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EXACT DIFFERENCE SCHEMES FOR THE SYSTEM OF ACOUSTIC EQUATIONS AND ANALYSIS OF RIEMANN PROBLEM

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Анотація. В роботі для рівняння переносу з постійними коефіцієнтами побудовані нові класи точних, монотонних та стійких фінітно-диференціальних схем з часовою та просторовою вагами. Приведено постановку відповідної задачі Рімана. Також побудовані нові класи точних зважених різницевих схем для системи рівнянь акустики та багатовимірного рівняння переносу. Виконано порівняльний аналіз запропонованих схем з добре відомими схемами Лакса та Лакса-Вендроффа.

Abstract. New classes of exact, monotone and stable finite-difference schemes with both time and space weights for the transport equation with constant coefficients were constructed. The mathematical description of Riemann problem was performed. New classes of exact weighted difference schemes for the system of acoustic equations and multidimensional transport equation were constructed. Comparative analysis of the proposed scheme with the well-known Lax and Lax-Wendroff schemes was performed.

MSC 2000: 65L12; 65N06; 65M25

1. Introduction

When constructing and investigating FDS for equations of mathematical physics, a primary concern is the accuracy of these methods. In other words, convergence of the solution y of a discrete problem to an exact solution u of a differential problem should be proved in some norm. It is natural to desire the maximum order of approximation for the minimum-stencil difference scheme.

Definition 1.1 *Exact difference scheme (EDS) is a FDS with zero error of approximation or* $y = u$ *at the grid nodes.*

EDSs represent interesting type of difference schemes. Moreover they can be used to construct efficient schemes.

Let us note some results on EDSs. In [9], under natural conditions the authors show that on an arbitrary grid there exists a unique two-point EDS, i.e., a difference scheme whose solution coincides with the projection onto the grid of the exact solution of the corresponding system of differential equations.

More difficult task is to construct EDSs for PDEs. For some systems of hyperbolic equations the exact difference schemes can be constructed on the basis of the characteristics method. It is worth to mention [6], in which an EDS for a nonlinear PDE with linear advection and an odd-cubic reaction term was constructed. The authors used a method based on the works by R. Mickens [7], [8].

EDS for some types of nonlinear ODEs, one-dimensional and multi-dimensional transport equations, some types of hyperbolic and parabolic equations were constructed in works [1]– [5]. In case of constant coefficients, such numerical methods can be constructed on rectangular grids, while in case of variable coefficients – on moving grids only.

Key words. Exact finite-difference scheme, initial-boundary value problem, transport equation, system of acoustic equations, decay of arbitrary discontinuity

The aim of the present work is construction of a new class of EDSs for the system of acoustic equations, multidimentional transport equation and investigation of Riemann problem in case of initial boundary value problem (IBVP). The advantage of the presented scheme is established in comparison with Lax and Lax-Wendroff schemes. When the Courant number (γ) is equal to 1 they coincide. If $\gamma = 0.1$ the practical shortcomings of conditional approximation of Lax scheme were demonstrated. The disadvantage of the Lax-Wendroff scheme is an appearance of oscillation in the approximate solution of Riemann problem. Presented EDS doesn't have these disadvantages.

2. Transport Equation

In this Section, we construct and analyze new classes of exact, monotone and stable schemes with both time and space weights for transport equation with constant coefficients.

2.1. Problem statement

In the domain $\overline{Q}_T = [0, l] \times [0, T]$ let us consider two IBVPs for linear homogeneous transport equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0,\ 0 < x \le l,\ 0 < t \le T,\ a > 0, \tag{2.1}$$

$$u(x, 0) = u_0(x),\ 0 \le x \le l,\ u(0, t) = \mu_1(t),\ 0 < t \le T \tag{2.2}$$

and

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0,\ 0 \le x < l,\ 0 \le t < T,\ a < 0, \tag{2.3}$$

$$u(x, 0) = u_0(x),\ 0 \le x \le l,\ u(l, t) = \mu_2(t),\ 0 < t \le T. \tag{2.4}$$

A weak solution of the problem (2.1)-(2.2) (or (2.3)-(2.4)) is piecewise continuously-differentiable function $u(x, t)$ that meets the initial and boundary conditions(2.2) (or (2.4)) and an integral conservation law [12, p. 505]

$$\oint\limits_{\partial Q'} udx - audt = 0, \tag{2.5}$$

$\forall\ Q' \subset \overline{Q}_T$. (Here $\partial Q'$ is the bound of the domain Q'.)

Since the solution of the problem (2.1)-(2.2) is constant along the characteristics $\frac{dx}{dt} = a$, so

$$u(x, t) = \begin{cases} u_0(x - at), & 0 < at \le x < l,\ t \le T, \\ \mu_1\left(t - \frac{x}{a}\right), & 0 < x < at \le aT,\ x < l, \end{cases} \quad a > 0. \tag{2.6}$$

Similarly for the solution of the problem (2.3)-(2.4) we get:

$$u(x, t) = \begin{cases} u_0(x - at), & 0 < -at \le l - x < l,\ t \le T, \\ \mu_2\left(t - \frac{x-l}{a}\right), & 0 < l - x < -at \le -aT,\ x > 0, \end{cases} \quad a < 0. \tag{2.7}$$

2.2. Weighted difference scheme

In the domain $\overline{Q}_T$ we introduce uniform rectangular grid

$$\overline{\omega} = \overline{\omega}_h \times \overline{\omega}_\tau,\quad \overline{\omega}_h = \left\{x_i = ih,\ i = \overline{0, N},\ hN = l\right\} = \omega_h \bigcup \{x_0 = 0,\ x_N = l\};$$

$$\overline{\omega}_\tau = \left\{t_n = n\tau,\ n = \overline{0, N_0},\ \tau N_0 = T\right\} = \omega_\tau \bigcup \{t_{N_0} = T\},\quad \omega = \omega_h \times \omega_\tau.$$

Approximating the problem (2.1)-(2.2) we construct a weighted finite difference scheme

$$y_{(\alpha)t} + a y_{\overline{x}}^{(\sigma)} = 0,$$
$$y_i^0 = u_0(x_i), \ i = \overline{0, N}, \ \ y_0^{n+1} = \mu_1(t_{n+1}), \ n = \overline{0, N_0 - 1}, \ a > 0, \tag{2.8}$$
where $\alpha, \sigma \in [0, 1]$.

Similarly to approximate the problem (2.3)-(2.4) we use the following scheme:

$$y_{(+1)(1-\alpha)t} + a y_{x}^{(\sigma)} = 0,$$
$$y_i^0 = u_0(x_i), \ i = \overline{0, N}, \ \ y_N^{n+1} = \mu_1(t_{n+1}), \ n = \overline{0, N_0 - 1}, \ a < 0, \tag{2.9}$$
where $\alpha, \sigma \in [0, 1]$.

Here and bellow we use standard notations of difference schemes theory [11, p.11]. In addition

$$y_i^n = y(x_i, t_n), \ \ y_{(\pm 1)} = y_{i\pm 1}^n, \ \ y_{(\alpha)} = \alpha y_i^n + (1-\alpha) y_{i-1}^n, \ \ y^{(\sigma)} = \sigma y_i^{n+1} + (1-\sigma) y_i^n, \ \ \gamma = \frac{a\tau}{h}.$$

It is easy to show that the schemes (2.8) and (2.9) are exact under the conditions

$$\alpha = 1 - \sigma, \ \ \gamma = 1. \tag{2.10}$$

2.2.1. Monotonicity

Rewrite difference scheme (2.8) in the canonical form [11, p. 243]:

$$(\alpha + \gamma\sigma) y_i^{n+1} = (\alpha + \gamma\sigma - \gamma) y_i^n + (\alpha + \gamma\sigma - 1) y_{i-1}^{n+1} + (1 - \alpha + \gamma(1-\sigma)) y_{i-1}^n. \tag{2.11}$$

For the monotonicity it is necessary to require the positivity of all the coefficients in the equation (2.11). It results in the following condition on weights and grid steps

$$\alpha + \gamma\sigma \geq \max(\gamma, 1),$$

which is equivalent to the following system of inequalities

$$\begin{cases} \gamma \leq 1, & \sigma = 0, \ \alpha = 1, \\ \frac{1-\alpha}{\sigma} \leq \gamma \leq \frac{\alpha}{1-\sigma}, & 0 < \sigma < 1, \ \alpha + \sigma \geq 1, \\ \gamma \geq 1 - \alpha, & \sigma = 1. \end{cases} \tag{2.12}$$

It is easy to show, that under (2.12) the solution of difference scheme (2.8) is stable in the uniform norm and the estimate

$$\|y^{n+1}\|_{\overline{C}} \leq \max\left\{ \max_{k=\overline{1, n+1}} |\mu_1(t_k)|, \ \|u_0\|_{\overline{C}} \right\}$$

holds, where

$$\|y^n\|_{\overline{C}} = \max_{i=\overline{0,N}} |y_i^n|.$$

Analysis of the scheme (2.9) can be done in the same way. When $\alpha = 1, \ \sigma = 0$ the scheme (2.8) is transformed into the well-known upstream scheme

$$y_t + a y_{\overline{x}} = 0$$

with the criterion of the stability $\gamma \leq 1$.

2.2.2. Stability

Fulfilment of the maximum principle and, as a consequence, conditions of monotonicity of the FDS requires strong restrictions on grid steps. Now we will get sufficient conditions of stability of the FDS in energy norm W_2^1 and the uniform norm with weaker restrictions.

Introduce the grid scalar product and norm in the grid functions space:

$$(y,\, v) = \sum_{i=1}^{N} h y_i v_i, \ \ \|y\|^2 = \sum_{i=1}^{N} h y_i^2.$$

Let us prove the following statement.

Theorem 2.1 *Suppose the following condition is true*

$$\alpha + \sigma\gamma \geq \frac{1+\gamma}{2}. \tag{2.13}$$

Then the difference scheme (2.8) *is uniformly stable with respect to initial data and for its solution the estimates*

$$\left\|y_{\overline{x}}^{n+1}\right\| \leq \left\|y_{\overline{x}}^{0}\right\|, \quad \left\|y^{n+1}\right\|_C \leq \sqrt{l}\left\|y_{\overline{x}}^{0}\right\| \tag{2.14}$$

hold.

Proof. Let us rewrite scheme (2.8) in the following form

$$y_{(0.5)t} + h\,(\alpha - 0.5)\,y_{t\overline{x}} + a y_{\overline{x}}^{(0.5)} + a\tau\,(\sigma - 0.5)\,y_{t\overline{x}} = 0$$

and take a scalar product with $2\tau y_{t\overline{x}}$. Since we study the stability with respect to initial data, we can set the boundary condition $y_0^{n+1} = 0$. Then we obtain

$$a\left\|y_{\overline{x}}^{n+1}\right\|^2 + \tau y_{t,N}^2 + 2\tau h\left((\alpha - 0.5) + \gamma\,(\sigma - 0.5)\right)\|y_{t\overline{x}}\|^2 = a\left\|y_{\overline{x}}^{n}\right\|^2.$$

Hence, under the condition (2.13), we get estimates

$$\left\|y_{\overline{x}}^{n+1}\right\|^2 \leq \left\|y_{\overline{x}}^{n}\right\|^2 \leq \ldots \leq \left\|y_{\overline{x}}^{0}\right\|^2.$$

Using these inequalities and embedding theorem ([11, p.118])

$$\|y\|_C \leq \sqrt{l}\,\|y_{\overline{x}}\|$$

the second estimate in (2.14) follows. □

2.3. Exact difference scheme for two-dimensional problem

Consider the Cauchy problem for two-dimensional linear homogeneous transport equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x_1} + b\frac{\partial u}{\partial x_2} = 0, \quad a, b > 0, \quad 0 < t \leq T, \tag{2.15}$$

$$u\,(x,\, 0) = u_0\,(x)\,, \quad x = (x_1,\, x_2) \in \mathbb{R}^2. \tag{2.16}$$

To approximate this equation we introduce the grid

$$\omega = \omega_{h_1} \times \omega_{h_2} \times \omega_\tau, \quad \omega_{h_k} = \{x_{k,i_k} = h_k \cdot i_k,\ i_k = 0, \pm 1, \pm 2, \ldots,\ k = 1, 2\}\,,$$

ω_τ – was defined before. On the grid ω we consider an explicit scheme with two weights

$$y_t + a y_{(\sigma_2)\overline{x}_1} + b y_{(\sigma_1)\overline{x}_2} = 0, \tag{2.17}$$

where

$$y_{(\sigma_1)} = \sigma_1 y_{i_1,i_2} + (1 - \sigma_1)\,y_{i_1-1,i_2}\,, \ \ y_{(\sigma_2)} = \sigma_2 y_{i_1,i_2} + (1 - \sigma_2)\,y_{i_1,i_2-1}\,,$$

$$y_{i_1,i_2}^n = y\,(x_{1,i_1},\, x_{2,i_2},\, t_n)\,, \qquad 0 \leq \sigma_1,\, \sigma_2 \leq 1.$$

Introduce the corresponding Courant numbers along each of the directions

$$\gamma_1 = \frac{ah_1}{\tau}, \qquad \gamma_2 = \frac{ah_2}{\tau}.$$

It is easy to show that the scheme is exact under the conditions

$$\gamma_1 = \gamma_2 = 1, \qquad \sigma_1 + \sigma_2 = 1$$

and conditionally stable in the uniform norm with

$$\gamma_1 \leq 1, \quad \gamma_2 \leq 1, \quad \sigma_1 + \sigma_2 \leq 1.$$

3. System of acoustic equations

In this Section, we introduce the exact solution of the IBVP for acoustic equations system with the boundary conditions of the arbitrary form. The mathematical nature of arbitrary discontinuity decay is analyzed. We propose a new class of exact under $\gamma = 1$ weighted difference schemes. It describes the behavior of discontinuous solution well with the Courant number considerably different from one. The comparative analysis of the scheme proposed with the well-known Lax [15, p.45] and Lax-Wendroff [14, p.74] schemes is presented.

3.1. Problem statement

In the rectangle $\overline{Q}_T$ we consider IBVP for the system of acoustic equations [15, p.45]

$$\frac{\partial v}{\partial t} = a\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial t} = a\frac{\partial v}{\partial x}, \quad a > 0, \tag{3.1}$$

$$\begin{aligned} & u(x,0) = u_0(x), \ v(x,0) = v_0(x), \quad 0 \leq x \leq l, \\ & \alpha_0 v(0,t) + \beta_0 u(0,t) = \mu_1(t), \qquad 0 < t \leq T, \\ & \alpha_l v(l,t) + \beta_l u(l,t) = \mu_2(t), \qquad 0 < t \leq T, \\ & (\alpha_0 - \beta_0)(\alpha_l + \beta_l) \neq 0. \end{aligned} \tag{3.2}$$

3.2. Riemann invariants and exact solution

Two families of characteristics of this system are determined by the ratio

$$\frac{dx}{dt} = \pm a.$$

Along the corresponding characteristics Riemann invariants

$$\begin{aligned} s &= v - u, \\ r &= v + u \end{aligned} \tag{3.3}$$

are constant.

Let us rewrite the problem (3.1)-(3.2) in terms of Riemann invariants:

$$\begin{aligned} \tfrac{\partial s}{\partial t} + a\tfrac{\partial s}{\partial x} &= 0, && 0 < x \leq l, \ 0 < t \leq T, \\ s(x,0) &= v_0(x) - u_0(x), && 0 < x \leq l, \\ s(0,t) &= \nu_1(t), && 0 < t \leq T, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \tfrac{\partial r}{\partial t} - a\tfrac{\partial r}{\partial x} &= 0, && 0 < x \leq l, \ 0 < t \leq T, \\ r(x,0) &= v_0(x) + u_0(x), && 0 < x \leq l, \\ r(l,t) &= \nu_2(t), && 0 < t \leq T, \end{aligned} \tag{3.5}$$

where the functions $\nu_1(t)$ and $\nu_2(t)$ are to be identified. According to (2.6)-(2.7) we get

$$s(x,t)=\begin{cases} v_0(x-at)-u_0(x-at), & 0<at\le x<l, \quad t\le T,\\ \nu_1\left(t-\frac{x}{a}\right), & 0<x<at\le aT,\ x<l\end{cases} \tag{3.6}$$

and

$$r(x,t)=\begin{cases} v_0(x+at)+u_0(x+at), & 0<at\le l-x<l, \quad t\le T,\\ \nu_2\left(t+\frac{x-l}{a}\right), & 0<l-x<at\le aT,\ x>0.\end{cases} \tag{3.7}$$

Now we are to determine $\nu_1(t)$ and $\nu_2(t)$. Let us write all known conditions on the bound

$$\nu_1=v(0,t)-u(0,t), \quad r(0,t)=v(0,t)+u(0,t), \quad \mu_1(t)=\alpha_0 v(0,t)+\beta_0 u(0,t),$$

$$\nu_2=v(l,t)+u(l,t), \quad s(l,t)=v(l,t)-u(l,t), \quad \mu_2(t)=\alpha_l v(l,t)+\beta_l u(l,t).$$

Hence

$$\begin{aligned}\nu_1(t)&=\tfrac{2\mu_1(t)-(\alpha_0+\beta_0)r(0,t)}{\alpha_0-\beta_0},\\ \nu_2(t)&=\tfrac{2\mu_2(t)-(\alpha_l-\beta_l)s(l,t)}{\alpha_l+\beta_l},\end{aligned} \tag{3.8}$$

where $s(l,t)$ and $r(0,t)$ can be found from (3.6) and (3.7) correspondingly.

Exposing the recurrent relation of the equation (3.6)-(3.8) for $\nu_1(t)$ and $\nu_2(t)$ we obtain the formulas

$$\begin{aligned}\nu_1(t)=\;&\sum_{k=0}^{\lfloor\frac{at}{2l}\rfloor}\left(\tfrac{\alpha_0+\beta_0}{\alpha_0-\beta_0}\cdot\tfrac{\alpha_l-\beta_l}{\alpha_l+\beta_l}\right)^k\tfrac{2}{\alpha_0-\beta_0}\mu_1\left(t-2k\tfrac{l}{a}\right)\\ &-\sum_{k=0}^{\lfloor\frac{at-l}{2l}\rfloor}\left(\tfrac{\alpha_0+\beta_0}{\alpha_0-\beta_0}\right)^{k+1}\left(\tfrac{\alpha_l-\beta_l}{\alpha_l+\beta_l}\right)^k\tfrac{2}{\alpha_l+\beta_l}\mu_2\left(t-(2k+1)\tfrac{l}{a}\right)\\ &+\left(\tfrac{\alpha_0+\beta_0}{\alpha_0-\beta_0}\right)^{\lfloor\frac{at+2l}{2l}\rfloor}\left(\tfrac{\alpha_l-\beta_l}{\alpha_l+\beta_l}\right)^{\lfloor\frac{at+l}{2l}\rfloor}f_{\lfloor\frac{at}{2l}\rfloor}(t),\end{aligned} \tag{3.9}$$

$$f_k(t)=\begin{cases}-(v_0(at-kl)-u_0(at-kl)), & k-\text{even},\\ v_0((k+1)l-at)+u_0((k+1)l-at), & k-\text{odd},\end{cases}$$

$$\begin{aligned}\nu_2(t)=\;&\sum_{k=0}^{\lfloor\frac{at}{2l}\rfloor}\left(\tfrac{\alpha_0+\beta_0}{\alpha_0-\beta_0}\cdot\tfrac{\alpha_l-\beta_l}{\alpha_l+\beta_l}\right)^k\tfrac{2}{\alpha_l+\beta_l}\mu_2\left(t-2k\tfrac{l}{a}\right)\\ &-\sum_{k=0}^{\lfloor\frac{at-l}{2l}\rfloor}\left(\tfrac{\alpha_0+\beta_0}{\alpha_0-\beta_0}\right)^{k}\left(\tfrac{\alpha_l-\beta_l}{\alpha_l+\beta_l}\right)^{k+1}\tfrac{2}{\alpha_0-\beta_0}\mu_1\left(t-(2k+1)\tfrac{l}{a}\right)\\ &+\left(\tfrac{\alpha_0+\beta_0}{\alpha_0-\beta_0}\right)^{\lfloor\frac{at+l}{2l}\rfloor}\left(\tfrac{\alpha_l-\beta_l}{\alpha_l+\beta_l}\right)^{\lfloor\frac{at+2l}{2l}\rfloor}g_{\lfloor\frac{at}{2l}\rfloor}(t),\end{aligned} \tag{3.10}$$

$$g_k(t)=\begin{cases}-(v_0((k+1)l-at)+u_0((k+1)l-at)), & k-\text{even},\\ v_0(at-kl)-u_0(at-kl), & k-\text{odd}.\end{cases}$$

Evaluating the solution of the problem (3.1)-(3.2) by the formulas $v=\frac{1}{2}(r+s)$ and $u=\frac{1}{2}(r-s)$ we get

$$v = \frac{1}{2}\begin{cases} v_0(x+at) + u_0(x+at) + \\ +v_0(x-at) - u_0(x-at), & 0 < t \le \min\left(\frac{x}{a}, \frac{l-x}{a}\right), \\ \nu_2\left(t + \frac{x-l}{a}\right) + v_0(x-at) - u_0(x-at), & 0 < \frac{l-x}{a} < t \le \frac{x}{a},\ t \le T, \\ v_0(x+at) + u_0(x+at) + \nu_1\left(t - \frac{x}{a}\right), & 0 < \frac{x}{a} < t \le \frac{l-x}{a},\ t \le T, \\ \nu_2\left(t + \frac{x-l}{a}\right) + \nu_1\left(t - \frac{x}{a}\right), & \max\left(\frac{x}{a}, \frac{l-x}{a}\right) < t \le T, \\ & 0 < x < l, \end{cases}$$

$$u = \frac{1}{2}\begin{cases} v_0(x+at) + u_0(x+at) + \\ -v_0(x-at) + u_0(x-at), & 0 < t \le \min\left(\frac{x}{a}, \frac{l-x}{a}\right), \\ \nu_2\left(t + \frac{x-l}{a}\right) - v_0(x-at) + u_0(x-at), & 0 < \frac{l-x}{a} < t \le \frac{x}{a},\ t \le T, \\ v_0(x+at) + u_0(x+at) - \nu_1\left(t - \frac{x}{a}\right), & 0 < \frac{x}{a} < t \le \frac{l-x}{a},\ t \le T, \\ \nu_2\left(t + \frac{x-l}{a}\right) - \nu_1\left(t - \frac{x}{a}\right), & \max\left(\frac{x}{a}, \frac{l-x}{a}\right) < t \le T, \\ & 0 < x < l, \end{cases} \tag{3.11}$$

where $\nu_1(t)$ and $\nu_2(t)$ can be found from (3.9) and (3.10).

Now we will show, that (3.11) expresses the weak solution of the problem (3.1)-(3.2) (meets an integral conservation law).

Taking half-sum of the equations obtained by using the law (2.5) for the solutions of the problems (3.4) and (3.5), where both of the integrals are taken on the one common arbitrary contour $\partial Q'$, we get:

$$0 = \frac{1}{2}\left(\oint\limits_{\partial Q'} sdx - asdt + \oint\limits_{\partial Q'} rdx + ardt\right) =$$

$$= \oint\limits_{\partial Q'} \frac{r+s}{2}dx + a\frac{r-s}{2}dt = \oint\limits_{\partial Q'} vdx + audt.$$

By analogy taking half-difference gives

$$\oint\limits_{\partial Q'} udx + avdt = 0.$$

Since these equalities are true for functions u and v from (3.11) for arbitrary contour $\partial Q'$, and they are obviously meet the initial and boundary conditions (3.2), so formula (3.11) gives the weak solution of the problem (3.1)-(3.2).

3.3. Riemann problem

Notice, that in transport equations for Riemann invariants (3.4) and (3.5) fluxes are directed to different sides. If the discountinuity in initial data is such that in the same point initial conditions for both of the Riemann invariants are discontinuous at the same time, then the discontinuity in the solution of the problem (3.1)-(3.2) will inevitably decay into two different discontinuities moving to opposite direction in every function. *Thereby, the behavior of the solution is fully determined by the transport of Riemann invariants, where there is no decay of discontinuity* [14, p.80]. It can be shown that in this case continuity of the initial data for one of the invariants is equivalent to fulfilment of Hugoniot conditions.

Hugoniot conditions for this problem can be defined as follows [12, p.507]

$$\begin{cases} D(v_+ - v_-) & = -a(u_+ - u_-), \\ D(u_+ - u_-) & = -a(v_+ - v_-), \end{cases} \tag{3.12}$$

where $f_+ = \lim\limits_{x\to+0} f(x)$, $f_- = \lim\limits_{x\to-0} f(x)$.

Multiplying one of the equations in (3.12) by D, and another by a we get the equivalent system:

$$\begin{cases} \left(D^2 - a^2\right)(v_+ - v_-) = 0, \\ \left(D^2 - a^2\right)(u_+ - u_-) = 0, \\ D(v_+ - v_-) = -a(u_+ - u_-). \end{cases}$$

Assuming that the discontituity exists $\left((v_+ - v_-)^2 + (u_+ - u_-)^2 \neq 0\right)$, we get

$$\begin{cases} |D| = a, \\ D(v_+ - v_-) = -a(u_+ - u_-), \end{cases}$$

that gives 2 different solutions

$$\left[\begin{array}{ll} v_+ + u_+ = v_- + u_-, & D = a, \\ v_+ - u_+ = v_- - u_-, & D = -a, \end{array} \right.$$

or

$$\left[\begin{array}{ll} r_+ = r_-, & D = a, \\ s_+ = s_-, & D = -a. \end{array} \right. \tag{3.13}$$

Hence one of the invariants is continuous and the discontinuity is moving to the flux direction of another invariant. Since all of the conversions were equivalent, so conditions(3.12) are equal to the fulfilment of one of the conditions from (3.13).

The nature of interaction of two discontinuities moving to the opposite directions can be described by simple addition. Discontinuities moving to the same direction cannot interact, because they have the same speed a.

On Fig. 1-3 the exact solution of the Riemann problem is presented. The initial data for this problem doesn't meet Hugoniot conditions.

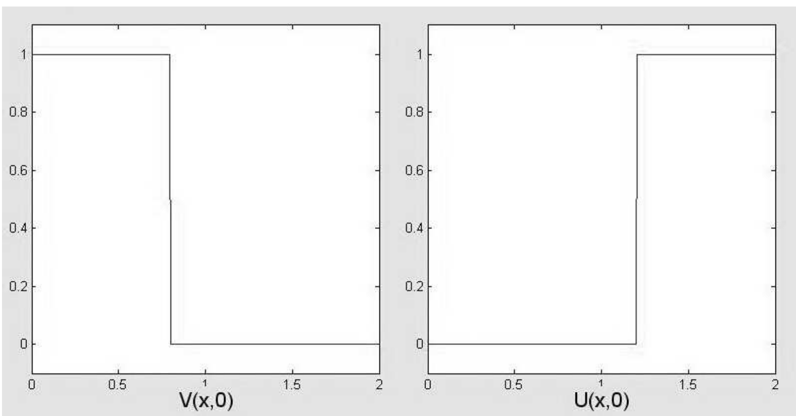

Fig. 1. Distribution of initial data with $t = 0$

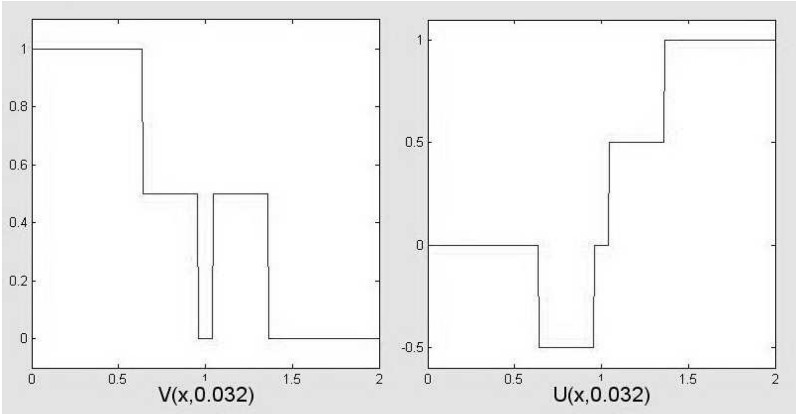

Fig. 2. Cutout of the solution at the moment $t = 0.032$

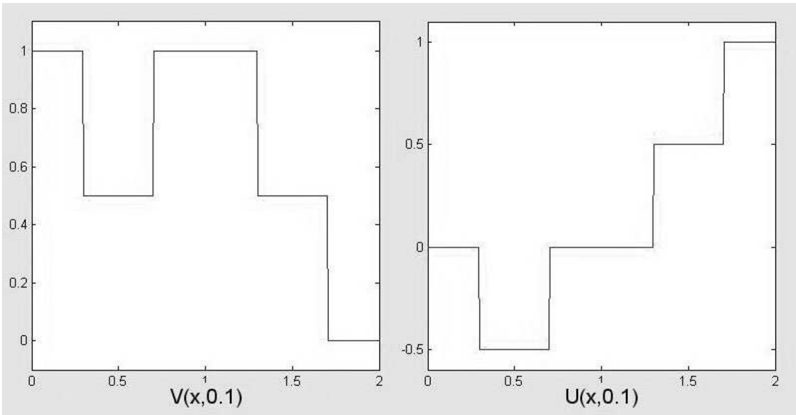

Fig. 3. Cutout of the solution at the moment $t = 0.1$

3.4. Exact difference schemes

3.4.1. Lax Scheme

In case of Cauchy problem Lax scheme is the most well-known

$$\frac{v_{h,i}^{n+1} - 0.5\left(v_{h,i+1}^{n} + v_{h,i-1}^{n}\right)}{\tau} = a\frac{u_{h,i+1}^{n} - u_{h,i-1}^{n}}{2h},$$

$$\frac{u_{h,i}^{n+1} - 0.5\left(u_{h,i+1}^{n} + u_{h,i-1}^{n}\right)}{\tau} = a\frac{v_{h,i+1}^{n} - v_{h,i-1}^{n}}{2h},$$

$$v_{h,i}^{0} = v_0\left(x_i\right),\ u_{h,i}^{0} = u_0\left(x_i\right),\ i \in \mathbb{Z},$$

which is exact under $\gamma = 1$ and stable under $\gamma \leq 1$. Disadvantage of this scheme is its conditional approximation $O\left(h^2 + \tau + \frac{h^2}{\tau}\right)$.

3.4.2. Lax-Wendroff scheme

For numerical solution of Cauchy problem Lax-Wendroff scheme is often used as well:

$$\begin{gathered} v_{h,t} = au_{h,\overset{\circ}{x}} + \frac{a^2\tau}{2}v_{h,\bar{x}x},\quad u_{h,t} = av_{h,\overset{\circ}{x}} + \frac{a^2\tau}{2}u_{h,\bar{x}x}, \\ u_{h,i}^{0} = u_0\left(x_i\right),\ v_{h,i}^{0} = u_0\left(x_i\right),\ \ i \in \mathbb{Z}, \end{gathered} \tag{3.14}$$

which is a scheme of second order of approximation $O\left(h^2 + \tau^2\right)$, exact and monotone only under $\gamma = 1$. Disadvantage of the scheme (3.14) is absence of the monotonicity under $\gamma \neq 1$. Same as all of the explicit difference schemes, it is stable under the fulfilment of the Courant criterion $\gamma \leq 1$.

3.4.3. Explicit scheme with viscosity

Bellow we introduce new classes of difference schemes for IBVP for the system of acoustic equations, which are exact under $\gamma = 1$ and save the property of stability and monotonicity under $\gamma \neq 1$.

Consider the scheme for the problem (3.1)-(3.2) on the grid $\overline{\omega}$:

$$v_{h,t} = au_{h,\overset{\circ}{x}} + \tfrac{ah}{2} v_{h,\overline{x}x}, \quad u_{h,t} = av_{h,\overset{\circ}{x}} + \tfrac{ah}{2} u_{h,\overline{x}x},$$
$$i = \overline{1, N-1}, \ n = \overline{0, N_0 - 1}, \tag{3.15}$$

$$u^0_{h,i} = u_0(x_i), \quad v^0_{h,i} = u_0(x_i), \ i = \overline{0, N}, \tag{3.16}$$

$$\alpha_0 v^{n+1}_{h,0} + \beta_0 u^{n+1}_{h,0} = \mu_1(t_{n+1}), \quad (v_h + u_h)_{0,t} - a(v_h + u_h)_{0,x} = 0,$$
$$\alpha_l v^{n+1}_{h,N} + \beta_l u^{n+1}_{h,N} = \mu_2(t_{n+1}), \quad (v_h - u_h)_{N,t} + a(v_h - u_h)_{N,\overline{x}} = 0. \tag{3.17}$$

Second and fourth of the equations in (3.17) were obtained by taking half-sum and half-difference of the equations (3.15) correspondingly.

Let us show that the difference scheme (3.15) for the problem (3.1)-(3.2) is exact under $\gamma = 1$. Rewrite first equality from (3.15) in the index form:

$$v^{n+1}_{h,i} - v^n_{h,i} = \frac{a\tau}{2h}\left(u^n_{h,i+1} - u^n_{h,i-1} + v^n_{h,i+1} + v^n_{h,i-1} - 2v^n_{h,i}\right).$$

Under the condition $\gamma = 1$ it gives

$$v^{n+1}_{h,i} = \frac{1}{2}\left(u^n_{h,i+1} - u^n_{h,i-1} + v^n_{h,i+1} + v^n_{h,i-1}\right).$$

Since for the exact solution $r^n_{i+1} = r^{n+1}_i$, $s^n_{i-1} = s^{n+1}_i$, so

$$u^n_{i+1} - u^n_{i-1} + v^n_{i+1} + v^n_{i-1} = v^{n+1}_i + u^{n+1}_i + v^{n+1}_i - u^{n+1}_i = 2v^{n+1}_i,$$

which means that the first equality of the scheme (3.15) is exact.

The exactness of the second equality from (3.15), second and fourth equalities from (3.17) can be shown the very same way.

Approximation, stability and monotonicity are important properties of the FDS. It is not hard to show that this scheme approximates the corresponding problem with the first order $O(h + \tau)$. Also from the maximum principle it follows that the scheme is uniformly stable and monotone under $\gamma \leq 1$.

3.4.4. Scheme, weighted along the time and space

Defining $\overset{\alpha}{\Lambda} u^n_i = \frac{1}{2}\left(2\alpha u^n_i + (1-\alpha) u^n_{i-1} + (1-\alpha) u^n_{i+1}\right)$, consider the scheme

$$\overset{\alpha}{\Lambda} v_{h,t} + (1-\alpha) h u_{h,t\overset{\circ}{x}} = a u^{(\sigma)}_{h,\overset{\circ}{x}} + \tfrac{ah}{2} v^{(\sigma)}_{h,\overline{x}x},$$
$$\overset{\alpha}{\Lambda} u_{h,t} + (1-\alpha) h v_{h,t\overset{\circ}{x}} = a v^{(\sigma)}_{h,\overset{\circ}{x}} + \tfrac{ah}{2} u^{(\sigma)}_{h,\overline{x}x}, \tag{3.18}$$
$$i = \overline{1, N-1}, \ n = \overline{0, N_0 - 1},$$

$$u^0_{h,i} = u_0(x_i), \quad v^0_{h,i} = u_0(x_i), \ i = \overline{0, N}, \tag{3.19}$$

$$\alpha_0 v^{n+1}_{h,0} + \beta_0 u^{n+1}_{h,0} = \mu_1(t_{n+1}), \quad (v_h + u_h)_{1,(1-\alpha)t} - a(v_h + u_h)^{(\sigma)}_{0,x} = 0,$$
$$\alpha_l v^{n+1}_{h,N} + \beta_l u^{n+1}_{h,N} = \mu_2(t_{n+1}), \quad (v_h - u_h)_{N,(\alpha)t} + a(v_h - u_h)^{(\sigma)}_{N,\overline{x}} = 0. \tag{3.20}$$

To find the numerical solution the matrix elimination method can be used [16, p. 106].

It is not hard to show that the scheme approximates the corresponding problem with the order $O\left(\left(\alpha-\frac{1}{2}\right)h+\left(\sigma-\frac{1}{2}\right)\tau+h^2+\tau^2\right)$. Also it can be shown that the scheme is exact under (2.10).

3.4.5. Uniform stability of the weighted schemes family

For simplicity consider IBVP (3.1)-(3.2) in the case, when the Riemann invariants are given on the bound ($\alpha_0=-\beta_0=\alpha_l=\beta_l=1$).

Theorem 3.1 *Let the following condition be true*

$$\alpha+\sigma\gamma\geq\max\left(1,\,\gamma\right). \tag{3.21}$$

Then the difference scheme (3.18)-(3.20) *is uniformly stable with respect to initial data and boundary conditions and for its solution a prior estimate*

$$\begin{aligned}&\max\left(\left\|v_h^{n+1}\right\|_{\overline{C}},\,\left\|u_h^{n+1}\right\|_{\overline{C}}\right)\leq\\&\leq\max\left(\left\|v_h^0\right\|_{\overline{C}}+\left\|u_h^0\right\|_{\overline{C}},\,\frac{1}{2}\left(\max_{k=\overline{1,n+1}}\left|\mu_1\left(t_k\right)\right|+\max_{k=\overline{1,n+1}}\left|\mu_2\left(t_k\right)\right|\right)\right)\end{aligned}$$

holds.

Proof. First we bound the norm of the solution by the norm of Riemann invariants from the same time layer:

$$\begin{aligned}\max\left(\left\|v_h^n\right\|_{\overline{C}},\,\left\|u_h^n\right\|_{\overline{C}}\right)&=\max\left(\left|v_{h,i'}^n\right|,\,\left|u_{h,i''}^n\right|\right)\\&=\max\left(\left|v_{h,\bar{i}}^n\right|,\,\left|u_{h,\bar{i}}^n\right|\right)\big|_{\bar{i}\in\{i',i''\}}\\&=0.5\left(\left|v_{h,\bar{i}}^n+u_{h,\bar{i}}^n\right|+\left|v_{h,\bar{i}}^n-u_{h,\bar{i}}^n\right|\right)\\&\leq0.5\left(\left\|v_h^n+u_h^n\right\|_{\overline{C}}+\left\|v_h^n-u_h^n\right\|_{\overline{C}}\right).\end{aligned}$$

For the Riemann invariants we have the following expression:

$$\left(\alpha+\gamma\sigma\right)r_i^{n+1}=\begin{cases}\left(\alpha+\gamma\sigma-\gamma\right)r_i^n+\left(\alpha+\gamma\sigma-1\right)r_{i+1}^{n+1}+\\+\left(1-\alpha+\gamma\left(1-\sigma\right)\right)r_{i+1}^n, & i=\overline{0,N-1},\\\left(\alpha+\gamma\sigma\right)\mu_2\left(t_{n+1}\right), & i=N,\end{cases}$$

$$\left(\alpha+\gamma\sigma\right)s_i^{n+1}=\begin{cases}\left(\alpha+\gamma\sigma-\gamma\right)s_i^n+\left(\alpha+\gamma\sigma-1\right)s_{i-1}^{n+1}+\\+\left(1-\alpha+\gamma\left(1-\sigma\right)\right)s_{i-1}^n, & i=\overline{1,N},\\\left(\alpha+\gamma\sigma\right)\mu_1\left(t_{n+1}\right), & i=0.\end{cases}$$

Hence, under the condition (3.21), the estimate follows

$$\left\|v_h^{n+1}+u_h^{n+1}\right\|_{\overline{C}}\leq\max\left\{\max_{k=\overline{1,n+1}}\mu_2\left(t_k\right),\,\left\|v_h^0+u_h^0\right\|_{\overline{C}}\right\},$$

$$\left\|v_h^{n+1}-u_h^{n+1}\right\|_{\overline{C}}\leq\max\left\{\max_{k=\overline{1,n+1}}\mu_1\left(t_k\right),\,\left\|v_h^0-u_h^0\right\|_{\overline{C}}\right\}.$$

It is obvious that

$$\frac{1}{2}\left(\left\|v_h^0-u_h^0\right\|_{\overline{C}}+\left\|v_h^0+u_h^0\right\|_{\overline{C}}\right)\leq\left\|v_h^0\right\|_{\overline{C}}+\left\|v_h^0\right\|_{\overline{C}}.$$

This completes the proof of the theorem. □

Remark 3.2 Under $\alpha=1$, $\sigma=0$ the scheme (3.18)-(3.20) transformes into the scheme (3.15)-(3.17), and monotonicity and stability condition (3.21) - into the Courant criterion $\gamma\leq1$.

3.5. Numerical experiments

Numerical experiments were made for the problem (3.1)-(3.2) with the initial data

$$u\,(x,\,0)=0,\;0\le x\le 2,\;v\,(x,\,0)=\begin{cases}1, & 0\le x<1,\\ 0, & 1<x\le 2,\end{cases} \tag{3.22}$$

$$v\,(0,\,t)=1,\;v\,(2,t)=0,\;0\le t\le 0.5,\;a=1.$$

Graphic presentation of the initial data is shown on the Fig. 4.

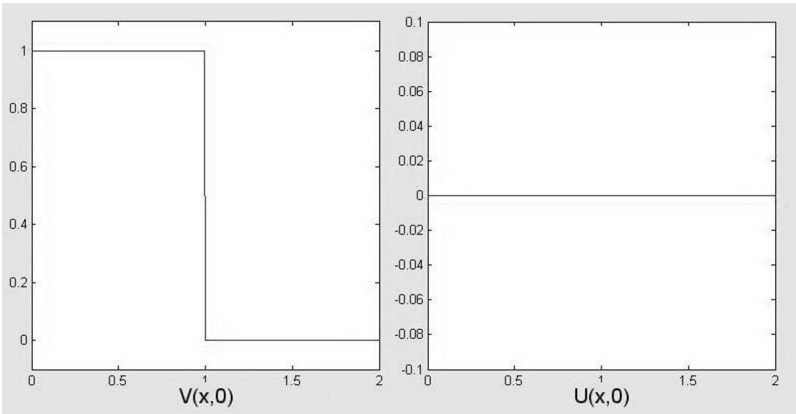

Fig. 4. Initial data

3.5.1. Lax scheme with $\gamma = 0.8$

Numerical experiment was made for the problem (3.1)-(3.2) with initial data (3.22) and grid steps

$$h=5\cdot 10^{-3},\;\tau=4\cdot 10^{-3},\;\text{with the Courant number }\gamma=0.8. \tag{3.23}$$

The result is shown on the Fig. 5.

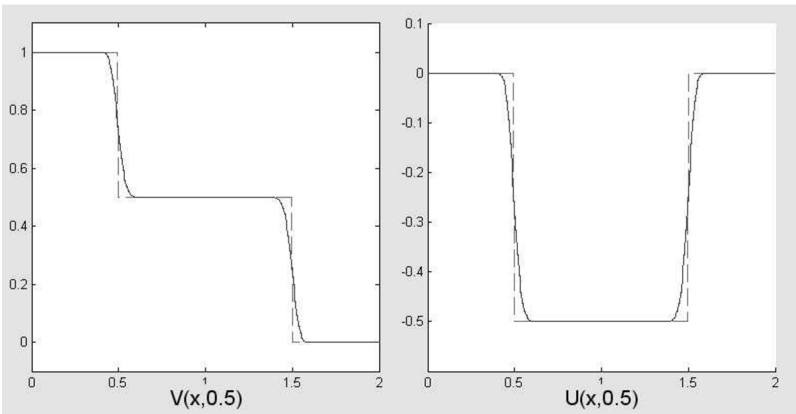

Fig. 5. The Lax scheme with the Courant number $\gamma = 0.8$

3.5.2. The Lax Scheme with $\gamma = 0.1$

Numerical experiment was made for the problem (3.1)-(3.2) with initial data (3.22) and grid steps

$$h=1\cdot 10^{-2},\;\tau=1\cdot 10^{-3},\;\text{with the Courant number }\gamma=0.1. \tag{3.24}$$

This experiment practicaly demonstrates the disadvantage of the conditional approximation. The result is shown on the Fig. 6.

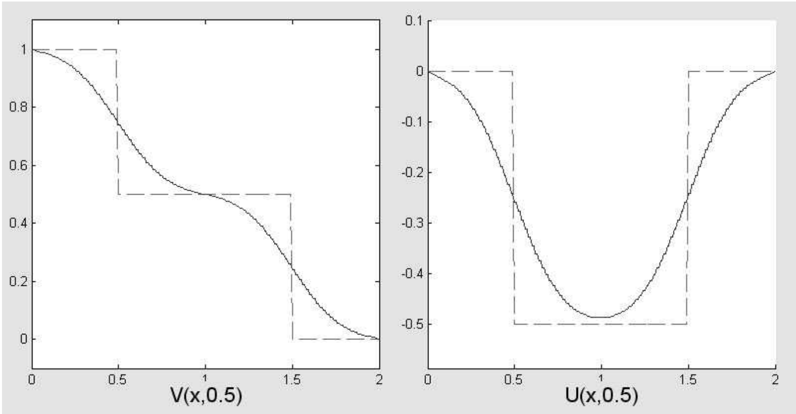

Fig. 6. The Lax scheme with the Courant number $\gamma = 0.1$

3.5.3. Lax-Wendroff scheme with $\gamma = 0.8$

Numerical experiment was made for the problem (3.1)-(3.2) with initial data (3.22) and grid steps (3.23). The result is shown on the Fig. 7.

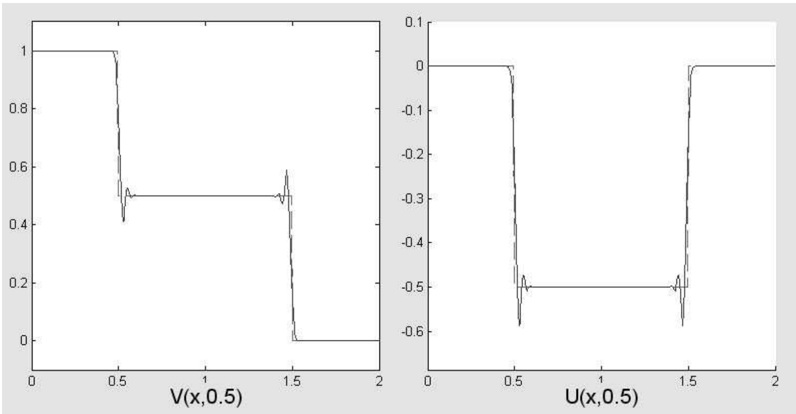

Fig. 7. The Lax-Wendroff scheme with the Courant number $\gamma = 0.8$

3.5.4. Explicit sceme (3.15) with viscosity with $\gamma = 0.8$

Numerical experiment was made for the problem (3.1)-(3.2) with initial data (3.22) and grid steps (3.23). The result is shown on the Fig. 8.

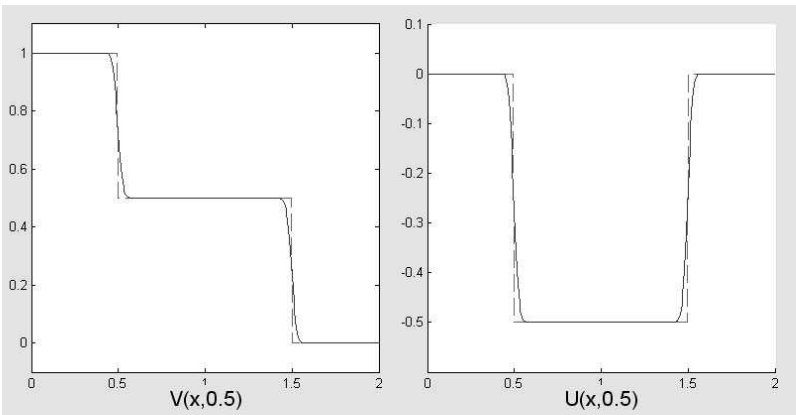

Fig. 8. The explicit sceme with viscosity with the Courant number $\gamma = 0.8$

3.5.5. Explicit sceme (3.15) with viscosity with $\gamma = 0.1$

Numerical experiment was made for the problem (3.1)-(3.2) with initial data (3.22) and grid steps (3.24). The result is shown on the Fig. 9.

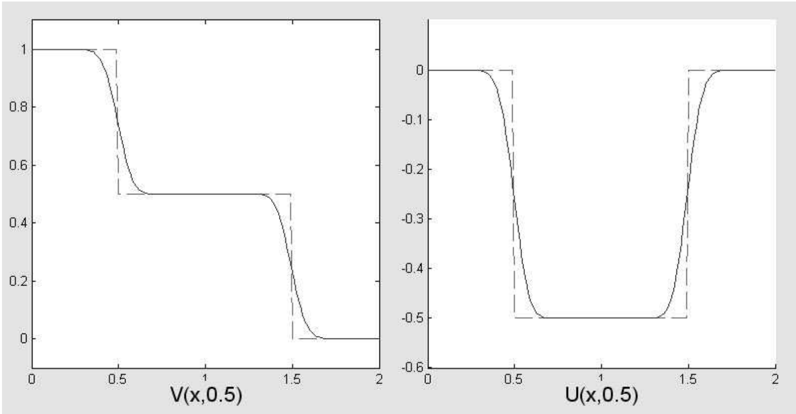

Fig. 9. The explicit sceme with viscosity with the Courant number $\gamma = 0.1$

3.5.6. Weighted scheme (3.18) with $\gamma = 1.25$

Numerical experiment was made for the problem (3.1)-(3.2) with initial data (3.22) and grid steps

$$h = 4 \cdot 10^{-3},\ \tau = 5 \cdot 10^{-3},\ \text{with the Courant number} \gamma = 1.25.$$

Parameters of the scheme $\sigma = 0.5$ and $\alpha = 0.625$ are taken so, that the condition of monotonicity (3.21) is true. The result is shown on the Fig. 10.

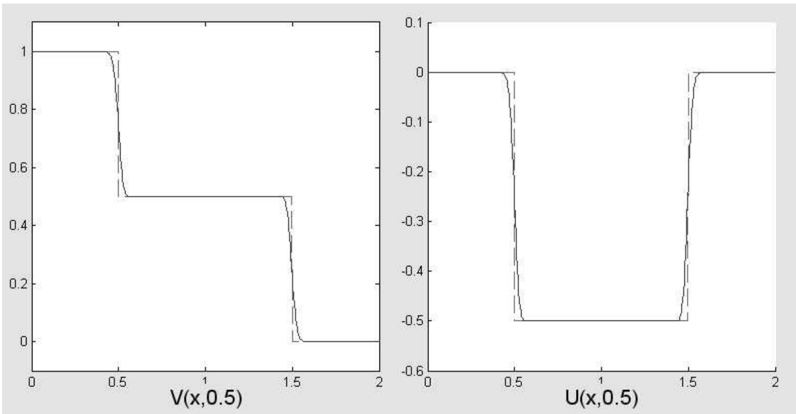

Fig. 10. Weighted scheme with the Courant number $\gamma = 1.25$

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