Журнал обчисл. прикл. матем.
Вип. 2 (105), 2011, стор. 98-110

J. Numer. Appl. Math.
No. 2 (105), 2011, pp. 98-110

FLAT VARIANT OF SUBSTANTIALLY SPATIAL PROBLEM OF ELECTROSTATICS AND SOME ASPECTS OF ITS SOLUTION, RELATED TO SPECIFICS OF INPUT INFORMATION

UDC 518:517.948

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Анотація. В роботі розглядається математична модель задачі, яка описує плоске електростатичне поле. Будується еквівалентне інтегральне рівняння першого роду. Досліджується проблема адитивної сталої у випадку наявності симетрії в граничних поверхнях. Приводяться чисельні розрахунки.

Abstract. Taking into account the specific characters of initial boundary value problem a mathematical model, which describes so-called flat electrostatic field is considered. In this connection the main attention is given to the equivalence of last one to the integral equation of the first kind and the problem of additive constant calculation. This constant appears in the integral representation of flat electrostatic field. There is shown that the constant mentioned above is easily calculated in the presence of symmetry disposition of boundary surface constituents. The proposed concept is illustrated by numerical solving of some model tasks.

1. Introduction

The problem of investigation of the electrostatic field, created by the system of charged electrodes, appears in the process of cathode-ray devices planning. If charged electrodes are modeled by infinitely long cylindrical surfaces, elements of which are infinitely thin and evenly charged on filament length and parallel to one of co-ordinate axes, then in a cut with an arbitrary plane, perpendicular to this axis, some number of the open circuit arcs appears. Under such conditions, the potential value of the investigated field in the arbitrary point of space does not depend on one co-ordinate. So necessary calculations are enough to conduct only in $\mathbb{R}^2$. Thus, the examined spatial problem should be interpreted as flat [1]. It is also noticed [2] that solving the problem in the substantially spatial setting at predominance of one geometrical constituent of surfaces over the other one the value of potential in the corresponding transversal cuts of the system, close to the central ones, slightly changes. Therefore, to show the high-quality picture of the field in the central transversal cuts of such electron-optical systems it is possible to confine the research only to the flat cuts of spatial constructions. Taking into account the concept mentioned above, we will concentrate our attention on the certain aspects of a mathematical model, which describes as so-called flat electrostatic field.

Let us suppose that $L := \bigcup\limits_{j=1}^{\nu} L_j$ is a combination of finite number of smooth, open-circuit and bounded arcs L_j on a plane $\mathbb{R}^2$, which do not have common points. We will designate x, y, ... as points in $\mathbb{R}^2$, $|x-y|$ as a distance between x and y, and x_m^* $(m = 2j-1, 2j;\ j = 1, 2, ..., \nu)$ as extreme points of the arc L_j. Let us also suppose that

$$\overline{L} := L \cup \{x_1^*, x_2^*, ..., x_{2\nu}^*\}.$$

Each of two sides of L we will consider as positive or negative, accordingly, depending on direction of normal to L . If, for example, a function $f(x)$ has finite limit, when $x \overline{\in} \overline{L}$ goes to a point x_0 on L from a positive(negative) side, then we will say that $f(x)$ is continuous from a positive(negative) side, and the limit we will designate as $f^+(x_0)$ $\left(f^-(x_0)\right)$.

Thus, at the mathematical modeling of the problem in the general setting it is necessary to find a function $U \in C^2\left(\mathbb{R}^2 \backslash \overline{L}\right)$, which together with the derivatives of the first order is continuous from a positive(negative) side L and satisfies

Key words. Initial boundary value problem, integral equation of the first kind, additive constant, symmetry disposition of boundary surface.

- two-dimensional Laplace equation

$$\Delta U = 0 \quad \text{in} \quad \mathbb{R}^2 \backslash \overline{L}; \tag{1.1}$$

- boundary conditions

$$U^{\pm}(x) = g(x), \quad x \in L, \tag{1.2}$$

where $g(x)$ is a known function, set on L, which, in our case, is constant;

- condition of the boundedness on infinity

$$U(\infty) = C, \tag{1.3}$$

and also

- "condition on an edge" [3]

$$\lim_{\rho \to 0} \sum_{m=1}^{2\nu} \int\limits_{C_m^*(\rho)} \left| \frac{\partial U(y)}{\partial \rho} \right| \mathrm{d}s_y = 0, \tag{1.4}$$

where $C_m^*(\rho) := \left\{ x \in \mathbb{R}^2 \middle| \ |x - x_m^*| = \rho \right\} \backslash L$.

Taking into account that $\Psi(x, y) := \dfrac{1}{2\pi} \ln \dfrac{1}{|x-y|}$ is a fundamental solution of two-dimensional Laplace equation (1.1), we will consider a point of the equivalence of the problem (1.1)–(1.4) to a certain integral equation.

Among the boundary problems of potential theory in $\mathbb{R}^2$ it is possible to distinguish the classes of problems, which have the Abelian group of symmetry of a certain finite order. Assuming that a boundary L owns the Abelian group of symmetry of finite order, in the process of numeral solving of the problem (1.1)–(1.4) on the basis of the integral equation method it is possible to use the apparatus of the groups theory [4,5]. In this context, the results about so-called additive constant are specified, which appears in integral presentation of the basic problem solution. For the calculation of the last one an effective formula is offered.

2. Equivalence of the problem to the integral equation

Let us formulate and prove a theorem on the equivalence of the initial problem (1.1)–(1.4) to integral equation of the first kind.

Theorem 2.1 *If the solution $U(x)$ of the problem (1.1)–(1.4) exists, then it can be shown as*

$$U(x) = \int\limits_L \Psi(x, y)\, \tau(y) ds_y + C, \quad x \in \mathbb{R}^2 \backslash \overline{L}, \tag{2.1}$$

where $\tau(y)$ satisfies such integral equation of the first kind as

$$\int\limits_L \Psi(x, y)\, \tau(y) ds_y = g(x) - C, \quad x \in L. \tag{2.2}$$

On the contrary, if the function $U(x)$ is defined by (2.1), where $\tau(y)$ and constant C satisfy the system

$$\begin{cases} \int\limits_L \Psi(x, y)\, \tau(y) ds_y = g(x) - C, \quad x \in L, \\ \int\limits_L \tau(y) ds_y = 0, \end{cases} \tag{2.3}$$

then $U(x)$ is a solution of the problem (1.1)–(1.4).

Proof. Let us use the methodology, introduced in the work [3]. Let us suppose that $z_j := \hat{c}^*_{2j-1}(\rho) \cup \hat{c}^*_{2j}(\rho) \cup L_j^{(+)} \cup L_j^{(-)}$ $\left(j = \overline{1, \nu}\right)$ is simple, closed contour which envelops L_j so that $\hat{c}^*_m(\rho)$ $(m = 2j-1,\, 2j)$ is the arc of circle of radius ρ with a center in a point x^*_m, and $L_j^{(+)}\left(L_j^{(-)}\right)$ is an open arc, placed in parallel to L_j in the distance $\rho' < \rho$ from a positive(negative) side L_j. Let us designate S_ρ as the union domains, bounded by the closed contours z_j, which contain L_j.

Let us suppose that x is the arbitrary fixed point in $\mathbb{R}^2 \backslash \overline{L}$, $T := \left\{y \in \mathbb{R}^2 \middle|\ |y-x| \le R\right\}$, where R is so large that S_ρ is fully contained in T. We will apply the second Green formula [6] for a function $U(y)$, which is a solution of the problem (1.1)–(1.4), and the function $\Psi(x,y)$ in the domain, bounded by $\bigcup\limits_{j=1}^{\nu} z_j$ and two circles $\Sigma_\varepsilon := \left\{y \in T \backslash \overline{S}_\rho \middle|\, |y-x| = \varepsilon\right\}$ and $\Sigma_R := \left\{y \in \mathbb{R}^2 \middle|\, |y-x| = R\right\}$. We should notice that $\varepsilon > 0$, according to the logic of proving, is an arbitrary small number.

Then, directing ρ' to zero we will get

$$\iint\limits_{T\backslash\left(K_\varepsilon \cup \overline{S}_\rho\right)} [\Psi(x,y)\Delta U(y) - \Delta\Psi_y(x,y)U(y)]\mathrm{d}\sigma_y = \int\limits_{\Sigma_R} \left[\Psi(x,y)\frac{\partial U(y)}{\partial n} - U(y)\times\right.$$

$$\left.\times\frac{\partial \Psi(x,y)}{\partial n_y}\right]\mathrm{d}s_y + \int\limits_{\Sigma_\varepsilon}\left[\Psi(x,y)\frac{\partial U(y)}{\partial n} - U(y)\frac{\partial \Psi(x,y)}{\partial n_y}\right]\mathrm{d}s_y + \sum_{m=1}^{2\upsilon} U^*_m(x,\rho) +$$

$$+\sum_{j=1}^{\upsilon}\int\limits_{L'_j}\Psi(x,y)\tau(y)\mathrm{d}s_y - \sum_{j=1}^{\upsilon}\int\limits_{L'_j}\frac{\partial\Psi(x,y)}{\partial n_y}\left[U^-(y) - U^+(y)\right]\mathrm{d}s_y, \tag{2.4}$$

where

$$K_\varepsilon := \left\{\, y \in T\backslash\overline{S}_\rho \middle|\, |y-x| \le \varepsilon\right\},\ L'_j := L_j \backslash \sum_{m=2j-1}^{2j}\left\{\, y \in L_j \middle|\, |y - x^*_m| \le \rho\right\},$$

$$U^*_m(x,\rho) := \int\limits_{c^*_m(\rho)}\left[\Psi(x,y)\frac{\partial U(y)}{\partial n} - U(y)\frac{\partial\Psi(x,y)}{\partial n_y}\right]\mathrm{d}s_y,$$

and $\tau(y) := \left(\dfrac{\partial U(y)}{\partial n}\right)^- - \left(\dfrac{\partial U(y)}{\partial n}\right)^+$.

In the relation (2.4) we will consider integrals, which depend on ε. It obvious that

$$\iint\limits_{T\backslash\left(K_\varepsilon\cup\overline{S}_\rho\right)} U(y)\Delta\Psi_y(x,y)\mathrm{d}\sigma_y = 0,$$

as $\Delta_y\Psi(x,y) = 0$ under condition $x \ne y$. In accordance with a definition of improper integral

$$\lim_{\varepsilon\to 0}\iint\limits_{T\backslash\left(K_\varepsilon\cup\overline{S}_\rho\right)}\Psi(x,y)\,\Delta U(y)\mathrm{d}\sigma_y = \iint\limits_{T\backslash\overline{S}_\rho}\Psi(x,y)\Delta U(y)\mathrm{d}\sigma_y = 0,$$

because $\Delta U(y) = 0$ at any $y \in \mathbb{R}^2\backslash\overline{L}$.

Then,

$$-\int\limits_{\Sigma_\varepsilon} U(y)\frac{\partial\Psi(x,y)}{\partial n_y}\mathrm{d}s_y = -\frac{1}{2\pi}\int\limits_{\Sigma_\varepsilon} U(y)\frac{\partial}{\partial n_y}\ln\frac{1}{|y-x|}\mathrm{d}s_y =$$

$$= -\frac{1}{2\pi\varepsilon}\int\limits_{\Sigma_\varepsilon} U(y)\mathrm{d}s_y = -\frac{1}{2\pi\varepsilon}\times 2\pi\varepsilon U^* = -U^*,$$

as $\dfrac{\partial}{\partial n_y}\ln\dfrac{1}{|y-x|} = -\dfrac{\partial}{\partial n_y}\ln|y-x| = \dfrac{\partial}{\partial r}\ln r = \dfrac{1}{r},\ \dfrac{1}{r}\Big|_{r=\varepsilon} = \dfrac{1}{\varepsilon}$, and U^* – mean value of function $U(y)$ on the circle Σ_ε. Taking into account the continuity of $U(y)$, when $\varepsilon \to 0$ $U^* \to U(x)$.

Similarly,

$$\int\limits_{\Sigma_\varepsilon} \Psi(x,y)\frac{\partial U(y)}{\partial n}\mathrm{d}s_y = \frac{1}{2\pi}\int\limits_{\Sigma_\varepsilon} \ln\frac{1}{|y-x|}\frac{\partial U(y)}{\partial n}\mathrm{d}s_y =$$

$$= -\frac{1}{2\pi}\ln\varepsilon\int\limits_{\Sigma_\varepsilon}\frac{\partial U(y)}{\partial n}\mathrm{d}s_y = -\frac{1}{2\pi}\ln\varepsilon\times 2\pi\varepsilon\left(\frac{\partial U}{\partial n}\right)^* = -\varepsilon\ln\varepsilon\left(\frac{\partial U}{\partial n}\right)^*,$$

where $\left(\frac{\partial U}{\partial n}\right)^*$ is mean value of function $U(y)$ normal derivative on the circle Σ_ε. At that time $\lim\limits_{\varepsilon\to 0}\varepsilon\ln\varepsilon\left(\frac{\partial U}{\partial n}\right)^* = 0$, as the first partial derivatives of function $U(y)$ are continuous in $\mathbb{R}^2\backslash\overline{L}$ and , therefore, a value $\left(\frac{\partial U}{\partial n}\right)^*$ is limited.

Let us estimate integrals in (2.4), which depend on R. It is obvious that

$$-\int\limits_{\Sigma_R} U(y)\frac{\partial\Psi(x,y)}{\partial n_y}\mathrm{d}s_y = -\frac{1}{2\pi}\int\limits_{\Sigma_R} U(y)\frac{\partial}{\partial n_y}\ln\frac{1}{|y-x|}\mathrm{d}s_y =$$

$$= -\frac{1}{2\pi}\int\limits_{\Sigma_R} U(y)\frac{\partial}{\partial R}\ln\frac{1}{R}\mathrm{d}s_y = \frac{1}{2\pi R}\int\limits_{\Sigma_R} U(y)\mathrm{d}s_y.$$

Under the condition of tending R to infinity and taking into account (1.3), it easy to notice, that $\frac{1}{2\pi R}\int\limits_{\Sigma_R} U(y)\mathrm{d}s_y \to C$.

Then, $\int\limits_{\Sigma_R}\Psi(x,y)\frac{\partial U(y)}{\partial n}\mathrm{d}s_y = -\frac{1}{2\pi}\int\limits_{\Sigma_R}\ln|y-x|\frac{\partial U(y)}{\partial n}\mathrm{d}s_y = -\frac{1}{2\pi}\ln R\int\limits_{\Sigma_R}\frac{\partial U(y)}{\partial n}\mathrm{d}S_R$. As $U(y)$ is a harmonic function on infinity, then in the vicinity of infinitely distant point of plane the estimation [6] is true

$$\left|\frac{\partial U(y)}{\partial n}\right| \le \frac{A}{R^2}, \quad R = |y-x|.$$

Then, under large enough R

$$\left|\int\limits_{\Sigma_R}\Psi(x,y)\frac{\partial U(y)}{\partial n}\mathrm{d}s_y\right| \le \frac{A\ln R}{R} \to 0, \quad R\to\infty.$$

From the formula (2.4), at $R\to\infty$, $\varepsilon\to 0$, basing on the information above and considering the conditions (1.2), we will get

$$U(x) = \sum_{j=1}^{\upsilon}\int\limits_{L'_j}\Psi(x,y)\tau(y)\mathrm{d}s_y + \sum_{m=1}^{2\upsilon} U^*_m(x,\rho) + C. \tag{2.5}$$

Let us consider the values $U^*_m(x,\rho)$, taking into account that $\rho > 0$ is an arbitrarily small number. As $\Psi(x,y)$ is continuously differentiable at $y\neq x$, then, $|\Psi(x,y)|$ and $\left|\frac{\partial\Psi(x,y)}{\partial n_y}\right|$ are bounded for any $y\in C^*_m(\rho)$. Therefor, $|U^*_m(x,\rho)| \le B\int\limits_{C^*_m(\rho)}\left[|U(y)| + \left|\frac{\partial U(y)}{\partial\rho}\right|\right]\mathrm{d}s_y$, where $B>0$ is a certain constant. At realization of the condition (1.4) $\lim\limits_{\rho\to 0} U^*_m(x,\rho) = 0$.

As a result of an assumption, related to the existence of the problem (1.1)–(1.4) solution, the expression in the right part (2.5), which exists at arbitrary ρ, will converge to the given $U(x)$, when $\rho\to 0$. Taking it into account, we will get

$$U(x) = \sum_{j=1}^{v} \int_{L_j} \Psi(x,y)\tau(y)\mathrm{ds}_y + C,$$

or using abridged notation,

$$U(x) = \int_L \Psi(x,y)\tau(y)\,\mathrm{ds}_y + C.$$

Finally, we will get an integral equation (2.2). For this purpose, in (2.1) we will direct a point x to the point $x_0 \in L$ from a positive or negative side. Taking into account the properties of a simple layer potential [6], we will get

$$U^{\pm}(x_0) = \int_L \Psi(x_0,y)\tau(y)\mathrm{ds}_y + C, \tag{2.6}$$

and the use of boundary conditions (1.2) results in

$$\int_L \Psi(x_0,y)\tau(y)\mathrm{ds}y = g(x_0) - C.$$

Thus, the integral equation is obtained and the first part of theorem is proven.

On the contrary, we will suppose that $U(x)$ is given by the right part (2.1), and $\tau(y)$ i C satisfy the system (2.3). Obviously, that $U(x)$ is a solution to (1.1). As boundary values of $U(x)$ are expressed by a formula (2.6), and $\tau(y)$ satisfies the system (2.3), so it is easy to notice that $U^{\pm}(x_0) = g(x_0)$.

As no restrictions are imposed on a constant C, we will choose it in the way that a condition (2.3) was performed.

For this purpose we will show $\tau(y)$ as $\tau(y) = \tau_1(y) - C\tau_2(y)$, where $\tau_1(y)$ i $\tau_2(y)$ are the solutions of the integral equations

$$\int_L \Psi(x,y)\tau_1(y)\mathrm{ds}_y = g(x)$$

and

$$\int_L \Psi(x,y)\tau_2(y)\mathrm{ds}_y = 1, \quad x \in L,$$

accordingly. If $C = \left(\int_L \tau_1(y)\mathrm{ds}_y\right)\left(\int_L \tau_2(y)\mathrm{ds}_y\right)^{-1}$, then $\tau(y)$ satisfies (2.3), which is easy to check. At the condition of $\int_L \tau(y)\mathrm{ds}_y = 0$ a function $\tau(y)$ is limited on infinity. In fact, if $x \overline{\in} \overline{L}$, and $\hat{y}$ is an arbitrary point on L, then, $U(x) = \int_L \Psi(x,y)\tau(y)\mathrm{ds}_y + C =$

$= \frac{1}{2\pi}\int_L \ln\frac{1}{|x-y|}\tau(y)\mathrm{ds}_y + \frac{1}{2\pi}\ln|x-\hat{y}|\int_L \tau(y)\mathrm{ds}_y = \frac{1}{2\pi}\int_L \ln\frac{|x-\hat{y}|}{|x-y|}\tau(y)\mathrm{ds}_y + C$. Applying a theorem about the middle, we will get such estimation as $\frac{1}{2\pi}\left|\int_L \ln\frac{|x-\hat{y}|}{|x-y|}\tau(y)\mathrm{ds}_y\right| \leq$ $\leq \frac{1}{2\pi}\int_L |\tau(y)|\left|\ln\frac{|x-\hat{y}|}{|x-y|}\right|\mathrm{ds}_y = \int_L |\tau(y)|\,\mathrm{ds}_y \frac{1}{2\pi}\left|\ln\frac{|x-\hat{y}|}{|x-y^*|}\right|$, where y^* is some point on L. Then, at $x \to \infty$ we will get $|x-\hat{y}|\,|x-y^*|^{-1} \to 1$, and $\ln\frac{|x-\hat{y}|}{|x-y^*|} \to 0$, that is $U(\infty) = C$. We will also notice that the value $\int_L |\tau(y)|\,\mathrm{ds}_y$ is limited, which is clear from the following considerations.

The realization of condition (1.4) remains to be checked. For this, it is enough to show that $|\partial U(y)/\partial\rho|$ at $y \in C_m^*(\rho)$ and $\rho \to 0$ is a value of $o(\rho^\alpha)$ order, where $\alpha > -1$. We will use the familiar presentation to the Cauchy integral [7]:

$$\Phi(z) := \frac{1}{2\pi i}\int\limits_L \frac{\varphi(t)}{t-z}\mathrm{d}t = v(x_1,\, x_2) + iu(x_1,\, x_2),$$

where $z := x_1 + ix_2 \overline{\in} L$, $\varphi(t)$ is the real-valued function limited on L, $v(x_1, x_2)$ is a potential of double layer, and $u(x_1, x_2)$is a logarithmic potential of simple layer up to constant. It easy to notice that

$$\Phi'(z) = \frac{1}{2\pi i}\int\limits_L \frac{\varphi(t)}{(t-z)^2}\mathrm{d}t.$$

On the other hand, taking into account the condition of Cauchy-Riman,

$$\Phi'(z) = \frac{\partial v}{\partial x_1} + i\frac{\partial u}{\partial x_1}, \quad \Phi'(z) = \frac{\partial u}{\partial x_2} - i\frac{\partial v}{\partial x_2}. \tag{2.7}$$

Without loss of generality, we will consider that L is the interval $(0, 1)$ of abscise axis of the Cartesian rectangular system of co-ordinates $0\mathrm{x}_1x_2$. We will build the circle of radius ρ with the center at the beginning of co-ordinates. The point z of the circle, which does not lie on L, belongs to the complex plane, that is why it can be shown as

$$z = x_1 + ix_2 = \rho e^{i\beta} = \rho(\cos\beta + i\sin\beta),$$

where β is an argument, ρ is a module of a complex number z.

The derivative of logarithmic potential U with respect to ρ can be calculated by the formula

$$\frac{\partial U}{\partial\rho} = \frac{\partial U}{\partial x_1}\cos\beta + \frac{\partial U}{\partial x_2}\sin\beta,$$

and the Cauchy integral derivative on the circle of radius ρin this case looks as

$$\Phi'(z) = \frac{1}{2\pi i}\int\limits_0^1 \frac{\varphi(s)}{(s\text{ - }\rho e^{i\beta})^2}\mathrm{d}s.$$

Separating real and imaginary parts in the last integral, and taking into account (2.7), we will get

$$\frac{\partial U}{\partial x_1} = -\frac{1}{2\pi}\int\limits_0^1 \frac{s^2 - 2\rho s\cos\beta + \rho^2\cos 2\beta}{(s^2+\rho^2-2\rho s\cos\beta)^2}\varphi(s)\mathrm{d}s,$$

$$\frac{\partial U}{\partial x_2} = \frac{\rho\sin\beta}{\pi}\int\limits_0^1 \frac{s-\rho\cos\beta}{(s^2+\rho^2-2\rho s\cos\beta)^2}\varphi(s)\mathrm{d}s.$$

Now formula for the calculation of the derivative U with respect to ρ can be shown as

$$\frac{\partial U}{\partial\rho} = \frac{1}{2\pi}\int\limits_0^1 \frac{2\rho s - (s^2+\rho^2)\cos\beta}{(s^2+\rho^2-2\rho s\cos\beta)^2}\varphi(s)\mathrm{d}s.$$

For estimating this integral after the value order we will use an obvious inequality $2\rho s \leq\leq s^2+\rho^2$, which is heed strictly at $s \neq \rho$. Then, we will get

$$2\rho s - (s^2+\rho^2)\cos\beta \leq (1-\cos\beta)(s^2+\rho^2),$$

$$(1-\cos\beta)^2(s^2+\rho^2)^2 \leq (s^2+\rho^2-2s\rho\cos\beta)^2, \quad 0 < \beta \leq \frac{\pi}{2},$$

$$\frac{\partial U}{\partial \rho} < \frac{1}{2\pi}\frac{\varphi(s^*)}{1-\cos\beta}\int\limits_0^1 \frac{\mathrm{d}s}{s^2+\rho^2} = o\left(\frac{1}{\rho}\right), \quad \rho \to 0, \quad s^* \in (0,\,1).$$

Strict realization of the last inequality and a property of monotonicity of exponential function allows to claim that

$$\frac{\partial U}{\partial \rho} = o(\rho^{\alpha}), \quad \alpha > -1.$$

Thus, the equivalence of the initial problem (1.1)–(1.4) to the integral equation of the first kind with a weak singularity in a kernel is proven.

Using the differential setting of the initial problem (1.1)–(1.4), the methodology introduced above and reasoning from opposite, it is easy to prove the unity of solving the corresponding integral equation. □

3. Additive constant theorem

To make more precise of the previous results we will formulate and prove such theorem.

Theorem 3.1 *If the initial boundary problem (1.1)–(1.4) owns the Abelian group of symmetry of finite k-order and the boundary values of potential on the separate sections of the bound take on arbitrary constant values $C_1, C_2, \ldots, C_k$, $|C_i| < +\infty$, $i = \overline{1,\,k}$, then the additive constant in the presentation of the solution (2.1) can be calculated by the following formula*

$$C = \frac{1}{k}\sum_{i=1}^{k} C_i.$$

Proof. Proving of the theorem 2.1 results in the fact that the additive constant in the integral expression of the solution to the task (1.1)–(1.4) looks as:

$$C = \left(\int\limits_L \tau_1(y)\,\mathrm{d}s_y\right)\left(\int\limits_L \tau_2(y)\,\mathrm{d}s_y\right)^{-1}.$$

Here $\tau_1(y)$ and $\tau_2(y)$ are the solutions to such integral equations

$$\int\limits_L \tau_1(y)\,\Psi(x,y)\,\mathrm{d}s_y = g(x), \quad x \in L, \tag{3.1}$$

$$\int\limits_L \tau_2(y)\,\Psi(x,y)\,\mathrm{d}s_y = 1, \quad x \in L, \tag{3.2}$$

accordingly. We will remind that $g(x) \equiv const$ on every constituent L, and $\Psi(x,\,y)$ is a fundamental solution to the Laplace equation in $\mathbb{R}^2$.

As the initial boundary problem (1.1)–(1.4) owns the Abelian group of symmetry of finite k-order, then a bound L assumes splitting in congruent constituents $L_i\,(i = \overline{1,\,k})$, where $L := \bigcup\limits_{i=1}^{k} L_i$, at $L_i \cap L_j = 0$ for $i \neq j$. Interpreting the sought densities $\tau_1(y)$ and $\tau_2(y)$ in accordance with such splitting L, we will show the integral equations (3.1) and (3.2) as

$$\sum_{i=1}^{k}\int\limits_{L_i} \tau_{1i}(y)\,\Psi(x,\,y)\,\mathrm{d}s_y = C_j, \quad x \in L_j, \quad j = \overline{1,\,k}, \tag{3.3}$$

$$\sum_{i=1}^{k}\int\limits_{L_i} \tau_{2i}(y)\,\Psi(x,\,y)\,\mathrm{d}s_y = 1, \quad x \in L_j, \quad j = \overline{1,\,k}, \tag{3.4}$$

where $\tau_{1i}(y)$, $\tau_{2i}(y)$ is narrowing of $\tau_1(y)$ and $\tau_2(y)$ on L_i, accordingly.

Carrying out the parameterization of the integral equations (3.3) and (3.4), we will get:

$$\sum_{i=1}^{k}\int_{a_i}^{b_i}\tau_{1i}(\alpha)\Phi\left[y_i(\alpha),x\right]\mathrm{d}\alpha=C_j,\quad x\in L_j,\quad j=\overline{1,k}\tag{3.5}$$

$$\sum_{i=1}^{k}\int_{a_i}^{b_i}\tau_{2i}(\alpha)\Phi\left[y_i(\alpha),x\right]\mathrm{d}\alpha=1,\quad x\in L_j,\quad j=\overline{1,k},\tag{3.6}$$

where

$$\tau_{1i}(\alpha):=\tau_1\left[y_i(\alpha)\right],\quad \tau_{2i}(\alpha):=\tau_2\left[y_i(\alpha)\right],\quad i=\overline{1,k};$$
$$\Phi\left[y_i(\alpha),x\right]:=\left|y_i(\alpha)\right|\Psi\left(y_i(\alpha),x\right),$$

where $y_i(\alpha)\left(a_i\le\alpha\le b_i,\ i=\overline{1,k}\right)$ are parametric presentations of L sections, and for notation of unknown densities the notations from (3.3), (3.4) are preserved. We also assume that vector-functions $y_i(\alpha)$ own a sufficient smoothness. As an operation of superposition of the elements of the group of symmetry, which are corresponding transformations of congruent constituents L from the aggregate of $\{L_i\}_{i=1}^{k}$, is transitive, then at elements σ_i of this group enumeration we will consider that $L_1=\sigma_iL_i$, $i=\overline{1,k}$, where σ_1 is identical transformation. Then, in (3.5), (3.6) we will do a transfer to a new base:

$$\tau'_{1i}(\alpha):=\tau_1\left[\tilde{\sigma}_i^{-1}y_1(\alpha)\right],\quad \tau'_{2i}(\alpha):=\tau_2\left[\tilde{\sigma}_i^{-1}y_1(\alpha)\right],\quad a_1\le\alpha\le b_1,\quad i=\overline{1,k}.$$

Here, $\tilde{\sigma}_i^{-1}$ are matrices, inverse to the matrices, which, in their turn, are the presentation of the elements of the group $\{\sigma_i\}_{i=1}^{k}$. Thus, "the replacements of variables which are introduced, allow us to pass from (3.5), (3.6) to such integral equations as

$$\sum_{i=1}^{k}\int_{a_1}^{b_1}\tau'_{1i}(\alpha)\Phi\left[\tilde{\sigma}_i^{-1}y_1(\alpha);\ \tilde{\sigma}_j^{-1}y_1(\bar{\alpha})\right]\mathrm{d}\alpha=C_j,\tag{3.7}$$

$$\sum_{i=1}^{k}\int_{a_1}^{b_1}\tau'_{2i}(\alpha)\Phi\left[\tilde{\sigma}_i^{-1}y_1(\alpha);\ \tilde{\sigma}_j^{-1}y_1(\bar{\alpha})\right]\mathrm{d}\alpha=1,\tag{3.8}$$

where $\bar{\alpha}\in[a_1,b_1]$, $j=\overline{1,k}$. Thus, at this stage of research we get two systems of the integral equations (3.7) and (3.8), in which the integration is carried out only on a congruent component L_1 of the bound L.

It is convenient to represent the systems (3.7) and (3.8) as such operator equations as

$$(AG_1)(\overline{\alpha})=\hat{C},\tag{3.9}$$

$$(AG_2)(\overline{\alpha})=I.\tag{3.10}$$

Here $A:=(A_{ji})_{j,i=1}^{k}$ is a matrix of operators; $G_1(\alpha):=[\tau'_{1i}(\alpha)]_{i=1}^{k}$, $G_2(\alpha):=[\tau'_{2i}(\alpha)]_{i=1}^{k}$ are column-functions; $\hat{C}:=(C_1,C_2,...,C_k)^T$; I is a unit column. In this connection each of the operators is calculated by formulas

$$\left(A_{ji}G_1^{(i)}\right)(\overline{\alpha}):=\int_{a_1}^{b_1}\tau'_{1i}(\alpha)\Phi\left[\tilde{\sigma}_i^{-1}y_1(\alpha);\ \tilde{\sigma}_j^{-1}y_1(\overline{\alpha})\right]\mathrm{d}\alpha,$$

$$\left(A_{ji}G_2^{(i)}\right)(\overline{\alpha}):=\int_{a_1}^{b_1}\tau'_{2i}(\alpha)\Phi\left[\tilde{\sigma}_i^{-1}y_1(\alpha);\ \tilde{\sigma}_j^{-1}y_1(\overline{\alpha})\right]\mathrm{d}\alpha,$$

where, $G_1^{(i)}$, $G_2^{(i)}$ – i components $\left(i=\overline{1,k}\right)$ of columns-functions $G_1(\alpha)$, $G_2(\alpha)$, accordingly. Further, using the theory of characters [4], we will build the matrix of Fourier transformation for the examined group of k-order which enables to show (3.9) and (3.10) as the split form:

$$\left(B_p\overline{G}_{1p}\right)(\overline{\alpha})=\overline{C}_p,\tag{3.11}$$

$$\left(B_p\overline{G}_{2p}\right)(\overline{\alpha}) = \overline{I}_p, \tag{3.12}$$

where $p = \overline{1,\, k}$; $a_1 \le \overline{\alpha} \le b_1$;

$$\overline{G}_{1p}(\alpha) := \sum_{i=1}^{k} F_{pi} G_1^{(i)}(\alpha); \quad \overline{G}_{2p}(\alpha) := \sum_{i=1}^{k} F_{pi} G_2^{(i)}(\alpha);$$

$$a_1 \le \alpha \le b_1; \quad \overline{C}_p := \sum_{i=1}^{k} F_{pi} C_i; \quad \overline{I}_p = \sum_{i=1}^{k} F_{pi},$$

and B_p are elements of diagonal matrix FAF^{-1}, with $F := \{F_{pi}\}_{p,i=1}^{k}$.

On the basis of the above-mentioned methodology of passage to k independent integral equations (3.11), (3.12) the sought additive constant can be calculated by such formula:

$$C = \left(\sum_{i=1}^{k} \int_{a_1}^{b_1} \tau'_{1i}(\alpha)\, \mathrm{d}\alpha\right) \Bigg/ \left(\sum_{i=1}^{k} \int_{a_1}^{b_1} \tau'_{2i}(\alpha)\, \mathrm{d}\alpha\right) =$$

$$= \left(\frac{1}{k} \int_{a_1}^{b_1} \sum_{p=1}^{k} \sum_{i=1}^{k} F_{ip} \overline{G}_{1i}(\alpha) \mathrm{d}\alpha\right) \Bigg/ \left(\frac{1}{k} \int_{a_1}^{b_1} \sum_{p=1}^{k} \sum_{i=1}^{k} F_{ip} \overline{G}_{2i}(\alpha)\, \mathrm{d}\alpha\right).$$

Taking into account the species of Fourier transformation matrix for the group of k-order, it is easy to notice that

$$C = \left(\int_{a_1}^{b_1} \overline{G}_{11}(\alpha)\, \mathrm{d}\alpha\right) \Bigg/ \left(\int_{a_1}^{b_1} \overline{G}_{21}(\alpha)\, \mathrm{d}\alpha\right),$$

where $\overline{G}_{11}(\alpha)$ and $\overline{G}_{21}(\alpha)$ are evaluated from such integral equations:

$$\left(B_1\overline{G}_{11}\right)(\overline{\alpha}) = \overline{C}_1, \tag{3.13}$$

$$\left(B_1\overline{G}_{21}\right)(\overline{\alpha}) = \overline{I}_1, \tag{3.14}$$

accordingly. Here

$$\overline{C}_1 = \sum_{i=1}^{k} F_{1i} C_i = \sum_{i=1}^{k} C_i,$$

and

$$\overline{I}_1 = \sum_{i=1}^{k} F_{1i} \cdot 1 = k, \quad a_1 \le \overline{\alpha} \le b_1.$$

As the initial problem (1.1)–(1.4) has a unique solution, there is an operator B_1^{-1}, that allows to show the solutions (3.13), (3.14) as $\overline{G}_{11} = B_1^{-1}\overline{C}_1$, $\overline{G}_{21} = B_1^{-1}\overline{I}_1$. As $\overline{C}_1$ are $\overline{I}_1$ are constants,

$$C = \left(\int_{a_1}^{b_1} \left(B_1^{-1}\overline{C}_1\right)(\alpha)\, \mathrm{d}\alpha\right) \Bigg/ \left(\int_{a_1}^{b_1} \left(B_1^{-1}\overline{I}_1\right)(\alpha)\, \mathrm{d}\alpha\right) = \frac{\overline{C}_1}{\overline{I}_1} = \frac{\sum\limits_{i=1}^{k} C_i}{k},$$

that we had to prove. □

This fact substantially simplifies the algorithm of the numerical solving the task (1.1)–(1.4), as at calculating $\tau(y)$ we can limit our research to only one integral equation of

$$\int_L \Psi(x, y)\, \tau(y) \mathrm{d}S_y = \hat{g}_i(x), \quad x \in L_i,$$

where $\hat{g}_i(x) = g_i(x) - C$, $g_i(x) \equiv C_i$, $i = \overline{1,\, N}$. Thus, the value of C can be found simply without additional difficult calculations.

Remark 3.2 Taking into account the complication of numerical realization of the corresponding algorithms based on the apparatus of the theory of groups, it is possible to be limited to the 16-order of the group of symmetry. A choice of the group with exactly such maximal order is, to our opinion, exhaustive from the point of view of presentation of the systems of electrodes, which are mostly used in the process of the real modeling of cathode-ray devices.

4. Numeral experiments

The problem of finding the electrostatic field of the flat electron-optical system is considered and shown in the fig. 1. As we can see, the information about geometry of the charged electrodes is shown as some aggregate of congruent smooth open circuit arcs L_1, L_2, L_3, L_4. The boundary values of potential $g_i(x) \equiv C_i$, $i = \overline{1, 4}$, are the arbitrary known values.

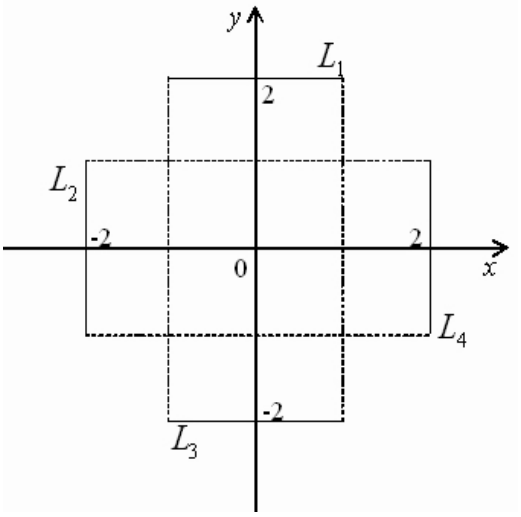

Fig. 1. The investigated flat electron-optical system

The numerical solution of such problem is carried out with the use of methodology, offered in the work [8]. It is taken into account that the boundary $L = \bigcup\limits_{i=1}^{4} L_i$ owns the Abelian group of symmetry of eighth order. For verification of authenticity of the results the additive constant C, which appears in the integral presentation of the field (2.1), is found numeral, without considering the results of the theorem 3.1. The approximate solution of integral equations is carried out by the method of collocation with approximation of the sought density by piecewise-constant base functions. The improper integrals, which we got, were calculated analytically. For the evident presentation of the electrostatic field the lines of equal potential and equipotential surfaces are used.

Example 4.1 In the table 4.1 the value of potential at some points of segment $[-5, 5]$ is shown with a step $h = 0,5$ under the anti-symmetric boundary values $g_1(x) = g_3(x) = 1$, $g_2(x) = g_4(x) = -1$. The number of points of collocation is $n = 100$. The general view of solution is shown by the lines of level (see fig. 2) and equipotential surface (see fig. 3).

Example 4.2 In the table 4.2 the value of potential at some points of segment $[-5, 5]$ is shown with a step $h = 0,5$ under the boundary values $g_1(x) = 1$, $g_2(x) = 10$, $g_3(x) = -1000$, $g_4(x) = 100$. The number of points of collocation is $n = 100$. The general view of solution is shown by the lines of level (see fig. 4) and equipotential surface (see fig. 5).

In the table 4.3 the calculated values of the additive constant are shown under the different boundary values of potential by the methodology which foresees the solution of two auxiliary integral equations.

Tabl. 4.1. Results at one point for the example 4.1.

x	y	u
-1.0	2.0	0.945862
0.0	2.0	0.995043
0.5	4.0	0.288362
0.0	1.5	0.561519
1.0	1.0	0.000000
1.5	0.5	-0.506728
-1.5	-0.5	-0.506728

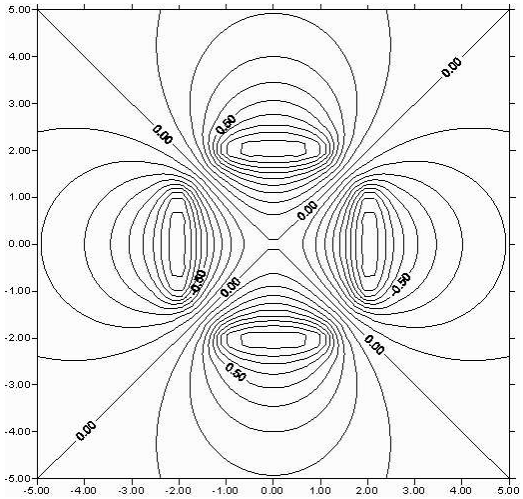

Fig. 2. Distribution of lines of level for the example 4.1

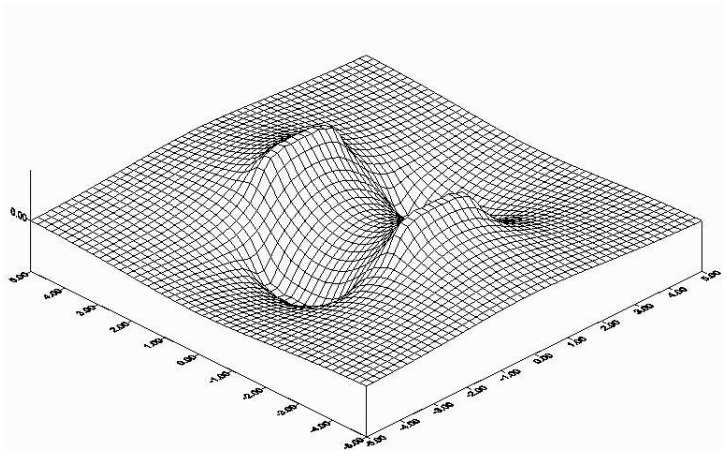

Fig. 3. Equipotential surface for the example 4.1

Tabl. 4.2. Results at one point for the example 4.2.

x	y	u
-1.0	-2.0	-998.868931
0.0	-2.0	-999.884148
-1.0	-2.5	-717.181022
-0.5	-0.5	-333.964404
0.5	1.0	-10.502750
0.0	0.0	-222.250000
0.0	2.0	0.996728

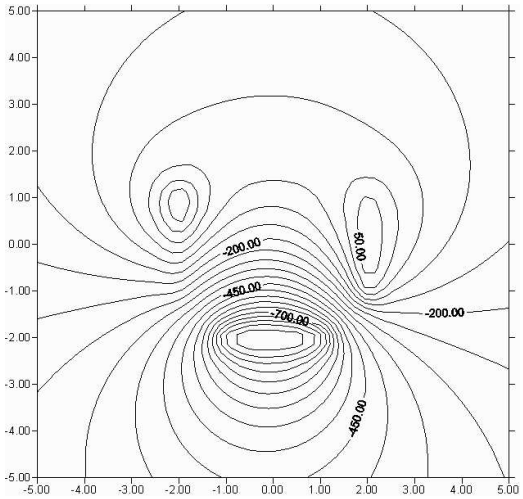

Fig. 4. Distribution of lines of level for the example 4.2

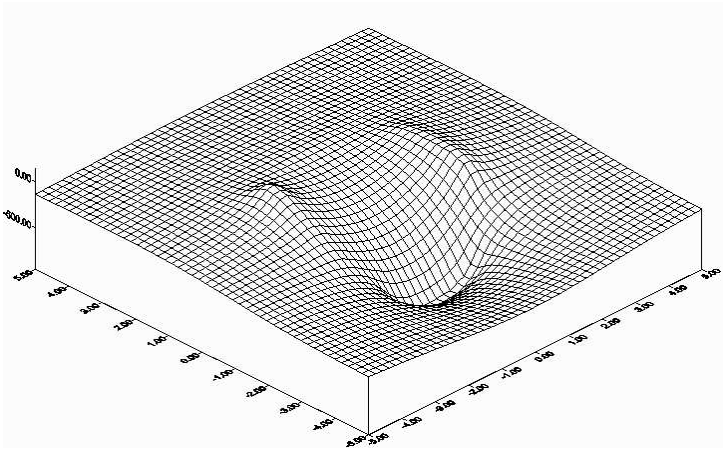

Fig. 5. Equipotential surface for the example 4.2

Tabl. 4.3. The value of constant C on the basis of numeral experiments

$g_1(x)$	$g_2(x)$	$g_3(x)$	$g_4(x)$	C
1	-1	1	-1	0.00
1	10	-1000	100	-222.25
10	20	10	-20	5.00
5	-100	-10	-1000	-276.25
50	100	500	1000	412.50

The analysis of the recent results shows that the effective formula for the calculation of the additive constant received in the theorem 3.1 is correct.

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Received 15.10.2010