)\xi}{V_1^{j-1+\alpha}(r_{j-1+\alpha})},&\xi\le r\le r_{j-1+\alpha}.\end{cases}\tag{3.6}$$

At $\alpha=1$ in view of (2.7), we have

$$\begin{aligned}\hat{u}(r)=&\frac{V_1^j(r)}{V_1^j(r_j)}\left[\int_0^R G(r_j,\xi)f(\xi,u(\xi))d\xi+\mu_2\right]+\\&+\frac{V_2^{j-1}(r)}{V_1^j(r_j)}\left[\int_0^R G(r_{j-1},\xi)f(\xi,u(\xi))d\xi+\mu_2\right],\quad r\in[r_{j-1},r_j].\end{aligned}$$

Since $V_1^j(r) + V_2^{j-1}(r) = V_1^j(r_j)$ we obtain

$$\hat{u}(r) = \frac{V_1^j(r)}{V_1^j(r_j)} \int\limits_0^R G(r_j, \xi) f(\xi, u(\xi)) d\xi + $$
$$+ \frac{V_2^{j-1}(r)}{V_1^j(r_j)} \int\limits_0^R G(r_{j-1}, \xi) f(\xi, u(\xi)) d\xi + \mu_2, \quad r \in [r_{j-1}, r_j].$$

Then equality (3.4), at $\alpha = 1$ we write in the form

$$Y_1^j(r,u) = \frac{V_1^j(r)}{V_1^j(r_j)} \int\limits_0^R G(r_j, \xi) f(\xi, u(\xi)) d\xi + \frac{V_2^{j-1}(r)}{V_1^j(r_j)} \int\limits_0^R G(r_{j-1}, \xi) f(\xi, u(\xi)) d\xi +$$
$$+ \int\limits_{r_{j-1}}^{r_j} \widetilde{G}^j(r,\xi) f(\xi, Y_1^j(\xi, u)) d\xi + u^{(0)}(r), \quad r \in [r_{j-1}, r_j],$$

$$Y_1^1(r,u) = \int\limits_0^R G(r_1, \xi) f(\xi, u(\xi)) d\xi + \int\limits_0^{r_1} \widetilde{G}^1(r,\xi) f(\xi, Y_1^1(\xi, u)) d\xi +$$
$$+ u^{(0)}(r), \quad r \in [0, r_1].$$

By virtue of equality $Y_2^j(r,u) = Y_1^{j+1}(r,u)$, we have

$$Y_2^j(r,u) = \frac{V_1^{j+1}(r)}{V_2^j(r_j)} \int\limits_0^R G(r_{j+1}, \xi) f(\xi, u(\xi)) d\xi + \frac{V_2^j(r)}{V_2^j(r_j)} \int\limits_0^R G(r_j, \xi) f(\xi, u(\xi)) d\xi +$$
$$+ \int\limits_{r_j}^{r_{j+1}} \widetilde{G}^{j+1}(r,\xi) f(\xi, Y_2^j(\xi, u)) d\xi + u^{(0)}(r), \quad r \in [r_j, r_{j+1}].$$

Consequently, the existence and uniqueness of the solution to problems (3.1), (3.2) is equivalent to a similar problem for equations

$$\begin{aligned} U_\alpha^j(r) = &\Im_\alpha^j(r, u, U_\alpha^j) = \frac{V_1^{j-1+\alpha}(r)}{V_\alpha^j(r_j)} \int\limits_0^R G(r_{j-1+\alpha}, \xi) f(\xi, u(\xi)) d\xi + \\ &+ \frac{V_2^{j-2+\alpha}(r)}{V_\alpha^j(r_j)} \int\limits_0^R G(r_{j-2+\alpha}, \xi) f(\xi, u(\xi)) d\xi + \\ &+ \int\limits_{r_{j-2+\alpha}}^{r_{j-1+\alpha}} \widetilde{G}^{j-1+\alpha}(r,\xi) f(\xi, U_\alpha^j(\xi, u)) d\xi + u^{(0)}(r), \\ &r \in [r_{j-2+\alpha}, r_{j-1+\alpha}], \quad j = 1, 2, \ldots, N+1-\alpha, \quad \alpha = 1, 2, \end{aligned} \tag{3.7}$$

$$\begin{aligned} U_1^1(r) = &\Im_1^1(r, u, U_1^1) = \int\limits_0^R G(r_1, \xi) f(\xi, u(\xi)) d\xi + \\ &+ \int\limits_0^{r_1} \widetilde{G}^1(r,\xi) f(\xi, U_1^1(\xi, u)) d\xi + u^{(0)}(r), \quad r \in [0, r_1]. \end{aligned} \tag{3.8}$$

Let's show that the operators $\Im_1^j(r,u,U_1^j)$, $\Im_1^1(r,u,U_1^1)$ map the sets $\Omega([r_{j-1},r_j],\rho)$ and $\Omega([0,r_1],\rho)$, respectively, onto itself. Suppose $U_1^j(\xi)\in\Omega([r_{j-1},r_j],\rho), U_1^1(\xi)\in\Omega([0,r_1],\rho)$. Then

$$\left|\Im_1^j(r,u,U_1^j)-u^{(0)}(r)\right|\le$$
$$\le K\left\{\frac{V_2^{j-1}(r)}{V_1^j(r_j)}\int\limits_0^R G(r_{j-1},\xi)d\xi+\frac{V_1^j(r)}{V_1^j(r_j)}\int\limits_0^R G(r_j,\xi)d\xi+\int\limits_{r_{j-1}}^{r_j}\widetilde{G}^j(r,\xi)d\xi\right\}=$$

$$=K\left\{\frac{V_2^{j-1}(r)}{V_1^j(r_j)}\left[\int\limits_{r_{j-1}}^R\xi V(\xi)d\xi+V(r_{j-1})\int\limits_0^{r_{j-1}}\xi d\xi\right]+\right.$$
$$+\frac{V_1^j(r)}{V_1^j(r_j)}\left[\int\limits_{r_j}^R\xi V(\xi)d\xi+V(r_j)\int\limits_0^{r_j}\xi d\xi\right]+$$
$$\left.+\frac{V_1^j(r)}{V_1^j(r_j)}\int\limits_r^{r_j}V_2^{j-1}(\xi)\xi d\xi+\frac{V_2^{j-1}(r)}{V_1^j(r_j)}\int\limits_{r_{j-1}}^r V_1^j(\xi)\xi d\xi\right\}=$$
$$=K\left\{\frac{V_2^{j-1}(r)}{V_1^j(r_j)}\left[V(r_{j-1})\int\limits_0^r\xi d\xi+\int\limits_r^R\xi V(\xi)d\xi\right]+\right.$$
$$+\frac{V_1^j(r)}{V_1^j(r_j)}\left[\int\limits_r^R\xi V(\xi)d\xi+V(r_j)\int\limits_0^r\xi d\xi\right]+$$
$$+\frac{V_1^j(r)}{V_1^j(r_j)}\left[\int\limits_r^{r_j}V_2^{j-1}(\xi)\xi d\xi+V(r_j)\int\limits_r^{r_j}\xi d\xi-\int\limits_r^{r_j}\xi V(\xi)d\xi\right]+$$
$$\left.+\frac{V_2^{j-1}(r)}{V_1^j(r_j)}\left[\int\limits_{r_{j-1}}^r V_1^j(\xi)\xi d\xi-V(r_{j-1})\int\limits_{r_{j-1}}^r\xi d\xi+\int\limits_{r_{j-1}}^r\xi V(\xi)d\xi\right]\right\}=$$
$$=K\left\{\left[\frac{V_2^{j-1}(r)V(r_{j-1})+V_1^j(r)V(r_j)}{V_1^j(r_j)}\right]\int\limits_0^r\xi d\xi+\right.$$
$$+\int\limits_r^R\xi V(\xi)d\xi+\frac{V_1^j(r)}{V_1^j(r_j)}\int\limits_r^{r_j}\left[V_2^{j-1}(\xi)+V(r_j)-V(\xi)\right]\xi d\xi+$$
$$\left.+\frac{V_2^{j-1}(r)}{V_1^j(r_j)}\int\limits_{r_{j-1}}^r\left[V_1^j(\xi)-V(r_{j-1})+V(\xi)\right]\xi d\xi\right\},$$

$$\left|\Im_1^1(r,u,U_1^1)-u^{(0)}(r)\right|\le$$
$$\le K\left\{\int\limits_r^{r_1}\xi V_2^0(\xi)d\xi+V_2^0(r)\int\limits_0^r\xi d\xi+\int\limits_{r_1}^R\xi V(\xi)d\xi+V(r_1)\int\limits_0^{r_1}\xi d\xi\right\}=$$
$$=K\left\{\left[V_2^0(r)+V(r_1)\right]\int\limits_0^r\xi d\xi+\int\limits_r^R\xi V(\xi)d\xi+\int\limits_r^{r_1}\xi\left[V_2^0(\xi)+V(r_1)-V(\xi)\right]d\xi\right\}.$$

Taking into account identities

$$V_2^{j-1}(\xi) + V(r_j) - V(\xi) = \int\limits_{\xi}^{r_j} \frac{dt}{tk(t)} + \int\limits_{r_j}^{R} \frac{dt}{tk(t)} - \int\limits_{\xi}^{R} \frac{dt}{tk(t)} = 0,$$

$$V_1^{j}(\xi) - V(r_{j-1}) + V(\xi) = \int\limits_{r_{j-1}}^{\xi} \frac{dt}{tk(t)} - \int\limits_{r_{j-1}}^{R} \frac{dt}{tk(t)} + \int\limits_{\xi}^{R} \frac{dt}{tk(t)} = 0,$$

$$\frac{V_2^{j-1}(r)V(r_{j-1}) + V_1^j(r)V(r_j)}{V_1^j(r_j)} =$$
$$= \frac{V_2^{j-1}(r)[V_1^j(r) + V(r)] + V_1^j(r)[V(r) - V_2^{j-1}(r)]}{V_1^j(r_j)} = V(r)$$

we obtain

$$\left|\Im_1^j(r,u,U_1^j) - u^{(0)}(r)\right| \le K\left\{V(r)\int\limits_0^r \xi d\xi + \int\limits_r^R \xi V(\xi) d\xi\right\} =$$
$$= K\int\limits_0^R G(r,\xi)d\xi, \quad j = 1, 2, \ldots, N.$$

Consequently

$$\|\Im_1^j(r,u,U_1^j) - u^{(0)}(r)\|^*_{1,\infty,[r_{j-1},r_j]} \le \frac{KR^2}{4c_1}\max(1, 2c_1) = \rho, \quad j = 1, 2, \ldots, N,$$

i.e., the operators $\Im_1^j(r,u,U_1^j)$, map the sets $\Omega([r_{j-1},r_j],\rho)$ onto itself.

Analogously, it is possible to show that if $U_2^j(r) \in \Omega([r_j, r_{j+1}],\rho)$, then

$$\|\Im_2^j(r,u,U_2^j) - u^{(0)}(r)\|^*_{1,\infty,[r_j,r_{j+1}]} \le \rho.$$

Moreover, for $\forall\ U_\alpha^j(r), \widetilde{U}_\alpha^j(r) \in \Omega([r_{j-2+\alpha}, r_{j-1+\alpha}],\rho)$ and $U_1^1(r), \widetilde{U}_1^1(r) \in \Omega([0,r_1],\rho)$ we have estimates

$$\begin{aligned}
&\|\Im_\alpha^j(r,u,U_\alpha^j) - \Im_\alpha^j(r,u,\widetilde{U}_\alpha^j)\|^*_{1,\infty,[r_{j-2+\alpha},r_{j-1+\alpha}]} \le \\
&\le \int\limits_{r_{j-2+\alpha}}^{r_{j-1+\alpha}} \left\|\widetilde{G}^{j-1+\alpha}(r,\xi)\right\|^*_{1,\infty,[r_{j-2+\alpha},r_{j-1+\alpha}]} \left|f(\xi, U_\alpha^j(\xi,u)) - f(\xi,\widetilde{U}_\alpha^j(\xi,u))\right| d\xi \le \\
&\le L\|U_\alpha^j - \widetilde{U}_\alpha^j\|^*_{1,\infty,[r_{j-2+\alpha},r_{j-1+\alpha}]} \int\limits_{r_{j-2+\alpha}}^{r_{j-1+\alpha}} \left\|\widetilde{G}^{j-1+\alpha}(r,\xi)\right\|^*_{1,\infty,[r_{j-2+\alpha},r_{j-1+\alpha}]} d\xi,
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
&\|\Im_1^1(r,u,U_1^1) - \Im_1^1(r,u,\widetilde{U}_1^1)\|^*_{1,\infty,(0,r_1)} \le \\
&\le \int\limits_0^{r_1} \left\|\widetilde{G}^1(r,\xi)\right\|^*_{1,\infty,[0,r_1]} \left|f(\xi,U_1^1(\xi,u)) - f(\xi,\widetilde{U}_1^1(\xi,u))\right| d\xi \le \\
&\le L\|U_1^1 - \widetilde{U}_1^1\|^*_{1,\infty,[0,r_1]} \int\limits_0^{r_1} \left\|\widetilde{G}^1(r,\xi)\right\|^*_{1,\infty,[0,r_1]} d\xi.
\end{aligned} \tag{3.10}$$

Since

$$\int\limits_{r_{j-2+\alpha}}^{r_{j-1+\alpha}} \left|\widetilde{G}^{j-1+\alpha}(r,\xi)\right| d\xi = \frac{V_1^{j-1+\alpha}(r)}{V_1^{j-1+\alpha}(r_{j-1+\alpha})} \int\limits_{r}^{r_{j-1+\alpha}} V_2^{j-2+\alpha}(\xi)\xi d\xi +$$

$$+ \frac{V_2^{j-2+\alpha}(r)}{V_1^{j-1+\alpha}(r_{j-1+\alpha})} \int\limits_{r_{j-2+\alpha}}^{r} V_1^{j-1+\alpha}(\xi)\xi d\xi =$$

$$= \frac{V_1^{j-1+\alpha}(r)}{V_1^{j-1+\alpha}(r_{j-1+\alpha})} \left[V_2^{j-2+\alpha}(\xi)\frac{\xi^2}{2}\Big|_r^{r_{j-1+\alpha}} + \int\limits_{r}^{r_{j-1+\alpha}} \frac{\xi^2}{2}\frac{1}{\xi k(\xi)} d\xi \right] +$$

$$+ \frac{V_2^{j-2+\alpha}(r)}{V_1^{j-1+\alpha}(r_{j-1+\alpha})} \left[V_1^{j-1+\alpha}(\xi)\frac{\xi^2}{2}\Big|_{r_{j-2+\alpha}}^{r} - \int\limits_{r_{j-2+\alpha}}^{r} \frac{\xi^2}{2}\frac{1}{\xi k(\xi)} d\xi \right] =$$

$$= \frac{V_1^{j-1+\alpha}(r)}{V_1^{j-1+\alpha}(r_{j-1+\alpha})} \left[-V_2^{j-2+\alpha}(r)\frac{r^2}{2} + \frac{1}{2}\int\limits_{r}^{r_{j-1+\alpha}} \frac{\xi}{k(\xi)} d\xi \right] +$$

$$+ \frac{V_2^{j-2+\alpha}(r)}{V_1^{j-1+\alpha}(r_{j-1+\alpha})} \left[V_1^{j-1+\alpha}(r)\frac{r^2}{2} - \frac{1}{2}\int\limits_{r_{j-2+\alpha}}^{r} \frac{\xi}{k(\xi)} d\xi \right] \le$$

$$\le \frac{V_1^{j-1+\alpha}(r)}{2V_1^{j-1+\alpha}(r_{j-1+\alpha})c_1}\frac{\xi^2}{2}\Big|_r^{r_{j-1+\alpha}} - \frac{V_2^{j-2+\alpha}(r)}{2V_1^{j-1+\alpha}(r_{j-1+\alpha})c_2}\frac{\xi^2}{2}\Big|_{r_{j-2+\alpha}}^{r} =$$

$$= \frac{V_1^{j-1+\alpha}(r)}{4V_1^{j-1+\alpha}(r_{j-1+\alpha})c_1}(r_{j-1+\alpha}^2 - r^2) - \frac{V_2^{j-2+\alpha}(r)}{4V_1^{j-1+\alpha}(r_{j-1+\alpha})c_2}(r^2 - r_{j-2+\alpha}^2) \le$$

$$\le \frac{(R^2-r^2)}{4c_1} \le \frac{R^2}{4c_1},$$

$$\int\limits_{r_{j-2+\alpha}}^{r_{j-1+\alpha}} \left| rk(r)\frac{\partial \widetilde{G}^{j-1+\alpha}}{\partial r} \right| d\xi = \frac{1}{V_1^{j-1+\alpha}(r_{j-1+\alpha})} \int\limits_{r}^{r_{j-1+\alpha}} \xi V_2^{j-2+\alpha}(\xi) d\xi +$$

$$+ \frac{1}{V_1^{j-1+\alpha}(r_{j-1+\alpha})} \int\limits_{r_{j-2+\alpha}}^{r} \xi V_1^{j-1+\alpha}(\xi) d\xi \le$$

$$\le \frac{V_2^{j-2+\alpha}(r)}{V_1^{j-1+\alpha}(r_{j-1+\alpha})}\frac{\xi^2}{2}\Big|_r^{r_{j-1+\alpha}} + \frac{V_1^{j-1+\alpha}(r)}{V_1^{j-1+\alpha}(r_{j-1+\alpha})}\frac{\xi^2}{2}\Big|_{r_{j-2+\alpha}}^{r} =$$

$$= \frac{V_2^{j-2+\alpha}(r)}{2V_1^{j-1+\alpha}(r_{j-1+\alpha})}(r_{j-1+\alpha}^2 - r^2) + \frac{V_1^{j-1+\alpha}(r)}{2V_1^{j-1+\alpha}(r_{j-1+\alpha})}(r^2 - r_{j-2+\alpha}^2) \le$$

$$\le \frac{V_2^{j-2+\alpha}(r) + V_1^{j-1+\alpha}(r)}{2V_1^{j-1+\alpha}(r_{j-1+\alpha})} R^2 = \frac{R^2}{2},$$

$$\int\limits_0^{r_1} |\widetilde{G}^1(r,\xi)| d\xi = V_2^0(r)\int\limits_0^r \xi d\xi + \int\limits_r^{r_1} \xi V_2^0(\xi) d\xi = V_2^0(r)\frac{\xi^2}{2}\Big|_0^r + V_2^0(\xi)\frac{\xi^2}{2}\Big|_r^{r_1} +$$

$$+ \frac{1}{2}\int\limits_r^{r_1} \frac{\xi^2}{\xi k(\xi)} d\xi = \frac{1}{2}\int\limits_r^{r_1} \frac{\xi d\xi}{k(\xi)} \le \frac{1}{2c_1}\int\limits_r^{r_1} \xi d\xi = \frac{{r_1}^2 - r^2}{4c_1} \le \frac{R^2}{4c_1},$$

$$\int\limits_0^{r_1} \left| rk(r)\frac{\partial \widetilde{G}^1}{\partial r} \right| d\xi = \int\limits_0^r \xi d\xi = \frac{r^2}{2} \le \frac{R^2}{2},$$

then

$$\int\limits_{r_{j-2+\alpha}}^{r_{j-1+\alpha}} \|\widetilde{G}^{j-1+\alpha}(r,\xi)\|^*_{1,\infty,[r_{j-2+\alpha},r_{j-1+\alpha}]}d\xi \le \frac{R^2}{4c_1}\max(1,2c_1), \tag{3.11}$$

$$\int\limits_{0}^{r_1} \|\widetilde{G}^{1}(r,\xi)\|^*_{1,\infty,[0,r_1]}d\xi \le \frac{R^2}{4c_1}\max(1,2c_1). \tag{3.12}$$

So, due to (3.9), (3.10), (3.11), (3.12) we have

$$\|\Im^j_\alpha(r,u,U^j_\alpha) - \Im^j_\alpha(r,u,\widetilde{U}^j_\alpha)\|^*_{1,\infty,[r_{j-2+\alpha},r_{j-1+\alpha}]} \le \frac{LR^2}{4c_1}\max(1,2c_1) = \\ = q\|U^j_\alpha - \widetilde{U}^j_\alpha\|^*_{1,\infty,[r_{j-2+\alpha},r_{j-1+\alpha}]},$$

$$\|\Im^1_1(r,u,U^1_1) - \Im^1_1(r,u,\widetilde{U}^1_1)\|^*_{1,\infty,[0,r_1]} \le q\|U^1_1 - \widetilde{U}^1_1\|^*_{1,\infty,[0,r_1]}.$$

Thus, for operators (3.7), (3.8) on the sets $\Omega([r_{j-2+\alpha},r_{j-1+\alpha}],\rho)$ and $\Omega([0,r_1],\rho)$, all conditions of the principle of contraction mapping are fulfilled and, hence, problems (3.1), (3.2) have a unique solution. □

Theorem 3.2 *Suppose that the assumptions of Theorem 2.1 are satisfied. Then, for problem (1.1), there exists an ETDS of the form*

$$\begin{aligned} &\frac{1}{r_j}(au_{\bar r})_{\hat r,j} = -\hat T^{r_j}(f(\xi,u(\xi))), \quad j=2,3,\dots,N-1,\\ &a_2u_{r,1}/\hbar_1 = -\hat T^{r_1}(f(\xi,u(\xi))), \quad u_N=\mu_2, \end{aligned} \tag{3.13}$$

the solution of which is the projection of the solution to problem (1.1) onto the grid $\hat\omega_h$, where

$$u_{\bar r,j} = \frac{u_j-u_{j-1}}{h_j}, \quad u_{r,j} = \frac{u_{j+1}-u_j}{h_{j+1}}, \quad u_{\hat r,j} = \frac{u_{j+1}-u_j}{\hbar_j}, \quad \hbar_j = \frac{h_j+h_{j+1}}{2},$$

$$a_j = a(r_j) = \left[\frac{1}{h_j}V^j_1(r_j)\right]^{-1}, \tag{3.14}$$

$$\hat T^{r_j}(w(\xi)) = \left[\hbar_j r_j V^j_1(r_j)\right]^{-1}\int\limits_{r_{j-1}}^{r_j}\xi V^j_1(\xi)w(\xi)d\xi + \left[\hbar_j r_j V^j_2(r_j)\right]^{-1}\int\limits_{r_j}^{r_{j+1}}\xi V^j_2(\xi)w(\xi)d\xi,$$

$$\hat T^{r_1}(w(\xi)) = \hbar_1^{-1}\int\limits_0^{r_1}\xi w(\xi)d\xi + \left[\hbar_1 V^1_2(r_1)\right]^{-1}\int\limits_{r_1}^{r_2}\xi V^1_2(\xi)w(\xi)d\xi,$$

$$V^j_1(r) = \int\limits_{r_{j-1}}^{r}\frac{dt}{tk(t)}, \quad V^j_2(r) = \int\limits_{r}^{r_{j+1}}\frac{dt}{tk(t)}, \quad j=1,2,\dots,N-1,$$

and the function $u(r)$ on the right-hand side of (3.13) is defined by formula (3.3) and only depends on $u(r_j),\ j=1,2,\dots,N$.

Proof. On the intervals $(0,r_2),(r_{j-1},r_{j+1}), j=2,3,\dots,N-1$ we consider the following boundary value problems accordingly

$$\begin{aligned} &\frac{1}{r}\frac{d}{dr}\left[rk(r)\frac{du}{dr}\right] = -f(r,u), \quad r\in(0,r_2),\\ &\lim_{r\to0} rk(r)\frac{du}{dr} = 0, \quad u(r_2)=u_2, \end{aligned} \tag{3.15}$$

$$\begin{aligned} &\frac{1}{r}\frac{d}{dr}\left[rk(r)\frac{du}{dr}\right] = -f(r,u), \quad r\in(r_{j-1},r_{j+1}),\\ &u(r_{j-1})=u_{j-1}, \quad u(r_{j+1})=u_{j+1}. \end{aligned} \tag{3.16}$$

Green's functions for problems (3.15),(3.16) have the forms

$$G^1(r,\xi) = \begin{cases} \xi V_2^1(\xi), & 0 \le r \le \xi, \\ \xi V_2^1(r), & \xi \le r \le r_2, \end{cases}$$

$$G^j(r,\xi) = \begin{cases} \dfrac{V_1^j(r)V_2^j(\xi)\xi}{V_1^j(r_{j+1})}, & r_{j-1} \le r \le \xi, \\ \dfrac{V_1^j(\xi)V_2^j(r)\xi}{V_1^j(r_{j+1})}, & \xi \le r \le r_{j+1}, \end{cases} \qquad j = 2, 3, \dots, N-1.$$

Let us construct an ETDS. With this purpose we write the obvious integrals consequences (3.15), (3.16) and obtain

$$\int_0^{r_2} G^1(r,\xi)\frac{1}{\xi}\frac{d}{d\xi}\left[\xi k(\xi)\frac{du}{d\xi}\right]d\xi = -\int_0^{r_2} G^1(r,\xi)f(\xi,u)d\xi, \tag{3.17}$$

$$\int_{r_{j-1}}^{r_{j+1}} G^j(r,\xi)\frac{1}{\xi}\frac{d}{d\xi}\left[\xi k(\xi)\frac{du}{d\xi}\right]d\xi = -\int_{r_{j-1}}^{r_{j+1}} G^j(r,\xi)f(\xi,u)d\xi. \tag{3.18}$$

Taking into account the boundary conditions (3.15), (3.16) and having integrated by parts the left-side of equalities (3.17), (3.18), we obtain

$$\int_0^{r_2} G^1(r,\xi)\frac{1}{\xi}\frac{d}{d\xi}\left[\xi k(\xi)\frac{du}{d\xi}\right]d\xi = V_2^1(r)\int_0^{r}\frac{d}{d\xi}\left[\xi k(\xi)\frac{du}{d\xi}\right]d\xi +$$

$$+\int_r^{r_2} V_2^1(\xi)\frac{d}{d\xi}\left[\xi k(\xi)\frac{du}{d\xi}\right]d\xi = V_2^1(r)rk(r)\frac{du}{dr} + V_2^1(\xi)\xi k(\xi)\frac{du}{d\xi}\bigg|_r^{r_2} + \int_r^{r_2}\frac{du}{d\xi}d\xi =$$

$$= u(r_2) - u(r),$$

$$\int_{r_{j-1}}^{r_{j+1}} G^j(r,\xi)\frac{1}{\xi}\frac{d}{d\xi}\left[\xi k(\xi)\frac{du}{d\xi}\right]d\xi = \frac{V_2^j(r)}{V_1^j(r_{j+1})}\int_{r_{j-1}}^{r}\xi V_1^j(\xi)\frac{1}{\xi}\frac{d}{d\xi}\left[\xi k(\xi)\frac{du}{d\xi}\right]d\xi +$$

$$+\frac{V_1^j(r)}{V_1^j(r_{j+1})}\int_r^{r_{j+1}}\xi V_2^j(\xi)\frac{1}{\xi}\frac{d}{d\xi}\left[\xi k(\xi)\frac{du}{d\xi}\right]d\xi =$$

$$=\frac{V_2^j(r)}{V_1^j(r_{j+1})}\left[V_1^j(\xi)\xi k(\xi)\frac{du}{d\xi}\bigg|_{r_{j-1}}^{r} - \int_{r_{j-1}}^{r}\frac{du}{d\xi}d\xi\right] +$$

$$+\frac{V_1^j(r)}{V_1^j(r_{j+1})}\left[V_2^j(\xi)\xi k(\xi)\frac{du}{d\xi}\bigg|_r^{r_{j+1}} - \int_r^{r_{j+1}}\frac{du}{d\xi}d\xi\right] =$$

$$=\frac{V_1^j(r)}{V_1^j(r_{j+1})}u(r_{j+1}) - u(r) + \frac{V_2^j(r)}{V_1^j(r_{j+1})}u(r_{j-1}).$$

Thus

$$u(r_2) - u(r) = -\int_r^{r_2}\xi V_2^1(\xi)f(\xi,u)d\xi - V_2^1(r)\int_0^{r}\xi f(\xi,u)d\xi, \tag{3.19}$$

$$
\begin{gathered}
\frac{V_1^j(r)}{V_1^j(r_{j+1})}u(r_{j+1}) - u(r) + \frac{V_2^j(r)}{V_1^j(r_{j+1})}u(r_{j-1}) = \\
= -\frac{V_2^j(rj)}{V_1^j(r_{j+1})}\int\limits_{r_{j-1}}^{r} \xi V_1^j(\xi)f(\xi,u)d\xi - \frac{V_1^j(r)}{V_1^j(r_{j+1})}\int\limits_{r}^{r_{j+1}} \xi V_2^j(\xi)f(\xi,u)d\xi.
\end{gathered}
\tag{3.20}
$$

Having multiplied the equality (3.19) at $r = r_1$ by $\frac{1}{V_2^1(r_1)}$, and the equality (3.20) at $r = r_j$ by $\frac{V_1^j(r_{j+1})}{\hbar_j r_j V_1^j(r_j)V_2^j(r_j)}$, we obtain

$$
\begin{gathered}
\frac{u_2 - u_1}{\hbar_1 V_2^1(r_1)} = -\hat{T}^{r_1}(f(\xi,u(\xi))), \\
\frac{1}{\hbar_j r_j}\left[\frac{u_{j+1} - u_j}{V_2^j(r_j)} - \frac{u_j - u_{j-1}}{V_1^j(r_j)}\right] = -\hat{T}^{r_j}(f(\xi,u(\xi))), \quad j = 2,3,\dots,N-1.
\end{gathered}
\tag{3.21}
$$

Then by virtue of (3.21) and taking into account the equality $V_2^j(r_j) = V_1^{j+1}(r_{j+1})$ we obtain (3.13). □

The existence of a solution to the ETDS (3.13) is proved in Theorem 3.2. The following lemma establishes the uniqueness.

Lemma 3.3 *Suppose that the assumptions of Theorem 3.2 are satisfied. Then* $\exists h_0 > 0$*, such that for* $|h| \le h_0$ *ETDS (3.13) has a unique solution* $(u_j)_{j=0}^N \in \Omega(\widehat{\overline{\omega}}_h, \rho)$*, which can be obtained by the method of successive approximations:*

$$
\begin{gathered}
\frac{1}{r_j}(au_{\bar{r}}^{(n)})_{\hat{r},j} = -\hat{T}^{r_j}(f(\xi,u^{(n-1)}(\xi))), \quad j = 2,3,\dots,N-1, \\
a_2 u_{r,1}^{(n)}/\hbar_1 = -\hat{T}^{r_1}(f(\xi,u^{(n-1)}(\xi))), \quad u_N^{(n)} = \mu_2,
\end{gathered}
\tag{3.22}
$$

$$
\begin{gathered}
u^{(n)}(r) = Y_\alpha^j(r,u^{(n)}), \quad r \in [r_{j-2+\alpha}, r_{j-1+\alpha}], \\
j = 3-\alpha, 4-\alpha, \dots, N+1-\alpha, \quad \alpha = 1,2, \\
u^{(n)}(r) = Y_1^1(r,u^{(n)}), \quad r \in [0,r_1], \quad u^{(0)}(r) = \mu_2
\end{gathered}
$$

with the error estimate

$$
\begin{gathered}
\left\|u^{(n)} - u\right\|_{1,\infty,\widehat{\omega}_h}^* = \\
= \max\left\{\left\|u^{(n)} - u\right\|_{0,\infty,\widehat{\omega}_h}, \left\|rk(r)\frac{du^{(n)}}{dr} - rk(r)\frac{du}{dr}\right\|_{0,\infty,\widehat{\omega}_h}\right\} \le Mq_1^n,
\end{gathered}
\tag{3.23}
$$

where $q_1 = q + M_1|h| < 1, M, M_1$ *are constants.*

Proof. Provided the scheme (3.13) is exact, its solution $\forall r \in \widehat{\overline{\omega}}_h$ can be written in the form

$$
\begin{gathered}
u(r) = \mathrm{Re}\ _h(r,(u_j)_{j=0}^N) = \int\limits_0^R G(r,\xi)f(\xi,u(\xi))d\xi + u^{(0)}(r) = \\
= \int\limits_0^{r_1} G(r,\xi)f(\xi,u(\xi))d\xi + \sum_{j=2}^{N}\int\limits_{r_{j-1}}^{r_j} G(r,\xi)f(\xi,u(\xi))d\xi + u^{(0)}(r),
\end{gathered}
\tag{3.24}
$$

where

$$
\begin{aligned}
&u(\xi) = Y_1^1(\xi,u), \quad \xi \in [0,r_1], \\
&u(\xi) = Y_\alpha^j(\xi,u), \quad \xi \in [r_{j-2+\alpha}, r_{j-1+\alpha}], \quad j = 3-\alpha, 4-\alpha, \dots, N+1-\alpha, \quad \alpha = 1,2.
\end{aligned}
$$

Let's examine the properties of the operator $\mathrm{Re}_h(r, (u_j)_{j=0}^N)$. The operator (3.24) maps the set $\Omega(\widehat{\overline{\omega}}_h, \rho)$ onto itself. Actually, let $(v_j)_{j=0}^N \in \Omega(\widehat{\overline{\omega}}_h, \rho)$. Then we obtain

$$v(r) = Y_1^1(r, v) \in \Omega([0, r_1]),$$
$$v(r) = Y_\alpha^j(r, v) \in \Omega([r_{j-2+\alpha}, r_{j-1+\alpha}]), \quad j = 3-\alpha, 4-\alpha, \dots, N+1-\alpha, \quad \alpha = 1, 2,$$

$$\left\| \mathrm{Re}_h(r, (v_j)_{j=0}^N) - u^{(0)}(r) \right\|_{1,\infty,\widehat{\omega}_h}^* \le K \left\| \int_0^R G(r,\xi) d\xi \right\|_{1,\infty,\widehat{\omega}_h}^* \le$$
$$\le \frac{KR^2}{4c_1} \max(1, 2c_1) = \rho, \quad \forall (v_j)_{j=0}^N \in \Omega(\widehat{\overline{\omega}}_h, \rho).$$

Moreover

$$\begin{aligned}
&\left\| \mathrm{Re}_h(r, (u_j)_{j=0}^N) - \mathrm{Re}_h(r, (v_j)_{j=0}^N) \right\|_{1,\infty,\widehat{\omega}_h}^* \le \\
&\le \int_0^R \|G(r,\xi)\|_{1,\infty,[0,R]}^* \left| f(\xi, u(\xi)) - f(\xi, v(\xi)) \right| d\xi \le \\
&\le L \int_0^R \|G(r,\xi)\|_{1,\infty,[0,R]}^* \left| u(\xi) - v(\xi) \right| d\xi \le \\
&\le L \|u - v\|_{1,\infty,[0,R]}^* \int_0^R \|G(r,\xi)\|_{1,\infty,[0,R]}^* d\xi \le \\
&\le \frac{LR^2}{4c_1} \max(1, 2c_1) \|u - v\|_{1,\infty,[0,R]}^* = q \|u - v\|_{1,\infty,[0,R]}^* \\
&\forall (u_j)_{j=0}^N, (v_j)_{j=0}^N \in \Omega(\widehat{\overline{\omega}}_h, \rho).
\end{aligned} \tag{3.25}$$

Let us show that

$$\|u - v\|_{1,\infty,[0,R]} \le (1 + M|h|) \|u - v\|_{1,\infty,\widehat{\omega}_h}^* . \tag{3.26}$$

We consider the following boundary value problems

$$\frac{1}{r} \frac{d}{dr} \left[r k(r) \frac{du}{dr} \right] = -f(r, u), \quad 0 < r < r_1,$$
$$\lim_{r \to 0} r k(r) \frac{du}{dr} = 0, \quad u(r_1) = u_1,$$

$$\frac{1}{r} \frac{d}{dr} \left[r k(r) \frac{du}{dr} \right] = -f(r, u), \quad r_{j-1} < r < r_j,$$
$$u(r_{j-1}) = u_{j-1}, \quad u(r_j) = u_j, \quad j = 2, 3, \dots, N.$$

The solution to this problems we write in the form

$$u(r) = \int_0^{r_1} \widetilde{G}^1(r,\xi) f(\xi, u(\xi)) d\xi + u_1, \quad r \in [0, r_1],$$
$$u(r) = \int_{r_{j-1}}^{r_j} \widetilde{G}^j(r,\xi) f(\xi, u(\xi)) d\xi + \hat{u}(r),$$
$$\hat{u}(r) = \frac{u(r_j) V_1^j(r) + u(r_{j-1}) V_2^{j-1}(r)}{V_1^j(r_j)}, \quad r \in [r_{j-1}, r_j], \quad j = 2, 3, \dots, N,$$

where the Green's functions $\widetilde{G}^1(r,\xi), \widetilde{G}^j(r,\xi)$ are defined by formulas (3.5), (3.6). By virtue of the Lipschitz condition

$$\|u-v\|_{1,\infty,[0,r_1]} \le |u_1-v_1| + L\,\|u-v\|_{1,\infty,[0,r_1]} \left\| \int\limits_0^{r_1} \widetilde{G}^1(r,\xi)d\xi \right\|_{1,\infty,[0,r_1]} \le$$
$$\le \|u-v\|_{0,\infty,\widehat{\omega}_h} + M_2|h|\,\|u-v\|_{1,\infty,[0,r_1]},$$

$$\|u-v\|_{1,\infty,[r_{j-1},r_j]} \le \|\hat{u}-\hat{v}\|_{1,\infty,[r_{j-1},r_j]} +$$
$$+ L\,\|u-v\|_{1,\infty,[r_{j-1},r_j]} \left\| \int\limits_{r_{j-1}}^{r_j} \widetilde{G}^j(r,\xi)d\xi \right\|_{1,\infty,[r_{j-1},r_j]} \le$$
$$\le \|\hat{u}-\hat{v}\|_{1,\infty,[r_{j-1},r_j]} + M_3|h|\,\|u-v\|_{1,\infty,[r_{j-1},r_j]}, \quad j=2,3,\ldots,N.$$

As far as

$$\|\hat{u}-\hat{v}\|_{0,\infty,[r_{j-1},r_j]} = \max_{r\in[r_{j-1},r_j]} \left\{ \frac{|u_j-v_j|V_1^j(r) + |u_{j-1}-v_{j-1}|V_2^{j-1}(r)}{V_1^j(r_j)} \right\} \le$$
$$\le \|u-v\|_{0,\infty,\widehat{\omega}_h}, \quad j=2,3,\ldots,N,$$

$$\left\| rk(r)\frac{d\hat{u}}{dr} - rk(r)\frac{d\hat{v}}{dr} \right\|_{0,\infty,[r_{j-1},r_j]} = \max_{r\in[r_{j-1},r_j]} \frac{h_j\,|u_{\overline{r},j}-v_{\overline{r},j}|}{V_1^j(r_j)} \le$$
$$\le (1+M_4|h|) \left\| rk(r)\frac{du}{dr} - rk(r)\frac{dv}{dr} \right\|_{0,\infty,\widehat{\omega}_h}, \quad j=2,3,\ldots,N,$$

we get estimates

$$\|u-v\|_{1,\infty,[0,r_1]} \le \|u-v\|^*_{1,\infty,\widehat{\omega}_h} + M_2|h|\,\|u-v\|_{1,\infty,[0,r_1]} \le$$
$$\le \frac{1}{1-M_2|h|}\,\|u-v\|^*_{1,\infty,\widehat{\omega}_h} \le (1+M|h|)\,\|u-v\|^*_{1,\infty,\widehat{\omega}_h},$$

$$\|u-v\|_{1,\infty,[r_{j-1},r_j]} \le \frac{1+M_4|h|}{1-M_3|h|}\,\|u-v\|^*_{1,\infty,\widehat{\omega}_h} \le$$
$$\le (1+M|h|)\,\|u-v\|^*_{1,\infty,\widehat{\omega}_h}, \; j=2,3,\ldots,N$$

that are followed by the inequality (3.26).

Taking into account (3.26) and in view of inequality (3.25), we obtain

$$\left\| \mathrm{Re}_{\,h}(r,(u_j)_{j=0}^N) - \mathrm{Re}_{\,h}(r,(v_j)_{j=0}^N) \right\|^*_{1,\infty,\widehat{\omega}_h} \le (q+M|h|)\|u-v\|^*_{1,\infty,\widehat{\omega}_h} =$$
$$= q_1\|u-v\|^*_{1,\infty,\widehat{\omega}_h}.$$

By virtue of (2.4) we have $q_1<1$ for a sufficiently small h_0 and operator (3.24)

$$\forall (u_j)_{j=0}^N, (v_j)_{j=0}^N \in \Omega(\widehat{\overline{\omega}}_h,\rho)$$

performs a contractive mapping.

Thus, according to the principle of contractive mappings for a sufficiency small h_0 ETDS (3.13) has a unique solution, which can be obtained by the method of successive approximations (3.22) with the error estimate (3.23). □

The existence and uniqueness of a solution to the auxiliary initial value problems in the following lemma are proved.

Lemma 3.4 *Suppose that assumption (2.1) is satisfied and there exist a constant $\Delta > 0$ such that assumptions (2.2) and (2.3) are satisfied in $\Omega([0,R],\rho+\Delta)$. Then $\exists h_0 > 0$ such that for $|h| \le h_0$ and $\forall (v_j)_{j=0}^N \in \Omega(\widehat{\overline{\omega}}_h, \rho)$ the problems*

$$\begin{gathered}\frac{1}{r}\frac{d}{dr}\left[rk(r)\frac{dY_1^1(r,v)}{dr}\right] = -f(r, Y_1^1(r,v)), \quad 0 < r < r_1,\\ Y_1^1(r_0,v) = v_0, \quad \lim_{r\to r_0} rk(r)\frac{dY_1^1(r,v)}{dr} = 0,\end{gathered} \tag{3.27}$$

$$\begin{gathered}\frac{1}{r}\frac{d}{dr}\left[rk(r)\frac{dY_\alpha^j(r,v)}{dr}\right] = -f(r, Y_\alpha^j(r,v)), \quad r_{j-2+\alpha} < r < r_{j-1+\alpha},\\ Y_\alpha^j(r_{j+(-1)^\alpha}, v) = v(r_{j+(-1)^\alpha}), \quad rk(r)\frac{Y_\alpha^j(r,v)}{dr}\bigg|_{r=r_{j+(-1)^\alpha}} = rk(r)\frac{dv}{dr}\bigg|_{r=r_{j+(-1)^\alpha}},\\ j = 3-\alpha, 4-\alpha, \dots, N+1-\alpha, \quad \alpha = 1,2\end{gathered} \tag{3.28}$$

will have a unique solution.

Proof. Since the problems (3.27), (3.28) are equivalent to the operator equations

$$U_1^1(r) = \mathrm{Re}\ {}_1^1(r,v,U_1^1) = \int_0^r \left[V_2^0(r) - V_2^0(\xi)\right]\xi f(\xi, U_1^1(\xi))d\xi + v_0, \quad r \in [0, r_1]$$

$$\begin{aligned}U_\alpha^j(r) &= \mathrm{Re}\ {}_\alpha^j(r,v,U_\alpha^j) =\\ &= \frac{1}{V_\alpha^j(r_j)}\int_{r_{j+(-1)^\alpha}}^r \left[V_1^{j-1+\alpha}(\xi)V_2^{j-2+\alpha}(r) - V_1^{j-1+\alpha}(r)V_2^{j-2+\alpha}(\xi)\right]\xi f(\xi, U_\alpha^j(\xi))d\xi +\\ &+ v_{j+(-1)^\alpha} + (-1)^{\alpha+1} rk(r)\frac{dv}{dr}\bigg|_{r=r_{j+(-1)^\alpha}} V_\alpha^j(r), \quad r \in [r_{j-2+\alpha}, r_{j-1+\alpha}],\\ j &= 3-\alpha, 4-\alpha, \dots, N+1-\alpha, \quad \alpha = 1,2\end{aligned}$$

we examine the properties of the operators $\mathrm{Re}\ {}_1^1(r,v,U_1^1)$, $\mathrm{Re}\ {}_\alpha^j(r,v,U_\alpha^j)$, $j = 3-\alpha, 4-\alpha, \dots, N+1-\alpha$, $\alpha = 1,2$. Note that

$$u^{(0)}(r) = \mu_2 = u_{j+(-1)^\alpha}^{(0)}, \quad rk(r)\frac{du^{(0)}}{dr}\bigg|_{r=r_{j+(-1)^\alpha}} = 0$$

and following inequalities are fulfilled

$$\begin{aligned}&V_1^j(\xi)V_2^{j-1}(r) - V_1^j(r)V_2^{j-1}(\xi) =\\ &= \int_{r_{j-1}}^{\xi}\frac{dt}{tk(t)}\int_r^{r_j}\frac{dt}{tk(t)} - \left(\int_{r_{j-1}}^{\xi}\frac{dt}{tk(t)} + \int_{\xi}^{r}\frac{dt}{tk(t)}\right)\left(\int_{\xi}^{r}\frac{dt}{tk(t)} + \int_r^{r_j}\frac{dt}{tk(t)}\right) =\\ &= -\int_{r_{j-1}}^{\xi}\frac{dt}{tk(t)}\int_{\xi}^{r}\frac{dt}{tk(t)} - \int_{\xi}^{r}\frac{dt}{tk(t)}\int_{\xi}^{r}\frac{dt}{tk(t)} - \int_{\xi}^{r}\frac{dt}{tk(t)}\int_{r}^{r_j}\frac{dt}{tk(t)} \le 0,\end{aligned}$$

$$\begin{aligned}&V_1^{j+1}(\xi)V_2^{j}(r) - V_1^{j+1}(r)V_2^{j}(\xi) =\\ &= \left(\int_{r_j}^{r}\frac{dt}{tk(t)} + \int_r^{\xi}\frac{dt}{tk(t)}\right)\left(\int_r^{\xi}\frac{dt}{tk(t)} + \int_{\xi}^{r_{j+1}}\frac{dt}{tk(t)}\right) - \int_{r_j}^{r}\frac{dt}{tk(t)}\int_{\xi}^{r_{j+1}}\frac{dt}{tk(t)} =\\ &= \int_{r_j}^{r}\frac{dt}{tk(t)}\int_r^{\xi}\frac{dt}{tk(t)} + \int_r^{\xi}\frac{dt}{tk(t)}\int_r^{\xi}\frac{dt}{tk(t)} + \int_r^{\xi}\frac{dt}{tk(t)}\int_{\xi}^{r_{j+1}}\frac{dt}{tk(t)} \ge 0.\end{aligned}$$

Let $U_1^1(r) \in \Omega([r_0, r_1], \rho+\Delta)$, $U_\alpha^j(r) \in \Omega([r_{j-2+\alpha}, r_{j-1+\alpha}], \rho+\Delta)$, then

$$\|\mathrm{Re}\ {}_1^1(r,v,U_1^1) - u^{(0)}\|_{1,\infty,[r_0,r_1]} \le \rho + K\int\limits_0^r \|V_2^0(r) - V_2^0(\xi)\|^*_{1,\infty,[r_0,r_1]}\xi d\xi,$$

$$\|\mathrm{Re}\ {}_\alpha^j(r,v,U_\alpha^j) - u^{(0)}\|_{1,\infty,[r_{j-2+\alpha},r_{j-1+\alpha}]} \le \rho +$$
$$+\frac{K(-1)^{(\alpha+1)}}{V_\alpha^j(r_j)}\int\limits_{r_{j+(-1)^\alpha}}^r \|V_1^{j-1+\alpha}(\xi)V_2^{j-2+\alpha}(r) - V_1^{j-1+\alpha}(r)V_2^{j-2+\alpha}(\xi)\|^*_{1,\infty,[r_{j-2+\alpha},r_{j-1+\alpha}]}\xi d\xi.$$

Since

$$\int\limits_0^r \left|V_2^0(r) - V_2^0(\xi)\right|\xi d\xi = \int\limits_0^r [V_2^0(\xi) - V_2^0(r)]\xi d\xi =$$
$$= [V_2^0(\xi) - V_2^0(r)]\frac{\xi^2}{2}\Big|_0^r + \frac{1}{2}\int\limits_0^r \frac{\xi d\xi}{k(\xi)} \le \frac{r^2}{4c_1} \le M_1|h|^2,$$

$$\int\limits_0^r \left|rk(r)\frac{dV_2^0(r)}{dr}\right|\xi d\xi = \int\limits_0^r \xi d\xi = \frac{r^2}{2} \le M_2|h|^2,$$

$$\frac{(-1)^{\alpha+1}}{V_\alpha^j(r_j)}\int\limits_{r_{j+(-1)^\alpha}}^r \left|V_1^{j-1+\alpha}(\xi)V_2^{j-2+\alpha}(r) - V_1^{j-1+\alpha}(r)V_2^{j-2+\alpha}(\xi)\right|\xi d\xi =$$
$$= \frac{1}{V_\alpha^j(r_j)}\int\limits_{r_{j+(-1)^\alpha}}^r \left[V_1^{j-1+\alpha}(r)V_2^{j-2+\alpha}(\xi) - V_1^{j-1+\alpha}(\xi)V_2^{j-2+\alpha}(r)\right]\xi d\xi =$$
$$= \frac{1}{V_\alpha^j(r_j)}\left[V_1^{j-1+\alpha}(r)V_2^{j-2+\alpha}(\xi) - V_1^{j-1+\alpha}(\xi)V_2^{j-2+\alpha}(r)\right]\frac{\xi^2}{2}\Big|_{r_{j+(-1)^\alpha}}^r +$$
$$+\int\limits_{r_{j+(-1)^\alpha}}^r \frac{\xi}{2k(\xi)}d\xi = (-1)^\alpha V_\alpha^j(r)\frac{r^2_{j+(-1)^\alpha}}{2} + \frac{1}{2}\int\limits_{r_{j+(-1)^\alpha}}^r \frac{\xi}{k(\xi)}d\xi \le M_3|h|,$$

$$\frac{1}{V_\alpha^j(r_j)}\int\limits_{r_{j+(-1)^\alpha}}^r \left|rk(r)\frac{\partial}{\partial r}\left[V_1^{j-1+\alpha}(\xi)V_2^{j-2+\alpha}(r) - V_1^{j-1+\alpha}(r)V_2^{j-2+\alpha}(\xi)\right]\right|\xi d\xi =$$
$$= \int\limits_{r_{j+(-1)^\alpha}}^r \xi d\xi \le M_4|h|$$

we obtain

$$\int\limits_0^r \|V_2^0(r) - V_2^0(\xi)\|^*_{1,\infty,[0,r_1]}\xi d\xi \le \max\{M_1, M_2\}|h|^2,$$

$$\int\limits_{r_{j+(-1)^\alpha}}^r \|V_1^{j-1+\alpha}(\xi)V_2^{j-2+\alpha}(r) - V_1^{j-1+\alpha}(r)V_2^{j-2+\alpha}(\xi)\|^*_{1,\infty,[r_{j-2+\alpha},r_{j-1+\alpha}]}\xi d\xi \le$$
$$\le \max\{M_3, M_4\}|h|.$$

Thus

$$\|\mathrm{Re}\ {}_1^1(r,v,U_1^1) - u^{(0)}\|_{1,\infty,[0,r_1]} \le \rho + K\max\{M_1,M_2\}|h|^2 = \rho+\Delta,$$
$$\|\mathrm{Re}\ {}_\alpha^j(r,v,U_\alpha^j) - u^{(0)}\|_{1,\infty,[r_{j-2+\alpha},r_{j-1+\alpha}]} \le \rho + K\max\{M_3,M_4\}|h| = \rho+\Delta,$$

i.e., the operators $\text{Re}\,{}_1^1(r,v,U_1^1)$, $\text{Re}\,{}_\alpha^j(r,v,U_\alpha^j), \alpha=1,2$ for $|h|\le h_0$, and $\forall (v_j)_{j=0}^N \in \Omega(\widehat{\overline{\omega}}_h,\rho)$ maps accordingly the sets $\Omega([0,r_1],\rho+\Delta), \Omega([r_{j-2+\alpha},r_{j-1+\alpha}],\rho+\Delta)$ onto itself. Besides

$$\begin{aligned}
&\|\text{Re}\,{}_1^1(r,v,U_1^1)-\text{Re}\,{}_1^1(r,v,\tilde{U}_1^1)\|_{1,\infty,[0,r_1]}\le\\
&\le L\|U_1^1-\widetilde{U}_1^1\|_{1,\infty,[0,r_1]}\int\limits_0^r \|V_2^0(r)-V_2^0(\xi)\|^*_{1,\infty,[r_0,r_1]}\xi d\xi\le\\
&\le L\max\{M_1,M_2\}|h|^2\|U_1^1-\widetilde{U}_1^1\|_{1,\infty,[0,r_1]}\le q\|U_1^1-\widetilde{U}_1^1\|_{1,\infty,[0,r_1]},
\end{aligned}$$

$$\begin{aligned}
&\|\text{Re}\,{}_\alpha^j(r,v,U_\alpha^j)-\text{Re}\,{}_\alpha^j(r,v,\widetilde{U}_\alpha^j)\|_{1,\infty,[r_{j-2+\alpha},r_{j-1+\alpha}]}\le\\
&\le L\|U_\alpha^j-\widetilde{U}_\alpha^j\|_{1,\infty,[r_{j-2+\alpha},r_{j-1+\alpha}]}\times\\
&\times\int\limits_{r_{j+(-1)^\alpha}}^r \|V_1^{j-1+\alpha}(\xi)V_2^{j-2+\alpha}(r)-V_1^{j-1+\alpha}(r)V_2^{j-2+\alpha}(\xi)\|^*_{1,\infty,[r_{j-2+\alpha},r_{j-1+\alpha}]}\xi d\xi\le\\
&\le L\max\{M_3,M_4\}|h|\|U_\alpha^j-\widetilde{U}_\alpha^j\|_{1,\infty,[r_{j-2+\alpha},r_{j-1+\alpha}]}\le\\
&\le q\|U_\alpha^j-\widetilde{U}_\alpha^j\|_{1,\infty,[r_{j-2+\alpha},r_{j-1+\alpha}]}.
\end{aligned}$$

Consequently, the operators $\text{Re}\,{}_1^1(r,v,U_1^1), \text{Re}\,{}_\alpha^j(r,v,U_\alpha^j), \alpha=1,2$ are accordingly a contraction operators on $\Omega([0,r_1],\rho+\Delta), \Omega([r_{j-2+\alpha},r_{j-1+\alpha}],\rho+\Delta)$. Therefore, $\forall (v_j)_{j=0}^N\in\Omega(\widehat{\overline{\omega}}_h,\rho)$ problems (3.27), (3.28) have a unique solution. □

Remark 3.5 From the equation (3.27) we have

$$\frac{d}{dr}\left[k(r)\frac{dY_1^1(r,v)}{dr}\right]+\frac{k(r)}{r}\frac{dY_1^1(r,v)}{dr}=-f(r,Y_1^1(r,v)). \tag{3.29}$$

Since $dY_1^1(r,v)/dr\to 0$ as $r\to 0$, we have

$$\frac{k(r)}{r}\frac{dY_1^1(r,v)}{dr}\to\frac{d}{dr}\left[k(r)\frac{dY_1^1(r,v)}{dr}\right]\quad \text{as } r\to 0$$

and from (3.29) follows identity

$$\frac{d}{dr}\left[k(r)\frac{dY_1^1(r,v)}{dr}\right]\Bigg|_{r=0}=-\frac{1}{2}f(0,Y_1^1(0,v)).$$

So, instead of a initial value problem (3.27) on the interval $(0,r_1)$ the problem can be considered

$$\begin{aligned}
&\frac{d}{dr}\left[k(r)\frac{dY_1^1(r,v)}{dr}\right]=\begin{cases}-f(r,Y_1^1(r,v))-\dfrac{k(r)}{r}\dfrac{dY_1^1(r,v)}{dr}, & r\neq 0,\\ -\dfrac{1}{2}f(0,Y_1^1(0,v)), & r=0,\end{cases}\\
&Y_1^1(0,v)=v_0,\quad k(r)\frac{dY_1^1(r,v)}{dr}\Bigg|_{r=0}=0.
\end{aligned}\tag{3.30}$$

4. Algorithmic implementation of the exact three-point difference scheme

First, we take into account that

$$\int\limits_0^{r_1}\xi f(\xi,u)d\xi=-r_1Z_1^1(r_1,u),$$

$$(-1)^{\alpha+1}\int\limits_{r_{j+(-1)^\alpha}}^{r_j}\xi V_\alpha^j(\xi)f(\xi,u)d\xi=(-1)^\alpha V_\alpha^j(r_j)Z_\alpha^j(r_j,u)+Y_\alpha^j(r_j,u)-u_{j+(-1)^\alpha},$$

where $Y_1^1(r_1,u), Z_1^1(r_1,u)$, $Y_\alpha^j(r_j,u), Z_\alpha^j(r_j,u)$, $\alpha = 1,2$ are solutions of following initial value problems

$$\begin{aligned}
&\frac{dY_1^1(r,u)}{dr} = \frac{Z_1^1(r,u)}{k(r)},\\
&\frac{dZ_1^1(r,u)}{dr} = \begin{cases} -f(r,Y_1^1(r,u)) - \dfrac{Z_1^1(r,u)}{r}, & r \neq 0,\\ -\dfrac{1}{2}f(0,Y_1^1(0,u)), & r = 0,\end{cases} \quad 0 < r < r_1,\\
&Y_1^1(0,u) = u_0, \quad Z_1^1(0,u) = 0,
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
&\frac{dY_\alpha^j(r,u)}{dr} = \frac{Z_\alpha^j(r,u)}{rk(r)}, \quad \frac{dZ_\alpha^j(r,u)}{dr} = -rf(r,Y_\alpha^j(r,u)), \quad r_{j-2+\alpha} < r < r_{j-1+\alpha},\\
&Y_\alpha^j(r_{j+(-1)^\alpha},u) = u_{j+(-1)^\alpha}, \quad Z_\alpha^j(r_{j+(-1)^\alpha},u) = rk(r)\frac{du}{dr}\Big|_{r=r_{j+(-1)^\alpha}},\\
&j = 3-\alpha, 4-\alpha, \ldots, N+1-\alpha, \quad \alpha = 1,2
\end{aligned} \tag{4.2}$$

and $\bar{V}_\alpha^j(r) = (-1)^{\alpha+1}V_\alpha^j(r)$ are solutions of such following initial value problems

$$\begin{aligned}
&\frac{d\bar{V}_\alpha^j(r)}{dr} = \frac{1}{rk(r)}, \quad r_{j-2+\alpha} < r < r_{j-1+\alpha},\\
&\bar{V}_\alpha^j(r_{j+(-1)^\alpha}) = 0, \quad j = 2-\alpha, 3-\alpha, ..., N+1-\alpha, \quad \alpha = 1,2.
\end{aligned} \tag{4.3}$$

Then the ETDS (3.13) can be written in the form

$$\begin{aligned}
&\frac{1}{r_j}(au_{\bar{r}})_{\hat{r},j} = -\varphi(r_j,u), \quad j = 2,3,\ldots,N-1,\\
&a_2u_{r,1}/\hbar_1 = -\varphi(r_1,u), \quad u_N = \mu_2,
\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}
&\varphi(r_1,u) = \hat{T}^{r_1}(f(\xi,u(\xi))) = \frac{1}{\hbar_1}\left[Z_2^1(r_1,u) - r_1Z_1^1(r_1,u) + \frac{Y_2^1(r_1,u)-u_2}{V_2^1(r_1)}\right],\\
&\varphi(r_j,u) = \hat{T}^{r_j}(f(\xi,u(\xi))) =\\
&\quad = \hbar_j^{-1}\sum_{\alpha=1}^{2}(-1)^\alpha\left[Z_\alpha^j(r_j,u) + (-1)^\alpha\frac{Y_\alpha^j(r_j,u)-u_{j+(-1)^\alpha}}{V_\alpha^j(r_j)}\right],\\
&\quad j = 2,3,\ldots,N-1.
\end{aligned} \tag{4.5}$$

So, in order to obtain the ETDS (3.14), (4.4), (4.5) $\forall r_j \in \hat{\omega}_h$, it is necessary to solve the four initial value problems, such as (4.1) or (4.2) and (4.3). If the problems (4.1)–(4.3) solved numerically, it could be possible to develop the truncated TDS.

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