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COMBINING NARIMANOV–MOISEEV' AND LUKOVSKY–MILES' SCHEMES FOR NONLINEAR LIQUID SLOSHING

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Анотація. Огляд презентує колекцію нелінійних асимптотичних модальних рівнянь теорії коливань рідини в баках, вивід яких комбінує варіаційні та асимптотичні методи. Акцент частково зроблено на класифікацію усталених резонансних розв'язків, які отримано за допомогою цих рівнянь.

Abstract. The survey collects asymptotic nonlinear modal equations of liquid sloshing theory whose derivation combines variational (Miles–Lukovsky's) and asymptotic (Narimanov-Moiseev's) methods. A particular emphasis is placed on classification of steady-state resonant solutions obtained by analyzing these equations.

1. Introduction

Modern mainstream in sloshing problems consists of employing the Computational Fluid Dynamics (CFD) associated with spatial-and-time discretization of the corresponding free-boundary problems (Ch. 10 of the book [8] and Rebouillat & Liksonov [42]). These numerical methods effectively solve the time-domain problem with different initial scenarios, but are rather limited for parameter studies of the frequency-domain problem dealing with steady-state regimes appearing on the long-time scale and, therefore, requiring long-time simulations. The CFD has also difficulties in coupling sloshing and structural dynamics. An alternative could be asymptotic *multimodal methods* whose idea consists of reducing the original free-boundary problem to compact systems of nonlinear ordinary differential equations, i.e. the so-called *modal equations*. These modal equations make it possible to solve both frequency and time domain problems as well as to identify chaotic liquid motions (see, [13, 14, 21, 34, 35] and references therein).

Systematic collections of experimental and engineering approaches to liquid sloshing dynamics in axial-symmetric tanks with non-shallow liquid depth (relevant to spacecraft systems) are given by Abramson [1] and Mikishev [32]. Marine and storage tank applications also deal with prismatic (rectangular shape) tanks which are considered by Faltinsen & Timokha [8]. Ibrahim [20] outlines the state-of-the-art for some of other applications. In summary, the engineering practice concentrates on *tank shapes* which are, basically, limited to upright cylindrical tanks of circular (sectored) and rectangular cross-sections, horizontal circular cylindrical tanks, spherical and conical tanks. A collection of asymptotic modal equations for upright cylindrical tanks is exhibited in the present survey for the case when these equations were derived being based on combining Narimanov-Moiseev' asymptotics and Lukovsky-Miles' variational method.

2. Multimodal method

2.1. Free-boundary problem

A mobile rigid tank partly filled by an inviscid incompressible liquid with irrotational flow is considered. Fig. 1 shows liquid domain $Q(t)$ bounded by the free surface $\Sigma(t)$ and the wetted tank surface $S(t)$, an absolute coordinate system $O'x'y'z'$, and non-inertial coordinate system $Oxyz$ rigidly fixed with the tank so that O lies in the geometric center of the mean free surface Σ_0 $(z = 0)$. The $Oxyz$-system moves relatively to $O'x'y'z'$ with the absolute translatory velocity $\mathbf{v}_0$ and the instant angular velocity ω. The gravity potential can be written as $U(x, y, z, t) = -\mathbf{g}\cdot\mathbf{r}'$; $\mathbf{r}' = \mathbf{r}'_0+\mathbf{r}$, where $\mathbf{r}'$ is the radius–vector of a point of the body–liquid system with respect to O', $\mathbf{r}'_0$ is the radius–vector of O with respect to O', $\mathbf{r}$ is the radius–vector with respect to O,

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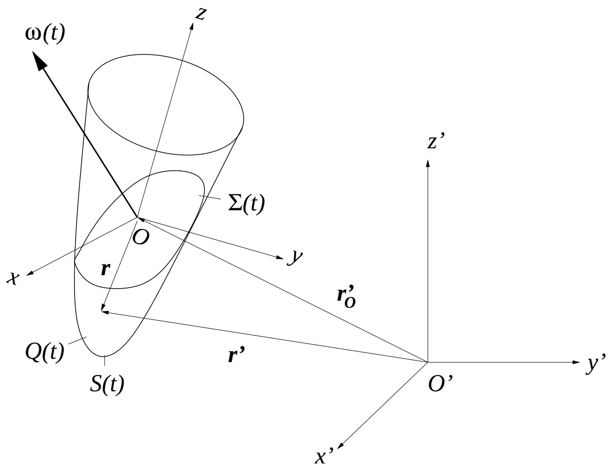

Fig. 1. Sketch of a moving tank. Nomenclature

and $\mathbf{g}$ is the gravity acceleration vector. The absolute velocity potential $\Phi(x,y,z,t)$ ($\mathbf{v}_a = \nabla\Phi$) and the free surface $\Sigma(t)$: $Z(x,y,z,t) = 0$ should be found from the following free-boundary problem [8, 29]

$$\begin{gathered}\Delta\Phi = 0 \quad \text{in } Q(t); \quad \frac{\partial\Phi}{\partial\nu} = \mathbf{v}_0 \cdot \mathbf{n} + \omega \cdot [\mathbf{r} \times \mathbf{n}] \quad \text{on } S(t), \\ \frac{\partial\Phi}{\partial n} = \mathbf{v}_0 \cdot \mathbf{n} + \omega \cdot [\mathbf{r} \times \mathbf{n}] + \frac{\partial Z/\partial t}{|\nabla Z|} \quad \text{on } \Sigma(t),\end{gathered} \tag{2.1a}$$

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2 - \nabla\Phi \cdot (\mathbf{v}_0 + \omega \times \mathbf{r}) + U = 0 \ \text{ on } \Sigma(t), \quad \int_{Q(t)} \mathrm{d}Q = \text{const}, \tag{2.1b}$$

where $\mathbf{n}$ is the outer normal. The free boundary problem (2.1) must be completed by either initial or periodicity conditions. Adding the initial conditions implies the time-domain problem. Using the periodicity conditions for the $T = 2\pi/\sigma$-periodic forcing terms $\mathbf{v}_0(t)$ and $\omega(t)$ corresponds to the frequency-domain problem.

2.2. Modal solution and its limitations

The multimodal methods are based on Fourier-type expressions for the free surface and velocity potential with time-dependent coefficients which are treated as *generalized coordinates* of the mechanical system (2.1). When tank walls are vertical in a neighborhood of $\Sigma(t)$, the modal representation of $\Sigma(t)$ takes the form

$$Z(x,y,z,t) = z - \zeta(x,y,t) = z - \sum_{i=1}^{\infty} \beta_i(t) f_i(x,y). \tag{2.2}$$

Here $f_i(x,y)$ is a complete orthogonal functional set satisfying the volume conservation condition $\int_{\Sigma_0} f_i(x,y)\mathrm{d}x\mathrm{d}y = 0$, where Σ_0 ($z = 0$) is the mean free surface and S_0 is the mean wetted tank surface.

In most general case, the multimodal methods express the velocity potential as

$$\Phi(x,y,z,t) = \mathbf{v}_0 \cdot \mathbf{r} + \omega \cdot \mathbf{\Omega} + \sum_{n=1}^{\infty} R_n(t)\,\varphi_n(x,y,z). \tag{2.3}$$

Here the complete functional set $\{\varphi_n(x,y,z)\}$ satisfies the Laplace equation and the vector–function $\mathbf{\Omega}(x,y,z) = (\Omega_1, \Omega_2, \Omega_3)$ (Ω_k are called the Stokes-Joukowski potentials) is solution of the following Neumann boundary problem

$$\begin{gathered}\Delta\mathbf{\Omega}=0 \quad \text{in} \quad Q(t); \quad \frac{\partial\Omega_1}{\partial n}|_{S(t)+\Sigma(t)}=y\mathbf{n}_3-z\mathbf{n}_2,\\ \frac{\partial\Omega_2}{\partial n}|_{S(t)+\Sigma(t)}=z\mathbf{n}_1-x\mathbf{n}_3; \ \frac{\partial\Omega_3}{\partial n}|_{S(t)+\Sigma(t)}=x\mathbf{n}_2-y\mathbf{n}_1,\end{gathered} \tag{2.4}$$

where $\mathbf{n}=(n_1,n_2,n_3)$. Being defined by (2.4), the Stokes-Joukowski potentials Ω_i are functions of generalized coordinates β_i.

Normally, φ_n and $f_n(x,y)=\varphi_n(x,y,0)$ are the natural sloshing modes defined by the spectral boundary problem

$$\nabla^2\varphi_n=0 \ \text{ in } \ Q_0; \quad \frac{\partial\varphi_n}{\partial n}=0 \ \text{ on } \ S_0; \quad \frac{\partial\varphi_n}{\partial z}=\kappa_n\varphi_n \ \text{ on } \ \Sigma_0; \quad \int_{\Sigma_0}\varphi_n \mathrm{d}y=0, \tag{2.5}$$

where Q_0 is the mean liquid domain. The natural sloshing frequencies are defined by the eigenvalues κ_i via $\sigma_n=\sqrt{g\kappa_n}$.

Substituting the modal solution (2.2) and (2.3) into (2.1) or its variational analogy [3, 8, 26, 29] transforms the free-boundary problem to a set of ordinary differential equations (modal equations) with respect to generalized coordinates β_i and R_n. According to Faltinsen & Timokha [8] there are the following limitations in using the modal solution:

- *Physical:* Original statement (2.1) and modal solution (2.2) and (2.3) neglect damping due to different physical mechanisms including roof impact and breaking waves. The damping can partly been accounted for in modal equations by incorporating additional terms. Ch. 6 in [8] discusses these terms for laminar boundary layer at the wetted tank surface, baffles, and screens. Furthermore, because of the normal free-surface representation (2.2), the modal solution makes it impossible to describe wave profiles modified by roof impact and overturning waves; the tank bottom has to be wet.
- *Geometrical:* Eq. (2.2) requires $\{f_n\}$ to be a Fourier basis whose domain of definition is time-independent and, generally speaking, coincides with Σ_0. Non-vertical walls make it impossible. A non-Cartesian parametrization is needed to revise the modal solution for tanks with non-vertical walls as reported in [16, 27, 29, 30, 39].
- *Numerical:* To be correctly substituted into eq. (2.1), modal solution (2.3) needs a harmonic functional set $\{\varphi_n\}$ defined outside of Q_0. Practical choice consists of natural modes (2.5) which are theoretically defined only in Q_0. An exception is upright cylindrical tanks of circular and rectangular cross-sections when the spectral problem (2.5) has analytical solution defined over Σ_0. For more general case, specific numerical methods are required [16, 17, 43] providing zero-Neumann condition on the time-dependent wetted tank surface $S(t)$. Otherwise, a numerical error on the mentioned part of the walls physically implies an inflow/outflow through $S(t)$.
- *Functional:* Harmonic functional set $\{\varphi_n\}$ should be complete for any instant $Q(t)$. The linear natural modes generally do not constitute such a complete functional basis. The latter fact was demonstrated by Timokha [46] for two-dimensional rectangular tank. In view of this functional problem, one can interpret the natural modes as an "asymptotic" harmonic basis, namely, assume that $Q(t)$ is to some extent asymptotically close to its unperturbed state Q_0. Within the framework of the asymptotic treatment, the completeness of $\{\varphi_n\}$ is only needed in Q_0.

One can distinguish *two analytical schemes* to derive nonlinear modal equations. The first scheme is associated with works by Narimanov [38] and Moiseev [36] employing asymptotic methods. The second one uses variational methods based on the Bateman-Luke [28, 33] or Lagrange [23–25] formulation of the free-boundary problem (2.1).

2.3. Asymptotic methods

As a rule, asymptotic methods in liquid sloshing problems with a *finite liquid depth* assume that the relative (with respect to tank size) forcing magnitude is small, of the order $O(\epsilon)$, and tank excitations are almost harmonic with the forcing frequency σ close to the lowest natural sloshing frequency σ_1. Furthermore, these methods assume the possibility of a third-order theory and consider the primarily excited mode being of the dominant, lowest asymptotic order, $O(\epsilon^{1/3})$. There are also generalized coordinates of the order $O(\epsilon^{2/3})$ and $O(\epsilon)$. Shallow liquid sloshing needs other approaches [8, 10].

One can classify two asymptotic approaches based on this third-order intermodal ordering. The first approach is associated with Narimanov-type asymptotic schemes and devoted to derivations of modal equations coupling generalized coordinates up to the order $O(\epsilon)$. The second asymptotic approach was proposed by Moiseev to derive a steady-state (periodic) resonant solution for purely harmonic excitations. This approach does not derive any modal equations but makes it possible to identify ordering of the generalized coordinates.

Narimanov-type asymptotic scheme. Similar asymptotic schemes for deriving asymptotic modal equations were proposed by Narimanov [38] and, later on, by Hutton [19] (interested readers can find results on these schemes in [5, 18, 29, 39]). In these schemes, asymptotic ordering of generalized coordinates is *postulated.* To obtain modal equations with respect to β_i, Narimanov assumed that generalized coordinates R_i take the form $R_i = F_i(\dot{\beta}_i, \beta_k)$ where functions F_i have a polynomial structure and can be found after substituting eqs. (2.2) and (2.3) into "kinematic" equations (2.1a) by using a Taylor expansion with respect to the mean free surface Σ_0 $(z = 0)$ leading to a series of Neumann' boundary-value problems. Having known R_i in (2.3) as explicit functions of β_k and $\dot{\beta}_k$ and substituting eqs. (2.2) and (2.3) into "dynamic" equations (2.1b), analogous Taylor expansion leads to the desirable modal equations with respect to β_n after neglecting the $o(\epsilon)$-order terms.

The Narimanov-type asymptotic scheme is straightforward but causes tedious analytical derivations. This explains why Narimanov et al. [39] found out arithmetic errors in [38] and Miles [35] detected errors in [19]. Furthermore, to minimize tedious derivations, many of the Narimanov-type nonlinear modal equations coupled only generalized coordinates responsible for dominant modes, but second- and third-order generalized coordinates were neglected (Stolbetsov [39, 44, 45] derived those simplified modal equations for upright rectangular and circular tanks). Lukovsky [29,39] showed that this simplification sufficiently effects results on steady-state sloshing which may not become consistent with experiments. Lukovsky [27] also generalized the Narimanov-type scheme for tanks of complex, non-cylindrical shapes.

Steady-state asymptotic solution by Moiseev. Moiseev [36] formally constructed a steady-state asymptotic solution of the original free-boundary problem (2.1). For finite liquid depth, he *proved* that, if the nondimensional forcing amplitude is small of the order $O(\epsilon)$, the primary-excited lowest natural sloshing modes *must* be of the order $O(\epsilon^{1/3})$ provided by $|\sigma^2 - \sigma_1^2|/\sigma_1^2 = O(\epsilon^{2/3})$. Moiseev's asymptotic scheme makes it possible to *detect* which of modes have second, $O(\epsilon^{2/3})$, and third $O(\epsilon)$ order of the smallness. This is in contrast to Narimanov' scheme which postulates the intermodal ordering.

Faltinsen [7] implemented Moiseev's asymptotic scheme to construct a steady-state solution for two-dimensional resonant sloshing in a rectangular tank due to either longitudinal or angular harmonic excitations. Di Maggio & Rehm [4] used similar asymptotic scheme to describe nonlinear free standing waves in an upright circular cylindrical tank.

2.4. Variational methods by Miles-Lukovsky

Variational multimodal methods were proposed by Miles [33] and Lukovsky [28]. Miles [33] demonstrated applicability of the Lagrange and Bateman-Luke principles, while Lukovsky [28] focused on Luke's variational formulation. These variational methods make it possible to derive nonlinear modal equations without any asymptotic relationships for generalized coordinates. The Lukovsky method leads to the following infinite-dimensional system of ordinary differential equations [29]

$$\sum_{k=1}^{\infty} \frac{\partial A_n}{\partial \beta_k} \dot{\beta}_k = \sum_{k=1}^{\infty} A_{nk} R_k, \quad n = 1, 2, \dots, \tag{2.6a}$$

$$\sum_n \dot{R}_n \frac{\partial A_n}{\partial \beta_i} + \frac{1}{2}\sum_n \sum_k \frac{\partial A_{nk}}{\partial \beta_i} R_n R_k + \dot{\omega}_1 \frac{\partial l_{1\omega}}{\partial \beta_i} + \dot{\omega}_2 \frac{\partial l_{2\omega}}{\partial \beta_i} + \dot{\omega}_3 \frac{\partial l_{3\omega}}{\partial \beta_i} + \omega_1 \frac{\partial l_{1\omega t}}{\partial \beta_i}$$
$$+ \omega_2 \frac{\partial l_{2\omega t}}{\partial \beta_i} + \omega_3 \frac{\partial l_{3\omega t}}{\partial \beta_i} - \frac{d}{dt}\left(\omega_1 \frac{\partial l_{1\omega t}}{\partial \dot{\beta}_i} + \omega_2 \frac{\partial l_{2\omega t}}{\partial \dot{\beta}_i} + \omega_3 \frac{\partial l_{3\omega t}}{\partial \dot{\beta}_i}\right)$$
$$+ (\dot{v}_{01} - g_1 + \omega_2 v_{03} - \omega_3 v_{02})\frac{\partial l_1}{\partial \beta_i} + (\dot{v}_{02} - g_2 + \omega_3 v_{01} - \omega_1 v_{03})\frac{\partial l_2}{\partial \beta_i}$$
$$+ (\dot{v}_{03} - g_3 + \omega_1 v_{02} - \omega_2 v_{01})\frac{\partial l_3}{\partial \beta_i} - \frac{1}{2}\omega_1^2 \frac{\partial J^1_{11}}{\partial \beta_i} - \frac{1}{2}\omega_2^2 \frac{\partial J^1_{22}}{\partial \beta_i} - \frac{1}{2}\omega_3^2 \frac{\partial J^1_{33}}{\partial \beta_i}$$
$$- \omega_1\omega_2 \frac{\partial J^1_{12}}{\partial \beta_i} - \omega_1\omega_3 \frac{\partial J^1_{13}}{\partial \beta_i} - \omega_2\omega_3 \frac{\partial J^1_{23}}{\partial \beta_i} = 0, \quad i = 1, 2, \ldots, \qquad (2.6b)$$

where

$$A_n = \rho \int_{Q(t)} \varphi_n \mathrm{d}Q; \quad A_{nk} = \rho \int_{Q(t)} \varphi_n \mathrm{d}Q; \quad J_{ij} = \rho \int_{S(t)+\Sigma(t)} \Omega_i \frac{\partial \Omega_j}{\partial n} \mathrm{d}S,$$
$$l_{k\omega} = \rho \int_{Q(t)} \Omega_k \mathrm{d}Q; \quad l_{k\omega t} = \rho \int_{Q(t)} \frac{\partial \Omega_k}{\partial t} \mathrm{d}Q; \quad l_3 = \rho \int_{Q(t)} z \mathrm{d}Q, \qquad (2.7)$$
$$l_1 = \rho \int_{Q(t)} x \mathrm{d}Q; \quad l_2 = \rho \int_{Q(t)} y \mathrm{d}Q$$

are nonlinear functions of β_i.

The nonlinear modal equations (2.6) are a full analogy of the original free-boundary problem (2.1) provided by normal representation of $\Sigma(t)$, i.e. $z = \zeta(y, z, t)$. Direct simulations by (2.6) imply the so-called Perko's numerical method [37,40]. Faltinsen & Timokha [9,10] remarked that these simulations can be very stiff for resonant sloshing and, therefore, certain numerical methods become numerically unstable unless the time step is taken to be extremely small. This unrealistic stiffness is caused by amplification of higher harmonics which are in the physical reality highly damped due to different physical mechanisms. La Rocca et al. [22] used truncated modal system (2.6) with additional linear damping terms for solving time-domain problems, however, they were not able to simulate strongly resonant sloshing since damping of higher harmonics are not well predicted by these linear terms. Alternative is to introduce asymptotic relationships between generalized coordinates and, thereby, exclude ("filter") the unrealistically high harmonics. These relationships can be taken from Moiseev's asymptotic analysis.

3. Asymptotic modal equations based on combining modal equations (2.6) and Moiseev's asymptotics

The aforementioned idea to use Moiseev-type ordering in the fully-nonlinear modal equations (2.6) was extensively used in [6, 11, 16, 29] for derivations of asymptotic modal equations. According to Moiseev's method, (i) the nondimensional excitations (associated with the six functions $(\eta_1(t), \eta_2(t), \eta_3(t))$ and $(\eta_4(t), \eta_5(t), \eta_6(t))$ defined by $\mathbf{v}_0 = (\dot{\eta}_1(t), \dot{\eta}_2(t), \dot{\eta}_3(t))$ and $\omega = (\dot{\eta}_4(t), \dot{\eta}_5(t), \dot{\eta}_6(t)))$ are of the highest asymptotic order, $O(\epsilon)$, (ii) the nondimensional generalized coordinates have different asymptotic ordering from $O(\epsilon^{1/3})$ to $o(\epsilon)$ for different tank shapes, and (iii) the $o(\epsilon)$-terms in the modal equations should be neglected. Algorithm for derivation of asymptotic modal equations contains the following steps:

1. Using *analytical relationships* between natural sloshing modes, one should identify the set of nondimensional generalized coordinates ordered by $O(\epsilon^{2/3})$ and $O(\epsilon)$ provided by the $O(\epsilon^{1/3})$-order for the generalized coordinates responsible for the lowest natural sloshing modes and the limit $\sigma \to \sigma_1$. This set, depending on the tank shape, can be finite or infinite. Other generalized coordinates should be considered of the order $o(\epsilon)$ and excluded from nonlinear modal analysis.
2. Using the Taylor expansion, one should find polynomial expressions (in terms of nondimensional generalized coordinates β_i) for $\partial A_n/\partial \beta_k$ and A_{nk} keeping up to the $O(\epsilon^{2/3})$-order and $\partial A_{nk}/\partial \beta_i$ keeping the $O(\epsilon^{1/3})$-terms.

3. We should find asymptotic solution $R_i = F(\beta_k, \dot{\beta}_k)$ from modal equations (2.6a) by substituting previously-found asymptotic expressions for $\partial A_n/\partial \beta_k$ and A_{nk}. This solution should neglect the $o(\epsilon)$-terms.
4. We should substitute expressions $R_i = F(\beta_k, \dot{\beta}_k)$ from the previous step into modal equations (2.6b) and keep up to the $O(\epsilon)$-terms. This will give the desirable asymptotic modal equations.

The algorithm is only realized for upright cylindrical tanks of circular and rectangular shape. Finite-dimensional asymptotic nonlinear modal equations were derived for rectangular base. The system of asymptotic modal equations is infinite-dimensional for circular base. The asymptotic modal equations remain quite accurate in describing both steady-state [8, 29] and transient [6, 15] waves.

3.1. Two-dimensional rectangular tank

The two-dimensional liquid sloshing is considered in the Oxz-plane. The tank has the width L_1 and forced horizontally or angularly with almost periodic excitation and σ is close to σ_1. The natural sloshing frequencies and the L_1-scaled natural modes are

$$f_i(x) = \cos\left(\pi i(x + \tfrac{1}{2})\right); \quad \varphi_i(x,z) = f_i(x)\frac{\cosh(\pi i(z+\bar{h}))}{\cosh(\pi i \bar{h})}; \quad \sigma_i^2 = g\pi i \tanh(\pi i \bar{h}), \tag{3.1}$$

where $\bar{h} = h/L_1$ is the nondimensional liquid depth.

According to analytical relationships for the trigonometric natural modes (3.1) (see, step 1.), the Moiseev ordering is [6, 7, 36] as follows

$$\beta_1 = O(\epsilon^{1/3}); \quad \beta_2 = O(\epsilon^{2/3}); \quad \beta_3 = O(\epsilon); \quad \beta_k = o(\epsilon),\ k \geq 4 \tag{3.2}$$

and, therefore, we arrive at the finite-dimensional system of nonlinear modal equations [6]

$$(\ddot{\beta}_1 + \sigma_1^2\beta_1) + d_1(\ddot{\beta}_1\beta_2 + \dot{\beta}_1\dot{\beta}_2) + d_2(\ddot{\beta}_1\beta_1^2 + \dot{\beta}_1^2\beta_1) + d_3\ddot{\beta}_2\beta_1 = K_1(t), \tag{3.3a}$$

$$(\ddot{\beta}_2 + \sigma_2^2\beta_2) + d_4\ddot{\beta}_1\beta_1 + d_5\dot{\beta}_1^2 = 0, \tag{3.3b}$$

$$(\ddot{\beta}_3 + \sigma_3^2\beta_3) + q_1\ddot{\beta}_1\beta_2 + q_2\ddot{\beta}_1\beta_1^2 + q_3\ddot{\beta}_2\beta_1 + q_4\dot{\beta}_1\dot{\beta}_2 + q_5\dot{\beta}_1^2\beta_1 = K_3(t); \tag{3.3c}$$

coupling β_1, β_2, and β_3. Higher generalized coordinates are formally of the order $o(\epsilon)$ and are governed by linear modal equations. The first two nonlinear equations of (3.3) couple β_1 with β_2 and do not depend on β_3. The third generalized coordinate is excited by rigid body motions and due to nonlinear terms in β_1 with β_2. The forcing terms are

$$K_i(t) = -P_{i,0}\left[\ddot{\eta}_1 + S_{i,0}\ddot{\eta}_5 + g/l\eta_5\right] \tag{3.4}$$

and the hydrodynamic coefficients d_i, q_i, $P_{i,0}$, $S_{i,0}$ are found analytically in [6] as functions of $\bar{h}$.

3.2. Three-dimensional nearly-square tank

Let us consider a three-dimensional rectangular tank whose cross-section has dimensions L_1 (along the Ox-axis) and L_2 (along the Oy-axis). The tank is filled with finite liquid depth h assuming that h/L_1 and h/L_2 are of the order $O(1)$. The L_1-scaled natural sloshing modes for three-dimensional rectangular tank are

$$\varphi_{i,j}(x,y,z) = f_i^{(1)} f_j^{(2)} \frac{\cosh(\lambda_{i,j}(z+\bar{h}))}{\cosh(\lambda_{i,j}\bar{h})},$$

$$\lambda_{i,j} = \pi\sqrt{i^2 + r_l^2 j^2}, \quad \sigma_{i,j}^2 = \frac{g}{L_1}\lambda_{i,j}\tanh(\lambda_{i,j}\bar{h}), \quad i,j \geq 0, \quad i+j \neq 0, \quad r_l = \frac{L_1}{L_2}. \tag{3.5}$$

Physically, the natural sloshing modes (3.5) can be classified in terms of three subclasses. The first class consists of two-dimensional Stokes waves in the Oxz and Oyz-planes. These are the same as the natural sloshing modes (3.1) implying two-dimensional, "planar" waves:

$$f_i^{(1)}(x) = \cos(\pi i(x + \tfrac{1}{2})), \quad i \geq 1; \quad f_j^{(2)}(y) = \cos(\pi j r_l(y + \tfrac{1}{2r_l})), \quad j \geq 1. \tag{3.6}$$

The second subclass deals with the three-dimensional wave patterns

$$f_i^{(1)}(x) \cdot f_j^{(2)}(y), \quad i, j \geq 1. \tag{3.7}$$

Finally, it is convenient to introduce the mixed modes

$$S_1^i(x, y) = A f_i^{(1)}(x) \pm \bar{B} f_i^{(2)}(y) \tag{3.8}$$

recombining the two modes (3.6) in perpendicular planes into three-dimensional patterns with non-zero weight coefficients, $A\bar{B} \neq 0$. These are called "diagonal".

The three-dimensional sloshing is expected due to passage to square geometry implying $\sigma_{i,j} \to \sigma_{j,i}$ ($r_l \to 1$) when natural modes $f_1^{(1)}$ and $f_1^{(2)}$ become degenerated (having equal natural frequencies). By using the trigonometric-type natural modes, Faltinsen et al. [11] showed that $L_1 \approx L_2$ leads to Moiseev' asymptotic relations

$$\beta_{1,0} \sim \beta_{0,1} = O(\epsilon^{1/3}); \quad \beta_{2,0} \sim \beta_{1,1} \sim \beta_{0,2} = O(\epsilon^{2/3}),$$
$$\beta_{3,0} \sim \beta_{2,1} \sim \beta_{1,2} \sim \beta_{0,3} = O(\epsilon); \quad \beta_{i,j} \lesssim O(\epsilon), \; i + j \geq 4 \tag{3.9}$$

in terms of the nondimensional forcing magnitude $O(\epsilon)$. They also derived the finite-dimensional system of modal equations coupling $\beta_{i,j}, \; i + j \leq 3$. Other modes ($i + j \geq 4$) are governed by the linear sloshing theory. It is the same as for the two-dimensional case in previous section. After re-denoting for brevity $\beta_{1,0} = a_1, \beta_{2,0} = a_2, \beta_{0,1} = b_1, \beta_{0,2} = b_2, \beta_{1,1} = c_1, \beta_{3,0} = a_3, \beta_{2,1} = c_{21}, \beta_{1,2} = c_{12}, \beta_{0,3} = b_3$ these nonlinear asymptotic modal equations take the form

$$\Big[\ddot{a}_1 + \sigma_{1,0}^2 a_1 + d_1(\ddot{a}_1 a_2 + \dot{a}_1 \dot{a}_2) + d_2(\ddot{a}_1 a_1^2 + \dot{a}_1^2 a_1) + d_3 \ddot{a}_2 a_1$$
$$+ P_{1,0}(\ddot{\eta}_1 - S_{1,0}\ddot{\eta}_5 - g\eta_5/L_1)\Big]$$
$$+ d_6 \ddot{a}_1 b_1^2 + \ddot{b}_1(d_7 c_1 + d_8 a_1 b_1) + d_9 \ddot{c}_1 b_1 + d_{10} \dot{b}_1^2 a_1 + d_{11} \dot{a}_1 \dot{b}_1 b_1 + d_{12} \dot{b}_1 \dot{c}_1 = 0, \tag{3.10a}$$

$$\Big[\ddot{b}_1 + \sigma_{0,1}^2 b_1 + \bar{d}_1(\ddot{b}_1 b_2 + \dot{b}_1 \dot{b}_2) + \bar{d}_2(\ddot{b}_1 b_1^2 + \dot{b}_1^2 b_1) + \bar{d}_3 \ddot{b}_2 b_1$$
$$+ P_{0,1}(\ddot{\eta}_2 + S_{0,1}\ddot{\eta}_4 + g\eta_4/L_1)\Big]$$
$$+ \bar{d}_6 \ddot{b}_1 a_1^2 + \ddot{a}_1(\bar{d}_7 c_1 + \bar{d}_8 a_1 b_1) + \bar{d}_9 \ddot{c}_1 a_1 + \bar{d}_{10} \dot{a}_1^2 b_1 + \bar{d}_{11} \dot{a}_1 \dot{b}_1 a_1 + \bar{d}_{12} \dot{a}_1 \dot{c}_1 = 0, \tag{3.10b}$$

$$\big[\ddot{a}_2 + \sigma_{2,0}^2 a_2 + d_4 \ddot{a}_1 a_1 + d_5 \dot{a}_1^2\big] = 0, \tag{3.10c}$$

$$\Big[\ddot{b}_2 + \sigma_{0,2}^2 b_2 + \bar{d}_4 \ddot{b}_1 b_1 + \bar{d}_5 \dot{b}_1^2\Big] = 0, \tag{3.10d}$$

$$\ddot{c}_1 + \hat{d}_1 \ddot{a}_1 b_1 + \hat{d}_2 \ddot{b}_1 a_1 + \hat{d}_3 \dot{a}_1 \dot{b}_1 + \sigma_{1,1}^2 c_1 = 0, \tag{3.10e}$$

$$\Big[\ddot{a}_3 + \sigma_{3,0}^2 a_3 + \ddot{a}_1(q_1 a_2 + q_2 a_1^2) + q_3 \ddot{a}_2 a_1 + q_4 \dot{a}_1^2 a_1 + q_5 \dot{a}_1 \dot{a}_2$$
$$+ P_{3,0}[\ddot{\eta}_1 - S_{3,0}\ddot{\eta}_5 - g\eta_5/L_1]\Big] = 0, \tag{3.11a}$$

$$\ddot{c}_{21} + \sigma_{2,1}^2 c_{21} + \ddot{a}_1(q_6 c_1 + q_7 a_1 b_1) + \ddot{b}_1(q_8 a_2 + q_9 a_1^2) + q_{10} \ddot{a}_2 b_1 + q_{11} \ddot{c}_1 a_1$$
$$+ q_{12} \dot{a}_1^2 b_1 + q_{13} \dot{a}_1 \dot{b}_1 a_1 + q_{14} \dot{a}_1 \dot{c}_1 + q_{15} \dot{a}_2 \dot{b}_1 = 0, \tag{3.11b}$$

$$\ddot{c}_{12} + \sigma_{1,2}^2 c_{12} + \ddot{b}_1(\bar{q}_6 c_1 + \bar{q}_7 a_1 b_1) + \ddot{a}_1(\bar{q}_8 b_2 + \bar{q}_9 b_1^2) + \bar{q}_{10} \ddot{b}_2 a_1 + \bar{q}_{11} \ddot{c}_1 b_1$$
$$+ \bar{q}_{12} \dot{b}_1^2 a_1 + \bar{q}_{13} \dot{a}_1 \dot{b}_1 b_1 + \bar{q}_{14} \dot{b}_1 \dot{c}_1 + \bar{q}_{15} \dot{a}_1 \dot{b}_2 = 0, \tag{3.11c}$$

$$\Big[\ddot{b}_3 + \sigma_{0,3}^2 b_3 + \ddot{b}_1(\bar{q}_1 b_2 + \bar{q}_2 b_1^2) + \bar{q}_3 \ddot{b}_2 b_1 + \bar{q}_4 \dot{b}_1^2 b_1 + \bar{q}_5 \dot{b}_1 \dot{b}_2 + P_{0,3}[\ddot{\eta}_2 + S_{0,3}\ddot{\eta}_4 + g\eta_4/L_1]\Big] = 0 \quad (3.11d)$$

with the hydrodynamic coefficients being functions of $\bar{h}$ and r_l. Analytical expressions for these coefficients are explicitly derived in [11]. We recognize in eqs. (3.10) and (3.11) that the terms in square brackets are associated with "planar" flows in either Oxz or Oyz planes. They are exactly the same as in section 3.1. Other terms and additional equations for c_{11}, c_{21} and c_{12} are due to three-dimensional intermodal interaction. Here the subsystem (3.10) does not depend on a_3, c_{21}, c_{12} and b_3 calculated from (3.11). The subsystem (3.11) is linear in a_3, c_{21}, c_{12} and b_3 and depends nonlinearly on a_1, b_1, a_2, b_2 and c_1.

Having based on modal equations (3.10), Faltinsen et al. [11,12] studied steady-state resonant sloshing occurring due to harmonic ($\eta_k = \eta_{k1}\cos(\sigma t)$, $\eta_{3a} = \eta_{6a} = 0$) longitudinal ($\eta_{2a} = \eta_{4a} = 0$) or diagonal ($\eta_{1a} = \eta_{2a}$, $\eta_{4a} = \eta_{5a}$) excitations. They showed that there are "planar "diagonal" and "swirling" regimes. If the forcing is along the Ox-axis, the following approximations of the steady-state elevation are possible

$$\zeta(x, y, t) = A f_1^{(1)} \cos(\sigma t) + o(A), \quad (3.12a)$$

$$\zeta(x, y, t) = [A f_i^{(1)}(x) \pm \bar{B} f_i^{(2)}(y)] \cos(\sigma t) + o(A, \bar{B}), \quad (3.12b)$$

$$\zeta(x, y, t) = A f_1^{(1)} \cos(\sigma t) \pm B f_i^{(2)}(y) \sin(\sigma t) + o(A, B) \quad (3.12c)$$

corresponding to these three regimes.

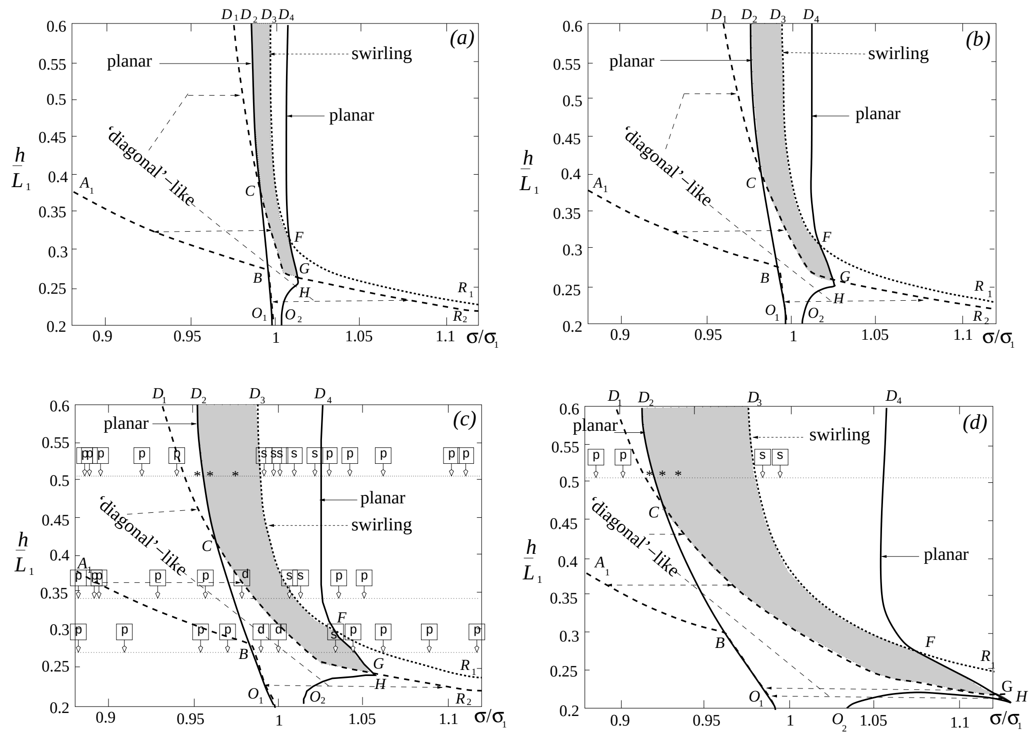

Fig. 2. Theoretical frequency domains of stable resonant steady-state sloshing as a function of the liquid depth-to-breadth ratio $\bar{h} = h/L_1$ ($L_1 = L_2$) versus σ/σ_1 for different forcing amplitudes. Longitudinal excitations with amplitudes (a) $\eta_{1a}/L_1 = 0.001$, (b) $\eta_{1a}/L_1 = 0.0025$, (c) $\eta_{1a}/L_1 = 0.0078$, and (d) $\eta_{1a}/L_1 = 0.025$. There are no stable steady-state waves and chaotic motions occur in the shaded area. Comparisons with experimental observations are made in (c) for the liquid depths $\bar{h} = 0.508, 0.34$ and 0.27 and in (d) for $\bar{h} = 0.508$. Experimental steady-state regimes are denoted as "p" – planar waves, "s" – swirling, "d" – nearly-diagonal and "*" – "chaos"

Fig. 2 summarizes frequency domains of *stable* steady-state regimes as function of $\bar{h}$ for square-base tank, $L_1 = L_2$, and four different forcing amplitudes. The figure shows that region of stable planar waves is always away from the main resonance $\sigma/\sigma_1 = 1$. The region where planar waves are unstable is denoted as $D_2D_4FGHO_2O_1$. This region becomes wider with increasing forcing amplitude. The two-dimensional analysis based on results in section 3.1. does not lead to this instability region. This confirms that it is caused by three-dimensional perturbations. Geometrically, $D_2D_4FGHO_2O_1$ falls into three sub-regions. The first one, O_1CGHO_2 corresponds to unstable planar regime, but diagonal waves are stable. This region appears only for smaller depths and is absent for fairly deep water. The second region, FD_4D_6, corresponds to the case when there is stable swirling, but planar waves are unstable (the region disappears at $\sigma/\sigma_1 = 1$ for small depths). No stable steady-state solutions exist and chaotic motions are possible for the region D_2D_6FHC which disappears for small liquid depths. Away from region $D_2D_4FGHO_2O_1$, the planar waves are stable and may co-exist with stable swirling (region R_1FD_4) or diagonal waves (region A_1BCD_1). There are no regions where stable swirling co-exist with stable diagonal waves. Swirling is stable right up to the border D_6FR_1. The region of stable swirling is away from $\sigma/\sigma_1 = 1$ for smaller $\bar{h}$, while effective domain of stable diagonal waves $A_1BO_1O_2HCD_1$ drifts left of the main resonance with increasing $\bar{h}$. The diagonal waves co-exist with planar waves when $\bar{h}$ is larger than the ordinate of C. Theoretically, initial conditions determine what kind of steady-state motion is realized after initial transients.

The theoretical effective frequency domains for diagonal excitations are summarized in Fig. 3. Obviously, planar regimes are then impossible. The instability of diagonal waves and swirling is between the solid and dashed lines, respectively. The instability zones become narrower with lower forcing amplitudes (note the abscissas in parts a-d). Both regimes are simultaneously unstable only when the zones overlap each other. This is only possible for smaller depths. Another interesting point is that two different stable steady-state regimes (diagonal and swirling) co-exist for $\bar{h} > 0.27$ in the vicinity of the main resonance. Depending on the transient/initial perturbation scenarios either diagonal or swirling waves are excited.

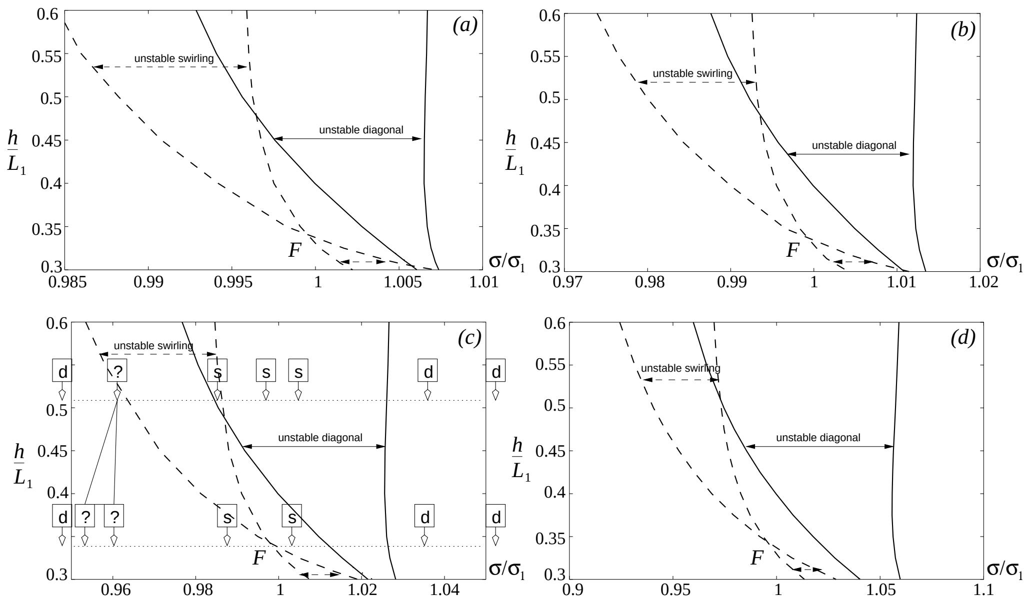

Fig. 3. Theoretical frequency domains of stable resonant steady-state motions caused by resonant diagonal excitations $\eta_{1a} = \eta_{2a} \neq 0$ and presented in the $(\sigma/\sigma_1, \bar{h})$-plane for different forcing amplitudes. The square-base tank and (a) $\sqrt{2\eta_{1a}^2}/L_1 = \epsilon = 0.001$, (b) $\epsilon = 0.0025$, (c) $\epsilon = 0.0078$, and (d) $\epsilon = 0.025$. The instability of diagonal waves and swirling is expected between solid and dashed lines, respectively. Comparisons with experimental data are made in part (c) for $\bar{h} = 0.508$ and 0.34. "d-stable diagonal waves, "s- stable swirling waves, ?= not clear identified

3.3. Upright circular cylindrical tank

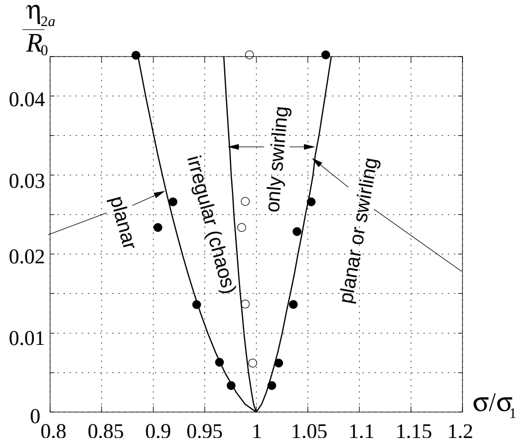

Fig. 4. Effective frequency domains for stable steady-state waves and chaos in the $(\sigma/\sigma_{1,1}, \eta_{2a}/R_0)$-plane for an upright circular cylindrical tank with $h/R_0 = 1.5$. Experimentally-predicted bounds are taken from Royon-Lebeaud et al. [41]: "solid circle" = experimental bounds for planar waves, "empty circle" = experimental bound for swirling

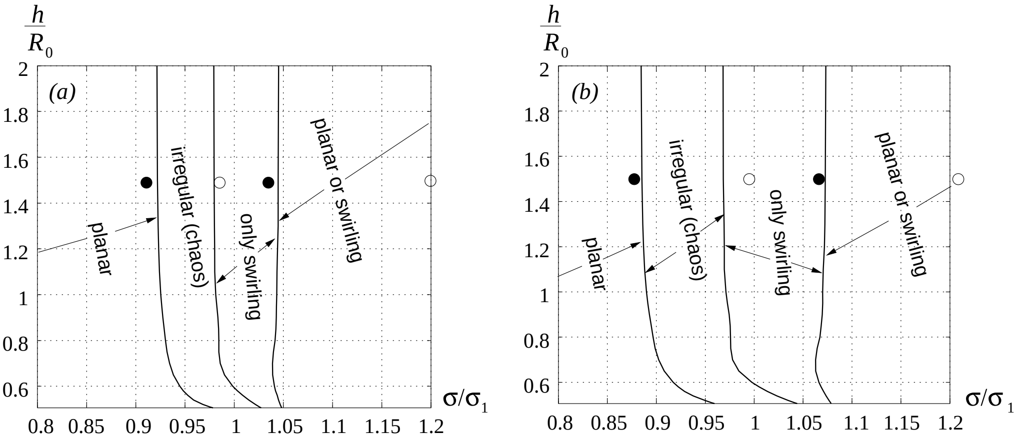

Fig. 5. Effective frequency domains for stable steady-state waves and chaos in the $(\sigma/\sigma_{1,1}, \eta_{2a}/R_0)$-plane for an upright circular cylindrical tank with $\eta_{2a}/R_0 = 0.23$ (case a) and 0.045 (case b). The bounds between the domains are obtained by using the modal theory (9.47)-(9.51), solid lines. Experimentally-predicted bounds are taken from Fig. 4

Consider an upright circular cylindrical tank of the radius R_0 with the liquid depth h. The R_0-scaled natural sloshing modes in cylindrical coordinate system (r, θ, z) are

$$\varphi_{m,n,1} = R_{m,n} Z_{m,n}(z) \cos(m\theta); \quad \varphi_{m,n,2} = R_{m,n} Z_{m,n}(z) \sin(m\theta), \quad m \geq 0, \ n \geq 1, \tag{3.13}$$

where $J'_m(k_{m,n}) = 0$, $\sigma^2_{m,n} = g k_{m,n} \tanh(k_{m,n} \bar{h})$ and

$$Z_{m,n}(z) = \frac{\cosh(k_{m,n}(z + \bar{h}))}{\cosh(k_{m,n}\bar{h})}; \quad R_{m,n}(r) = \frac{J_m(k_{m,n} r)}{J_m(k_{m,n}\bar{h})}.$$

The natural modes (3.13) possess the trigonometric algebra by angular coordinate θ and the Bessel functions J_m describe wave patterns in radial direction. The latter causes an infinite number of generalized coordinates corresponding to the second and third order

$$\beta_{1,1,j} = O(\epsilon^{1/3}), \ \beta_{0,n} = O(\epsilon^{2/3}), \ \beta_{2,n,j} = O(\epsilon^{2/3}), \ \beta_{3,n,j} = O(\epsilon), \ j = 1, 2, \ n \geq 1. \tag{3.14}$$

Adopting the notations $\beta_{1,1,1} = p_{1,1}$, $\beta_{1,1,2} = r_{1,1}$, $\beta_{0,n} = p_{0,n}$, $\beta_{2,n,1} = p_{2,n}$, $\beta_{2,n,2} = r_{2,n}$, $\beta_{3,n,1} = p_{3,n}$, $\beta_{3,1,2} = r_{3,n}$, Lukovsky [31] derived the corresponding infinite-dimensional system. The equations for the dominant generalized coordinates and the second-order terms take the form

$$\begin{aligned}\mu_{1,1}(\ddot{p}_{1,1} + \sigma_{1,1}^2 p_{1,1}) + p_{1,1}\sum_{n=1}^{\infty} d_{0,n}^{(2)}\ddot{p}_{0,n} + \sum_{n=1}^{\infty} d_{0,n}^{(3)}(\ddot{p}_{1,1}p_{0,n} + \dot{p}_{1,1}\dot{p}_{0,n})\\
+ d_1(p_{1,1}^2\ddot{p}_{1,1} + p_{1,1}\dot{p}_{1,1}^2 + r_{1,1}p_{1,1}\ddot{r}_{1,1} + p_{1,1}\dot{r}_{1,1}^2)\\
+ d_2(r_{1,1}^2\ddot{p}_{1,1} + 2r_{1,1}\dot{r}_{1,1}\dot{p}_{1,1} - r_{1,1}p_{1,1}\ddot{r}_{1,1} - 2p_{1,1}\dot{r}_{1,1}^2) + \sum_{n=1}^{\infty} d_{2,n}^{(2)}(p_{1,1}\ddot{p}_{2,n} + r_{1,1}\ddot{r}_{2,n})\\
+ \sum_{n=1}^{\infty} d_{2,n}^{(3)}(\ddot{p}_{1,1}p_{2,n} + \ddot{r}_{1,1}r_{2,n} + \dot{p}_{1,1}\dot{p}_{2,n} + \dot{r}_{1,1}\dot{r}_{2,n})\\
= -P_1(\ddot{\eta}_1 - g\eta_5/R_0 - S_1\ddot{\eta}_5),\end{aligned} \tag{3.15a}$$

$$\begin{aligned}\mu_{1,1}(\ddot{r}_{1,1} + \sigma_{1,1}^2 r_{1,1}) + r_{1,1}\sum_{n=1}^{\infty} d_{0,n}^{(2)}\ddot{p}_{0,n} + \sum_{n=1}^{\infty} d_{0,n}^{(3)}(\ddot{r}_{1,1}p_{0,n} + \dot{r}_{1,1}\dot{p}_{0,n})\\
+ d_1(r_{1,1}^2\ddot{r}_{1,1} + r_{1,1}\dot{r}_{1,1}^2 + r_{1,1}p_{1,1}\ddot{p}_{1,1} + r_{1,1}\dot{p}_{1,1}^2)\\
+ d_2(p_{1,1}^2\ddot{r}_{1,1} + 2p_{1,1}\dot{r}_{1,1}\dot{p}_{1,1} - r_{1,1}p_{1,1}\ddot{p}_{1,1} - 2p_{1,1}\dot{p}_{1,1}^2) + \sum_{n=1}^{\infty} d_{2,n}^{(2)}(p_{1,1}\ddot{r}_{2,n} - r_{1,1}\ddot{p}_{2,n})\\
+ \sum_{n=1}^{\infty} d_{2,n}^{(3)}(\ddot{p}_{1,1}r_{2,n} - \ddot{r}_{1,1}p_{2,n} + \dot{p}_{1,1}\dot{r}_{2,n} - \dot{r}_{1,1}\dot{p}_{2,n})\\
= -P_1(\ddot{\eta}_2 + g\eta_4/R_0 + S_1\ddot{\eta}_4),\end{aligned} \tag{3.15b}$$

$$\mu_{2,0}(\ddot{p}_{0,n} + \sigma_{0,n}^2 p_{0,n}) + d_{0,n}^{(1)}(\dot{p}_{1,1}^2 + \dot{r}_{1,1}^2) + d_{0,n}^{(2)}(\ddot{p}_{1,1}p_{1,1} + \ddot{r}_{1,1}r_{1,1}) = 0, \tag{3.15c}$$

$$\mu_{2,n}(\ddot{p}_{2,n} + \sigma_{2,n}^2 p_{2,n}) + d_{2,n}^{(1)}(\dot{p}_{1,1}^2 - \dot{r}_{1,1}^2) + d_{2,n}^{(2)}(\ddot{p}_{1,1}p_{1,1} - \ddot{r}_{1,1}r_{1,1}) = 0, \tag{3.15d}$$

$$\mu_{2,n}(\ddot{r}_{2,n} + \sigma_{2,n}^2 r_{2,n}) + 2d_{2,n}^{(1)}\dot{r}_{1,1}\dot{p}_{1,1} + d_{2,n}^{(2)}(\ddot{p}_{1,1}r_{1,1} + \ddot{r}_{1,1}p_{1,1}) = 0. \tag{3.15e}$$

Analyzing the modal equations, one can find two different steady-state regimes

$$\zeta(r,\theta,t) = AR_{1,1}\cos\theta\cos(\sigma t) + o(A), \tag{3.16a}$$

$$\zeta(r,\theta,t) = R_{1,1}[A\cos\theta\cos(\sigma t) + B\sin\theta\sin(\sigma t)] + o(A,B). \tag{3.16b}$$

called "planar" and "swirling". Effective frequency domains associated with stable steady-state regimes and the possibly of "chaos" (versus forcing amplitude and liquid depth) are illustrated in Fig. 4 and 5. The results are supported by Royon-Lebeaud et al.' [41] model tests. The range of "chaos" increases with increasing excitation amplitude, but Fig. 5 shows that the frequency domain for "chaos" decreases with decreasing $\bar{h}$.

3.4. Non-cylindrical tank shapes

Lukovsky [27] proposed a non-conformal mapping technique to extend Narimanov's asymptotic scheme for tanks with non-vertical walls. Later on, the same technique was generalized in [16, 29] to derive an analogy of modal systems (2.6). However, because of difficulties in finding higher natural sloshing modes satisfying the Laplace equation and the zero-Neumann condition for any instant $S(t)$, this technique was realized to derive asymptotic modal equations only for circular conical tanks [16, 30]. Further, the first step of the algorithm in section 3. suggests establishing the second- and third-order generalized coordinates. This task becomes quite difficult for complex tank shapes and explains why the modal equations in [16, 30] (conical tank) are not complete and include, along with two dominants, only three second-order generalized coordinates whose number must be theoretically infinite.

There were made some preliminary steps towards derivations of modal equations for a spherical tank. For $h/R_0 < 1$ (R_0 is the radius), the natural sloshing modes to be used in algorithm from section 3. were constructed [2]. For higher liquid depths, the main difficulty is the singular behavior of these modes at the contact line formed by Σ_0 and S_0.

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