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A POSTERIORI ERROR ESTIMATOR FOR DIFFUSION-ADVECTION-REACTION BOUNDARY VALUE PROBLEMS: PIECEWISE LINEAR APPROXIMATIONS ON TRIANGLES

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Анотація. Метою цієї праці є побудова апостеріорних оцінок похибок (АОП) частинами лінійних апроксимацій методу скінченних елементів (МСЕ) для розв'язків двовимірних крайових задач з еліптичними рівняннями другого порядку. Шуканий АОП сконструйовано із незалежних частин на кожному трикутнику поділу у вигляді квадратичного полінома, який набуває нульових значень у вершинах і є розв'язком задачі про лишок апроксимації МСЕ на цьому скінченному елементі. За допущення, що дані рівняння дифузії-адвекції-реакції є сталими на кожному трикутнику, знайдено розподіли точкових значень в центрах ваг та енергетичних норм АОП на розрахункових тріангуляціях. Характеристики запропонованого АОП доповнено результатами обчислювальних експериментів.

Abstract. The purpose of this paper is to construct an a posteriori error estimator (AEE) for piecewise linear approximations of finite element method (FEM) for 2D boundary value problems (BVP) with linear elliptic equations of the second order. The AEE is constructed on each triangle as a quadratic polynomial with zero values at the triangle vertices and is a solution of the residual problem. By assuming equation data to be constant on each triangle, distributions of point values in the mass centers and energy norms of AEE are found. Characteristics of the proposed AEE are supplemented by the results of numerical experiments.

1. Introduction

A design of a posteriori error estimates for a finite element approximation is an actual problem of finite element technology, see for example [3,7,12]. In papers [1,2] there was proposed a method for construction of an a posteriori error estimator (AEE) for piecewise linear approximations of the finite element method (FEM) for boundary values problems with ordinary differential equation of the second order. The main feature of this piecewise defined AEE is a possibility of calculating approximate error values of previously found FEM approximation by sequential solution of local residual variational problems on each individual finite element. Till this time for elliptic boundary value problems of greater dimensionality similar results were obtained starting from quadratic FEM approximations, see [8,9] and [5,6].

In terms of computer modeling objective of this paper is to construct a piecewise defined a posteriori error estimator for piecewise linear approximations on a triangular mesh of a 2D domain in such a way that its value could be calculated independently on a separate finite element.

In connection with this fact the paper is organized as follows. In section 2 we are formulating model boundary value problem and a corresponding variational problem for diffusion-advection-reaction equation. To simplify the statement we only consider the boundary value problem with homogeneous Dirichlet condition. However the results of our AEE construction remain correct for well-posed problems with either Neumann conditions or combined boundary conditions. We consider sufficient requirements for a variational problem to be well-posed and draw attention to

Key words. Diffusion-advection-reaction boundary value problem, finite element method, piecewise linear approximation, element residual equation, a posteriori error estimator, convergence rate.

the most important particular cases when this problem becomes singularly perturbed, in other words becomes complicated for effective solving by the classic FEM schemes. In sections 3-5 after applying Galerkin method we consider the general scheme of variational problem discretization for finding FEM approximation and the error of this approximation. Also it is showed that energy norm of the approximation error is similar to the norm of the residual functional of the calculated approximation. And again after applying Galerkin method here we form the discrete problem for finding the approximate values of the exact FEM approximation error. In section 6 we complete the forming of general scheme for approximate calculating of an a posteriori error estimate without any limitations on choice of the approximation space basis and their supplements, and in section 7 we give detailed calculations of the quadratic AEE for linear FEM approximation on a triangular mesh. Here we give the main results of the paper — the rules to obtain the AEE value in the mass center of the triangle and the rules to calculate the energetic norms on every element of the triangulation. Section 8 supplements the theoretical basis of the paper by the results of numerical experiments that affirm reliability and effectiveness of the proposed a posteriori error estimator for FEM approximation.

The results of this paper were shortly announced in [4].

2. Problem formulation and auxiliary results

We assume that Ω is a domain in euclidian space $\mathbb{R}^d$ with a continuous Lipschitz boundary $\Gamma \equiv \partial\Omega$.

We consider a boundary value problem for diffusion-advection-reaction equation

$$\begin{cases} \textit{find } u = u(x) \textit{ such that} \\ -\nabla.[\mu\nabla u] + \beta.\nabla u + \sigma u = f \textit{ in } \Omega \subset \mathbb{R}^d, \\ \qquad\qquad\qquad u = 0 \textit{ on } \Gamma, \end{cases} \tag{2.1}$$

variational formulation of which is as follows

$$\begin{cases} \textit{given } V := H_0^1(\Omega) = \{v \in H^1\Omega : v = 0 \textit{ on } \Gamma\} \textit{ and} \\ \qquad a_\Omega(w,v) := \int_\Omega [\mu\nabla w.\nabla v + v\beta.\nabla w + \sigma wv]dx, \\ \qquad \langle l_\Omega, v\rangle := \int_\Omega fvdx \quad \forall v, w \in V; \\ \textit{find } u \in V \textit{ such that} \\ \qquad a_\Omega(u,v) = \langle l_\Omega, v\rangle \quad \forall v \in V. \end{cases} \tag{2.2}$$

Here $\nabla u := \{\partial u/\partial x_i\}_{i=1}^d$ and $\mu = \{\mu_{ij}(x)\}_{i,j=1}^d$, $\beta = \{\beta_i(x)\}_{i=1}^d$, $\sigma = \sigma(x)$, $f = f(x)$ are given functions such that

$$\begin{cases} \mu_{ij}(x) = \mu_{ji}(x), \\ \displaystyle\sum_{i,j=1}^d \mu_{ij}(x)\xi_i\xi_j \ge \mu_0 \sum_{i=1}^d \xi_i^2, \ \mu_0 = const \quad \forall \xi_i \in \mathbb{R}, \\ \nabla.\beta(x) := \displaystyle\sum_{i=1}^d \partial\beta_i(x)/\partial x_i = 0 \qquad \textit{almost everywhere in} \quad \Omega, \end{cases} \tag{2.3}$$

$$\mu_{ij},\ \beta_i,\ \sigma \in L^\infty(\Omega), \quad f \in L^2(\Omega). \tag{2.4}$$

Under conditions (2.3),(2.4) bilinear form $a_\Omega(.,.) : V \times V \to \mathbb{R}$ is continuous, V-elliptic and generates energy norm [10, 11]

$$\|v\|_V := \sqrt{a_\Omega(v,v)} \quad \forall v \in V. \tag{2.5}$$

Norm (2.5) is equivalent to the norm $|v|_{1,\Omega} := \sqrt{(\nabla v, \nabla v)}$, more precisely

$$\mu_0|v|_{1,\Omega}^2 \le \|v\|_V^2 \le \mu_0 \left[\frac{\|\mu_0\|_\infty}{\mu_0} + Pe(1+Sh)\right]|v|_{1,\Omega}^2 \quad \forall v \in V, \tag{2.6}$$

where

$$Pe := \frac{\|\beta\|_\infty \text{diam}\Omega}{\mu_0}, \quad Sh := \frac{\|\sigma\|_\infty \text{diam}\Omega}{\|\beta\|_\infty}, \quad Fo := \frac{\mu_0}{\|\sigma\|_\infty (\text{diam}\Omega)^2} = \frac{1}{PeSh} \tag{2.7}$$

are well known Peclet, Struchal and Fourier criteria of similarity,

$$\|\sigma\|_\infty := ess \sup_{x \in \Omega} |\sigma(x)|, \quad \|\beta\|_\infty := \sqrt{\sum_{i=1}^{d} \|\beta_i\|_\infty^2}, \quad \|\mu\|_\infty := \sqrt{\sum_{i,j=1}^{d} \|\mu_{ij}\|_\infty^2}.$$

Remark 2.1 Criteria (2.7) naturally appear in BVP formulation (2.1) if new variables x_i' are introduced as follows

$$x_i' := x_i/\text{diam}\Omega, \quad i = 1, ..., d.$$

Then after scaling of data according to the rules

$$\mu_{ij}' = \mu_{ij}/\mu_0, \quad \beta_i' := \beta_i/\|\beta\|_\infty, \quad \sigma' := \sigma/\|\sigma\|_\infty, \quad f' := f/\|\sigma\|_\infty$$

the problem (2.1) will be of the following form:

$$\begin{cases} \textit{find } u = u(x' \text{diam}\, \Omega) \textit{ such that} \\ -\nabla.[\mu' \nabla u] + Pe[\beta'.\nabla u + Sh(\sigma' u - f')] = 0 \quad in \quad \Omega' \subset \mathbb{R}^d, \\ \qquad\qquad\qquad\qquad\qquad\qquad u = 0 \quad on \quad \Gamma' \equiv \partial\Omega'. \end{cases} \tag{2.8}$$

Hence, the BVP (2.8) becomes singularly perturbed if $Pe \to \infty$ or/and $Sh \to \infty$.

According to assumption (2.4), the linear functional $l : V \to \mathbb{R}$ is continuous. That's why according to Lax-Milgram-Vishik theorem problem (2.2) is well-posed, in other words, has unique solution $u \in V$ and

$$\|u\|_V \leq \|l\|_*,$$

where $\| \,.\, \|_*$ is the norm of the adjoint space V'.

3. Discrete problem

Here we assume that $\Omega \subset \mathbb{R}^d$ is divided into finite elements K in such way that triangulation $\mathrm{T}_h = \{K\}$, $h := \max_{K \in \mathrm{T}_h} h_K$, $h_K := \text{diam}K$, has following properties:

$$\Omega = \bigcup_{K \in \mathrm{T}_h} K; \tag{3.1a}$$

$$K \bigcap K' = \emptyset \quad \forall K, K' \in \mathrm{T}_h : \; K \neq K'; \tag{3.1b}$$

$$\overline{K} \bigcap \overline{K'} = \begin{cases} S := \{\textit{common edge of } K \textit{ and } K'\}, \\ A := \{\textit{common vertex of } K \textit{ and } K'\}, \\ \emptyset. \end{cases} \tag{3.1c}$$

After building in some way finite dimensional approximation space $V_h \subset V$, $\dim V_h = N(h) = N < +\infty$, on triangulation $\mathrm{T}_h = \{K\}$, we replace the problem (2.2) with the following problem:

$$\begin{cases} \textit{given } \mathrm{T}_h = \{K\} \textit{ and } V_h \subset V, \dim V_h = N(h) = N < +\infty; \\ \textit{find } u_h \in V_h \textit{ such that} \\ a_\Omega(u_h, v) = \langle l_\Omega, v \rangle \quad \forall v \in V_h. \end{cases} \tag{3.2}$$

Now choosing the basis $\{\phi_i(x)\}_{i=1}^N$ of approximation space V_h we concretize the problem (3.2) to the following algebraical form:

$$\begin{cases} \textit{given } \mathrm{T}_h = \{K\} \textit{ and basis } \{\phi_i(x)\}_{i=1}^N \textit{ of the space } V_h \subset V; \\ \textit{find coefficients } q_m \in \mathbb{R} \textit{ of approximation } u_h(x) := \sum_{m=1}^{N(h)} q_m \phi_m(x) \in V_h \\ \textit{such that satisfy the system of linear algebraic equations} \\ \sum_{m=1}^{N(h)} a_\Omega(\phi_m, \phi_i) q_m = \langle l_\Omega, \phi_i \rangle \quad i = 1, ..., N(h). \end{cases} \tag{3.3}$$

Thus, the Galerkin procedure provides us the constructive way of finding approximation $u_h \in V_h$ of the problem (2.2) solution $u \in V$, and the finite element technology of building the basis $\{\phi_i(x)\}_{i=1}^N$ gives possibilities for effective computations.

After obtaining the approximation $u_h \in V_h$ the analysis of its error

$$e(x) := u(x) - u_h(x) = u(x) - \sum_{m=1}^{N(h)} q_m \phi_m(x) \quad \forall x \in \Omega. \tag{3.4}$$

is important. Modern methods of such analysis results in a posteriori error estimators which give constructive criteria of accuracy improvement. Below we discuss the problem of AEE construction and its solution in the context of results [8, 9].

4. Approximation error problem

With respect to the variational problem (2.2) and its discretized version (3.2) we come to the following problem:

$$\begin{cases} \textit{given partition } \mathrm{T}_h = \{K\}, \textit{corresponding approximation } u_h \in V_h; \\ \textit{find error } e := u - u_h \in E := V \backslash V_h \textit{such that} \\ a_\Omega(e, v) = \langle \rho(u_h), v \rangle \qquad \forall v \in E, \\ \textit{where functional of error sources (residual)} \\ \langle \rho(w), v \rangle := \langle l_\Omega, v \rangle - a_\Omega(w, v) \qquad \forall w, v \in V. \end{cases} \tag{4.1}$$

The residual $\rho(u_h) \in V'$ can be written as follows

$$\langle \rho(u_h), v \rangle = \langle l_\Omega, v \rangle - a_\Omega(u_h, v) = a_\Omega(u, v) - a_\Omega(u_h, v) = a_\Omega(u - u_h, v) \qquad \forall v \in V.$$

Once can prove, see [9], that

$$\langle \rho(u_h), v \rangle = a_\Omega(u_h - u, v) = 0 \qquad \forall v \in V_h \subset V; \tag{4.2a}$$

$$\langle \rho(u_h), u - u_h \rangle = a_\Omega(u - u_h, u - u_h) = \|u - u_h\|_V^2; \tag{4.2b}$$

$$\|\rho(u_h)\|_* = \sup_{0 \neq v \in V} \frac{|\langle \rho(u_h), v \rangle|}{\|v\|_V} = \sup_{0 \neq v \in V} \frac{|a_\Omega(u - u_h, v)|}{\|v\|_V} \leq \|u - u_h\|_V. \tag{4.2c}$$

Properties (4.2b) and (4.2c) show that the residual functional norm equals to the energy norm of FEM approximation error:

$$\|\rho(u_h)\|_* = \|u - u_h\|_V \qquad \forall h > 0. \tag{4.3}$$

5. A posteriori error estimator problem

To solve the problem (4.1) we will apply Galerkin procedure by using the certain finite dimensional subspace $E_h \subset E$:

$$\begin{cases} \textit{given partition } \mathrm{T}_h = \{K\}, \textit{corresponding approximation } u_h \in V_h \\ \qquad \textit{and subspace } E_h \subset E := V \backslash V_h, \quad \dim E_h = M(h) < +\infty; \\ \textit{find error estimator } e_h \in E_h \textit{such that} \\ \qquad a_\Omega(e_h, v) = \langle \rho(u_h), v \rangle \qquad \forall v \in E_h. \end{cases} \tag{5.1}$$

The problem (5.1) is well-posed and its solution $e_h \in E_h$ (known as an a posteriori error estimator for u_h) has the following property

$$\|e_h\|_V^2 = a_\Omega(e_h, e_h) = \langle \rho(u_h), e_h \rangle \le \|\rho(u_h)\|_* \|e_h\|_V \le \|u - u_h\|_V \|e_h\|_V,$$

therefore

$$\|e_h\|_V \le \|u - u_h\|_V \qquad \forall h > 0. \tag{5.2}$$

Now, by selecting the most convenient basis $\{\phi_i(x)\}_{i=1}^{M(h)} \subset E_h$ we concretize the problem (5.1) to the algebraic representation

$$\begin{cases} \textit{given } \mathrm{T}_h = \{K\} \textit{ and basis } \{\phi_i(x)\}_{i=1}^{M(h)} \textit{ of the space } E_h \subset E; \\ \textit{find coefficients } \lambda_m \in \mathbb{R} \textit{ of linear combination } e_h(x) := \displaystyle\sum_{m=1}^{M(h)} \lambda_m \phi_m(x) \in E_h \\ \textit{such that } \displaystyle\sum_{m=1}^{M(h)} a_\Omega(\phi_m, \phi_i)\lambda_m = \langle \rho(u_h), \phi_i \rangle \qquad i = 1, \dots, M(h). \end{cases} \tag{5.3}$$

It is obvious that the quality of obtained estimator $e_h \in E_h$ will depend on fullness of the subspace $E_h \subset E = V \backslash V_h$ and on effectiveness of computations. This effectiveness is provided by the finite element procedure which makes it possible to construct close to orthogonal bases. Further we introduce a numerical scheme meeting these criteria.

6. The solution of the AEE problem

To solve the problem (5.1) in effective way we will build the linear independent orthogonal system of functions $\{\phi_K\}_{K \in \mathrm{T}_h}$ according to the next rules:

$$\begin{cases} \phi_K \in V \backslash V_h, \\ \operatorname{supp} \phi_K := K \qquad \forall K \in \mathrm{T}_h. \end{cases} \tag{6.1}$$

Taking into consideration that each function $\phi_K \notin V_h$ we will set them as the basis of the subspace $E_h, \dim E_h := M(h) = \operatorname{card} \mathrm{T}_h$. Thus, the structure of solution $e_h \in E_h$ of the problem (5.3) can be naturally characterized by expression

$$e_h(x) := \sum_{K \in \mathrm{T}_h} \lambda_K \phi_K(x) \qquad \forall x \in \Omega \tag{6.2}$$

with unknown coefficients $\lambda_K \in \mathbb{R}$. They are obtained by solution of linear equations (5.3) with a diagonal matrix. In other words,

$$a_\Omega(\phi_K, \phi_K)\lambda_K = \langle \rho(u_h), \phi_K \rangle \qquad \forall K \in \mathrm{T}_h \tag{6.3}$$

and therefore

$$\lambda_K = \frac{\langle \rho(u_h), \phi_K \rangle}{a_\Omega(\phi_K, \phi_K)} = \frac{\langle l_\Omega, \phi_K \rangle - a_\Omega(u_h, \phi_K)}{a_\Omega(\phi_K, \phi_K)} \qquad \forall K \in \mathrm{T}_h. \tag{6.4}$$

Thus, the following statement is true.

Theorem 6.1 *Let* $\mathrm{T}_h = \{K\}$, $h_K := \mathit{diam}\, K$, $h := \mathit{max}_{K \in \mathrm{T}_h} h_K$ *and let us introduce the following definitions:*

(i) restriction of u_h *on element* K *as*

$$u_K(x) := u_h(x) \qquad \forall x \in \overline{K} \qquad \forall K \in \mathrm{T}_h; \tag{6.5}$$

(ii) decomposition of bilinear form and of linear functional of the following form

$$\begin{cases} a_\Omega(w, v) = \displaystyle\sum_{K \in \mathrm{T}_h} \int_K [\mu \nabla w . \nabla v + w \beta . \nabla u + \sigma w v]\, dx =: \sum_{K \in \mathrm{T}_h} a_K(w, v), \\ \langle l_\Omega, v \rangle = \displaystyle\sum_{K \in \mathrm{T}_h} \int_K f v dx =: \sum_{K \in \mathrm{T}_h} \langle l_K, v \rangle \qquad \forall v, w \in V. \end{cases} \tag{6.6}$$

In addition to this, let the error $e := u - u_h \in E := V \backslash V_h$ be approximated by the following linear combination

$$e_h(x) := \sum_{K \in \mathrm{T}_h} \lambda_K \phi_K(x) =: \sum_{K \in \mathrm{T}_h} e_K(x) \qquad \forall x \in \Omega \tag{6.7}$$

of functions $\{\phi_K\}_{K \in \mathrm{T}_h}$ with properties (6.1).

Then coefficients λ_K can be calculated according to the rule

$$\lambda_K = \frac{\langle \rho(u_h), \phi_K \rangle}{a_\Omega(\phi_K, \phi_K)} = \frac{\langle l_\Omega, \phi_K \rangle - a_\Omega(u_h, \phi_K)}{\|\phi_K\|_V^2} \qquad \forall K \in \mathrm{T}_h. \tag{6.8}$$

Moreover, the distribution of AEE energy norms along the elements of triangulation is characterized by expressions

$$\begin{aligned} \|e_K\|_V &= \langle \rho(u_h), \|\phi_K\|_V^{-1} \phi_K \rangle = \\ &= \left| \langle l_\Omega, \|\phi_K\|_V^{-1} \phi_K \rangle - a_\Omega \left(u_h, \|\phi_K\|_V^{-1} \phi_K \right) \right| \qquad \forall K \in \mathrm{T}_h. \end{aligned} \tag{6.9}$$

Proof. Since according to (6.1) each basis function ϕ_K of the error approximations subspace E_h takes non-zero values only at finite element K, in view of (6.4)-(6.6) statement (6.8) is correct.

Moreover, the sequence of calculations by using (6.8)

$$\begin{aligned} \|e_h\|_V^2 = \left\| \sum_{K \in \mathrm{T}_h} e_K \right\|_V^2 &= \sum_{K \in \mathrm{T}_h} \|e_K\|_V^2 = \sum_{K \in \mathrm{T}_h} \lambda_K^2 \|\phi_K\|_V^2 \\ &= \sum_{K \in \mathrm{T}_h} \frac{\left| \langle l_K, \phi_K \rangle - a_K(u_h, \phi_K) \right|^2}{\|\phi_K\|_V^2} \\ &= \sum_{K \in \mathrm{T}_h} \left| \langle l_K, \|\phi_K\|_V^{-1} \phi_K \rangle - a_K \left(u_h, \|\phi_K\|_V^{-1} \phi_K \right) \right|^2 \end{aligned} \tag{6.10}$$

proves (6.9). □

Now on assumption that we consider 2D case ($d = 2$) of the problem (2.1) we will introduce a method of AEE construction.

7. AEE refined structure on triangle

7.1. Linear FEM approximations on the triangle

Let us assume that domain $\Omega \subset \mathbb{R}^2$ of points $x = (x_1, x_2)$ is partitioned into finite elements $K = A_i A_j A_k$ so that the resulting triangulation $\mathrm{T}_h = \{K\}$ has properties (3.1) and let us denote by $A_h = \{A_i\}_{i=1}^n$ and $E_h = \{S_m\}$ its sets of vertices $A_i := (x_1^i, x_2^i)$ and of edges S_m correspondingly. Also, let us assume that there is obtained approximation $u_h \in V_h := \{ v \in C(\Omega) : v|_K \in P_1(K) \quad \forall K \in \mathrm{T}_h \}$ for the problem (2.2) solution $u \in V = H_0^1(\Omega)$ of the kind

$$u_h(x) \,|_K = \sum_{m=i,j,k} u_m L_m(x) \qquad \forall x = (x_1, x_2) \in K \qquad \forall K \in \mathrm{T}_h, \tag{7.1}$$

where $P_1(K)$ – is a set of arbitrary polynomials of the first order determined on triangle K and $L_m = L_m(x)$ is barycentric coordinate system of triangle $K = A_i A_j A_k$,

$$\begin{cases} L_i(x) := \dfrac{a_i + b_i x_1 + c_i x_2}{2\,|K|}, \\ a_i := x_1^k x_2^j - x_1^j x_2^k, \; b_i := x_2^j - x_2^k, \; c_i := -x_1^j + x_1^k, \qquad i \to j \to k \to i. \end{cases} \tag{7.2}$$

7.2. A posteriori error estimator on the triangle

For estimation of the accuracy of approximation (7.1) on arbitrary triangle K we find AEE (6.7) as a quadratic polynomial

$$e(x)\Big|_K \simeq e_K(x) := \lambda_K \phi_K(x) \tag{7.3}$$

of such a structure

$$\begin{cases} \operatorname{supp}\phi_K := K, \\ \phi_K(x) := 3\left[L_i(x)L_j(x) + L_j(x)L_k(x) + L_k(x)L_i(x)\right] \qquad \forall x \in K \qquad \forall K \in \mathrm{T}_h. \end{cases} \tag{7.4}$$

Here $\lambda_K \in \mathbb{R}$ is unknown factor which is equal to the value of AEE in the mass center of triangle $x^K = \left(\frac{1}{3}(x_1^i + x_1^j + x_1^k), \frac{1}{3}(x_2^i + x_2^j + x_2^k)\right)$, in other words,

$$\lambda_K = e_K\left(x^K\right) \qquad \forall K \in \mathrm{T}_h. \tag{7.5}$$

Remark 7.1 Since $\phi_K(A_m) = \phi_K(x_1^m, x_2^m) = 0, \quad m = i, j, k$, a choice of AEE structure (7.4) assumes that node values $u_m = u_h(x_1^m, x_2^m) = 0, \quad m = i, j, k$, are calculated with the sufficient accuracy.

7.3. Calculation of AEE value on the triangle

Now, by taking (6.8) and (7.1) into account, we are ready to calculate coefficients λ_K of estimator $e_h(x)$ for each $K \in \mathrm{T}_h$

$$\begin{aligned} \lambda_K &= \frac{\langle l_K, \phi_K\rangle - a_K(u_h, \phi_K)}{a_K(\phi_K, \phi_K)} = \frac{\langle l_K, \phi_K\rangle - \sum\limits_{m=i,j,k} a_K(L_m, \phi_K)u_m}{a_K(\phi_K, \phi_K)} \\ &= \frac{\int\limits_K f\phi_K dx_1dx_2 - \sum\limits_{m=i,j,k} u_m \int\limits_K \left[\mu\nabla L_m.\nabla\phi_K + \phi_K\beta.\nabla L_m + \sigma L_m\phi_K\right] dx_1dx_2}{\int\limits_K \left[\mu\nabla\phi_K.\nabla\phi_K + \phi_K\beta.\nabla\phi_K + \sigma\phi_K\phi_K\right] dx_1dx_2}. \end{aligned} \tag{7.6}$$

Important properties of basis functions of subspaces V_h and E_h printed out below.

Lemma 7.2 *About integral characteristics of AEE basis function.*
Let basis function ϕ_K on triangle $K = A_iA_jA_k$ be described by (7.3).
Then the following integral characteristics will be true:

1. $\int\limits_K \phi_K dx_1dx_2 = \frac{3}{4}|K|$;
2. $\int\limits_K \phi_K^2 dx_1dx_2 = \frac{9}{15}|K|$;
3. $\int\limits_K L_m\phi_K dx_1dx_2 = \frac{3}{12}|K| \qquad m = i, j, k.$

Proof. Proof is done by the direct application of a well-known rule of integral computation by using barycentric coordinates

$$\int\limits_K L_i^r L_j^s L_k^t dx_1dx_2 = 2|K|\frac{r!\,s!\,t!}{(r+s+t+2)!}, \quad i \neq j \neq k \neq i, \quad r, s, t \geq 0. \tag{7.7}$$

□

Lemma 7.3 *about integral characteristics of AEE basis function derivatives.*

If all conditions of the lemma 7.2 are satisfied, then the following will be correct:

$$\int\limits_K \frac{\partial}{\partial x_1}\phi_K dx_1 dx_2 = \int\limits_K \frac{\partial}{\partial x_2}\phi_K dx_1 dx_2 = 0, \tag{7.8}$$

$$\int\limits_K |\nabla \phi_K|^2 \, dx_1 dx_2 = \int\limits_K \left[\left(\frac{\partial}{\partial x_1}\phi_K \right)^2 + \left(\frac{\partial}{\partial x_2}\phi_K \right)^2 \right] dx_1 dx_2$$

$$= \frac{9}{48\,|K|} \sum_{m=i,j,k} \left(b_m^2 + c_m^2 \right), \tag{7.9}$$

$$\int\limits_K \phi_K \frac{\partial}{\partial x_1}\phi_K \, dx_1 dx_2 = \int\limits_K \phi_K \frac{\partial}{\partial x_2}\phi_K dx_1 dx_2 = 0 \qquad \forall K \in \mathrm{T}_h. \tag{7.10}$$

Proof. Explicit calculations based on expression (7.3) show that

$$\frac{\partial}{\partial x_1}\phi_K = 3\frac{\partial}{\partial x_1}\left(L_i L_j + L_j L_k + L_k L_i \right) = -\frac{3}{2\,|K|} \sum_{m=i,j,k} b_m L_m,$$

$$\frac{\partial}{\partial x_2}\phi_K = -\frac{3}{2\,|K|} \sum_{m=i,j,k} c_m L_m,$$

since

$$\sum_{m=i,j,k} b_m = 0 = \sum_{m=i,j,k} c_m.$$

Thus, according to (7.7), for example,

$$\int\limits_K \frac{\partial}{\partial x_1}\phi_K dx_1 dx_2 = -\frac{3}{2\,|K|} \sum_{m=i,j,k} b_m \int\limits_K L_m dx_1 dx_2$$

$$= -\frac{3}{2\,|K|} \sum_{m=i,j,k} b_m \frac{2\,|K|}{3!} = -\frac{1}{2} \sum_{m=i,j,k} b_m = 0.$$

Similarly we obtain

$$\int\limits_K \left(\frac{\partial}{\partial x_1}\phi_K \right)^2 dx_1 dx_2 = \left(\frac{3}{2\,|K|} \right)^2 \int\limits_K \left(\sum_{m=i,j,k} b_m L_m \right)^2 dx_1 dx_2$$

$$= \frac{9}{2\,|K|}\frac{1}{4!}\left(2b_i^2 + 2b_i b_j + 2b_j^2 + 2b_j b_m + 2b_m^2 + 2b_m b_i \right) = \frac{9}{48\,|K|}\left(\sum_{m=i,j,k} b_m^2 \right).$$

Then, by using the integration by parts we have

$$\int\limits_K \phi_K \frac{\partial}{\partial x_1}\phi_K \, dx_1 dx_2 = \frac{1}{2}\int\limits_K \frac{\partial}{\partial x_1}\phi_K^2 \, dx_1 dx_2 = \frac{1}{2}\int\limits_{\partial K} \phi_K^2 \cos(n, x_1)\, d\gamma$$

$$= -\frac{b_k}{|A_i A_j|}\int\limits_{A_i}^{A_j} \phi_K\Big|_{A_i A_j} d\gamma - \frac{b_i}{|A_j A_k|}\int\limits_{A_j}^{A_k} \phi_K\Big|_{A_j A_k} d\gamma - \frac{b_j}{|A_k A_i|}\int\limits_{A_k}^{A_i} \phi_K\Big|_{A_k A_i} d\gamma = \tag{7.11}$$

$$= -\frac{b_k}{|A_i A_j|}\int\limits_{A_i}^{A_j} L_i L_j d\gamma - \frac{b_i}{|A_j A_k|}\int\limits_{A_j}^{A_k} L_j L_k d\gamma - \frac{b_j}{|A_k A_i|}\int\limits_{A_k}^{A_i} L_k L_i d\gamma = 0,$$

where $n = (n_1, n_2)$ — unit vector of outer normal to the edge ∂K; e.g., on the side $A_i A_j \in \partial K$ its components are found according to the rule

$$n_1 := \cos(n, x_1)\Big|_{A_i A_j} = -\frac{b_k}{|A_i A_j|}, \quad n_2 := \cos(n, x_2)\Big|_{A_i A_j} = -\frac{c_k}{|A_i A_j|}. \tag{7.12}$$

After applying the formulae

$$\int_{A_i}^{A_j} L_i L_j d\gamma = \frac{1}{6} |A_i A_j|$$

to (7.11) we come to the result declared in the statement (7.10) of the lemma. □

Now based upon the above theorems we to the main result of this paper.

Theorem 7.4 *about AEE for FEM piecewise linear approximations.*

If satisfied conditions of the theorem 6.1 and of the lemma 7.2 and in addition to it on each triangle the approximation $u_h = u_h(x)$ is calculated according to (7.1).

Then coefficients λ_K of FEM approximation error estimator (6.7) is obtained separately for each triangle $K \in \mathrm{T}_h$ according to the rule

$$\begin{aligned} \lambda_K \equiv e_h(x_K) = \frac{\langle l_\Omega, \phi_K\rangle - a_\Omega(u_h, \phi_K)}{\|\phi_K\|_V^2} \cong \\ \cong 2|K| \left. \frac{6|K| f - \sum_m u_m [3(\beta_1 b_m + \beta_2 c_m) + 2|K|\sigma]}{\mu \sum_m (b_m^2 + c_m^2) + \frac{16}{5}|K|^2} \right|_{x=x^K} \quad \forall K \in \mathrm{T}_h. \end{aligned} \tag{7.13}$$

Moreover, the distribution of AEE energy norms along the elements of triangulation is characterized by expressions

$$\begin{aligned} \|e_K\|_V^2 = \langle \rho(u_h), \|\phi_K\|_V^{-1} \phi_K \rangle \cong \\ \cong 3|K| \left. \frac{\left\{|K| f - \frac{1}{6}\sum_m u_m [3(\beta_1 b_m + \beta_2 c_m) + 2|K|\sigma]\right\}^2}{\mu \sum_m (b_m^2 + c_m^2) + \frac{16}{5}|K|^2} \right|_{x=x^K} \quad \forall K \in \mathrm{T}_h. \end{aligned} \tag{7.14}$$

Proof. Based on the notation of the AEE coefficients in (7.6), the mean value theorem and lemma 7.2, 7.3 we have

$$\begin{aligned} \langle l_K, \phi_K \rangle = \int_K f \phi_K dx_1 dx_2 \cong f\left(x^K\right) \int_K \phi_K dx_1 dx_2 = \\ = \frac{1}{4}|K| f\left(x_1^K, x_2^K\right) \equiv \frac{1}{4}|K| f\big|_{x=x^K}. \end{aligned} \tag{7.15}$$

Similarly,

$$\begin{aligned} \int_K \mu \nabla L_m . \nabla \phi_K dx_1 dx_2 \cong \mu\left(x^K\right) \int_K \nabla L_m . \nabla \phi_K dx_1 dx_2 = \\ = \mu\left(x^K\right) \frac{1}{2|K|} \left[b_m \int_K \frac{\partial}{\partial x} \phi_K dx_1 dx_2 + c_m \int_K \frac{\partial}{\partial x} \phi_K dx_1 dx_2 \right] = 0, \end{aligned}$$

$$\int_K \sigma L_m \phi_K dx_1 dx_2 \cong \sigma\left(x^K\right) \int_K L_m \phi_K dx_1 dx_2 = \frac{1}{12}|K| \sigma\left(x^K\right),$$

$$\begin{aligned} \int_K \phi_K \beta . \nabla L_m dx_1 dx_2 \equiv \frac{1}{2|K|} \left[\beta_1 b_m \left(x^K\right) \int_K \phi_K dx_1 dx_2 + \beta_2 c_m \left(x^K\right) \int_K \phi_K dx_1 dx_2 \right] = \\ = \frac{1}{2|K|} \left[\beta_1 b_m \left(x^K\right) + \beta_2 c_m \left(x^K\right) \right] \frac{1}{4}|K| = \\ = \frac{1}{8} \left[\beta_1 b_m \left(x^K\right) + \beta_2 c_m \left(x^K\right) \right]. \end{aligned}$$

Summarizing the above calculations we come to conclusions

$$\begin{aligned} a_K(L_m, \phi_K) = \int_K [\mu \nabla L_m . \nabla \phi_K + \phi_K \beta . \nabla L_m + \sigma L_m \phi_K] dx_1 dx_2 \cong \\ \cong \frac{1}{8} \left[\beta_1 b_m \left(x^K\right) + \beta_2 c_m \left(x^K\right) \right] + \frac{1}{12}|K| \sigma\left(x^K\right) \end{aligned} \tag{7.16}$$

and

$$
\begin{aligned}
a_K(\phi_K,\phi_K) \equiv \|\phi_K\|_V^2 = \int_K [\mu\nabla\phi_K.\nabla\phi_K + \phi_K\beta.\nabla\phi_K + \sigma\phi_K\phi_K]\, dx_1 dx_2 \cong \\
\cong \frac{1}{|48K|}\mu\left(x^K\right)\sum_m \left(b_m^2 + c_m^2\right) + \frac{1}{15}|K|\,\sigma\left(x^K\right).
\end{aligned} \tag{7.17}
$$

Finally, applying (7.15)-(7.17) to (6.8) and (6.9) after a little algebra we will come to the equations (7.13) and (7.14). □

Remark 7.5 If diffusion-advection-reaction equation (2.1) is constant on the finite element K, then values of the a posteriori error estimator in (7.13) and (7.14) are computed precisely.

8. Numerical experiments

In our experiments the model problems are Dirichlet BVPs for:

1. *Helmholtz equation* ($\sigma < 0$), which describes the amplitude of stationary constrained vibrations with given circle frequency $\omega = \sqrt{-\sigma}$; if such frequencies take great values (greater than first resonant frequency) then solutions of these problems have a complicated structure that can be reproduced only on very dense triangulations of finite elements; moreover, stability condition for used piecewise linear FEM approximations is of the following form:

$$
-\sigma h \leq 1; \tag{8.1}
$$

2. *Diffusion-reaction equation* ($\sigma > 0$), which especially describes chemical reactions of the first order (for example the decay of radionuclides); on condition that $\sigma \to \infty$, solutions of such problems have the so called boundary layers — thin boundary areas where gradients of these solutions reach the highest values.

Thus, both the classic FEM schemes and suggested a posteriori approximation error estimators are a priori tested by the solving selected BVPs.

Therefore firstly we compute FEM approximations on the sequence of uniformly refined meshes $\mathrm{T}_h = \{K\}$ of triangular elements, and after that find convergence rates of approximations characteristics. For example, convergence rate of AEE e_h in the norm of the space $H^1(\Omega)$ we compute using the following rule:

$$
P\left[H^1(\Omega), e_{h/2}\right] := \log_2 \frac{\|e_h\|_{1,\Omega} - \|e_{h/2}\|_{1,\Omega}}{\|e_{h/2}\|_{1,\Omega} - \|e_{h/4}\|_{1,\Omega}}. \tag{8.2}
$$

Besides if AEE e_n and e_{n+m} are found on triangulations with diameters h_n and h_{n+m} accordingly then we assume that their norms are of the form

$$
\|e_n\|_V = Ch_n^p, \quad \|e_{n+m}\|_V = Ch_{n+m}^p \tag{8.3}
$$

with unknown constant $C = \text{const} > 0$ and with unknown rate p of convergence. Then

$$
\frac{\|e_n\|_V}{\|e_{n+m}\|_V} = \frac{h_n^p}{h_{n+m}^p},
$$

whence it appears a simple rule for calculating the convergence rate of a posteriori FEM approximations error estimators:

$$
P[V, e_{n+m}] := \frac{\ln\|e_n\|_V - \ln\|e_{n+m}\|_V}{\ln h_n - \ln h_{n+m}}. \tag{8.4}
$$

Tabl. 8.1. Input data of the boundary value problem for Helmholtz equation

μ	$\beta = (\beta_1, \beta_2)$	σ	$f(x, y)$	Ω
$\{\delta_{ij}\}_{i,j=1}^2$	$(0, 0)$	-10	$100x^2y^2$	$(-1, +1) \times (-1, +1)$

8.1. Singularly perturbed problem for Helmholtz equation

Below we provide the numerical results, see Tables 8.2, 8.3, that were obtained by solving the boundary value problem (2.1) with input data given in Table 8.1.

Below we give the results of calculations starting from uniform triangulation T_h that contains 256 triangles, card $\mathrm{T}_h = 256$, $h = 1/8$. It was built by uniform partition of domain Ω into 8×8 similar squares which are divided into four triangles by its diagonals; refinements of triangulations were obtained by the dividing of each triangle into four triangles of similar area. Initial triangulations with lower quantity of elements have not assured monotonic convergence of the mean-square norm of FEM approximation. This fact indicates the disability of less dimensional spaces of piecewise linear approximations V_h to qualitatively reproduce the structure of sought BVP solution.

Data given in Table 8.2 characterize the convergence of FEM approximations norms and affirm that veritable values of their convergence rates are those that

$$P\left[L^2(\Omega), u_h\right] \cong P\left[H^1(\Omega), u_h\right] \cong 2. \tag{8.5}$$

Remark 8.1 In the norm of space $H := L^2(\Omega)$ a priori estimates of FEM theory provide the second order of convergence of piecewise linear FEM approximations sequence. In the norm of space $W := H^1(\Omega)$ as opposed to (8.5) this theory provides the first order of convergence and this fact requires additional explanation.

Tabl. 8.2. Convergence of piecewise linear approximations computed on the sequence of uniformly refined triangulations; here $H := L^2(\Omega)$, $W := H^1(\Omega)$

card T_h	nod T_h	$\|u_h\|_H$	$\|u_h\|_W$	$P[H, u_h]$	$P[W, u_h]$
256	145	1.53647	6.28932	–	–
1024	545	1.54065	6.28932	–	–
4096	2113	1.54364	6.63797	0.48	1.69
16384	8321	1.54451	6.65992	1.78	1.91
65536	33025	1.54473	6.66551	1.98	1.97

Tabl. 8.3. Convergence of the AEE for piecewise linear approximations computed on the sequence of uniformly refined triangulations

card T_h	$\|e_h\|_V$	$\max_{K \in \mathrm{T}_h} \|e_K\|_V$	$P[V, e_h]$
256	3.07937	0.67813	–
1024	1.48825	0.20113	–
4096	0.73723	0.05487	1.08
16384	0.36773	0.01433	1.02
65536	0.18375	0.01433	1.01

At the same time Table 8.3 shows that convergence rates of a posteriori error estimators stably approach to the unit. This tendency totally meets our expectations and demonstrates the reliability of suggested AEE.

8.2. Singularly perturbed problem for diffusion-reaction equation

Further we analyze the computation results obtained by solving the problem (2.1) with input data given in Table 8.4. This problem is singularly perturbed with $Sh = 10^3$.

Tabl. 8.4. Input data of the boundary value problem for diffusion-advection equation

μ	$\beta = (\beta_1, \beta_2)$	σ	$f(x,y)$	Ω
$10^{-3} \times I$	$(0,0)$	1	1	$(-1,+1) \times (-1,+1)$

Tables 8.5,8.6 contain the convergence characteristics of the norms of both the piecewise linear approximations and a posteriori error estimators for these approximations.

Tabl. 8.5. Convergence of piecewise linear approximations computed on the sequence of uniformly refined triangulations; here $H := L^2(\Omega)$, $W := H^1(\Omega)$

card T_h	nod T_h	$\|u_h\|_H$	$\|u_h\|_W$	$P[H, u_h]$	$P[W, u_h]$
256	145	1.86567	9.76355	–	–
1024	545	1.89128	10.32196	–	–
4096	2113	1.90172	10.75257	1.29	0.37
16384	8321	1.90455	11.04272	1.88	0.57
65536	33025	1.90522	11.14424	2.08	1.52

Tabl. 8.6. Convergence of the AEE for piecewise linear approximations computed on the sequence of uniformly refined triangulations

card T_h	$\|e_h\|_V$	$\max_{K \in \mathrm{T}_h} \|e_K\|_V$	$P[V, e_h]$
256	3.18029	0.22297	–
1024	2.27892	0.07607	–
4096	1.05555	0.01700	1.11
16384	0.33491	0.00269	1.66
65536	0.08976	0.00036	1.90

9. Conclusion

In this paper we give theoretical basis of the conception for constructing piecewise defined AEE for FEM approximations of the diffusion-advection-reaction BVP solutions. The main attribute of this conception is the ability of classic Galerkin scheme to make the analysis of the approximation error on the selected finite element apart from others elements of the triangulation. This approach of the AEE construction is in detail demonstrated for the 2D case when required AEE has the form of the quadratic polynomial, which vanishes in the vertices of the triangular finite element, and is FEM approximation residual problem solution on this finite element. On assumption that the data of diffusion-advection-reaction equation is constant on every triangle we found the distributions of the energy norms on each finite element and point values of the AEE in the mass centers of these finite elements. The characteristics of the proposed AEE are supplemented with the results of numerical experiments.

Finally, let us admit that introduced method of constructing the AEE for FEM approximations can be easily used to estimate the approximation error of the solutions for 3D elliptic boundary value problems.

Bibliography

1. Abramov Y. h-adaptive finite element method for onedimensional boundary value problems / Y. Abramov, H. Shynkarenko // 18th Internat. Conf. on Computer Methods in Mechanics: Short papers.– Zielona Góra: Zielona Góra University Press.– 2009.– P. 107-109.
2. Abramov Y. The numerical analysis of h-adaptive FEM schemes for piecewise linear approximations / Y. Abramov, O. Lipina, H. Shynkarenko, A. Yamelynets // Visnyk Lviv Univ. Ser. Appl. Math. Comp. Sci.– 2006.– No 11.– P. 3-18. (in Ukrainian).
3. Ainsworth M. A Posteriori Error Estimation in Finite Element Analysis / M. Ainthworth, J. T. Oden.– New York: Wiley, 2000.
4. Borovyy R. V. A posteriori error estimator of finite element method for advection-diffusion-reaction problems: piecewise linear approximations on triangles / R. V. Borovyy, O. Y. Ostapov, H. A. Shynkarenko // 19th Internat. Conf. on Computer Methods in Mechanics: Short papers.– Warsaw: Warsaw University of Technology Press, 2011.– P. 143-144.
5. Chaban F. Constructing of h-adaptive finite element method for piezoelectricity problem / F. Chaban, H. Shynkarenko // J. Numer. Appl. Math.– 2009.– Vol. 1 (97).– P. 1-9.
6. Chaban F. The construction and analysis of a posteriori error estimators for piezoelectricity stationary problems / F. Chaban, H. Shynkarenko // Operator Theory: Advances and Application.– 2009.– Vol. 191.– P. 291-304.
7. Eriksson K. Introduction to adaptive methods for differential equations / K. Eriksson, D. Estep, P. Hansbo, C. Johnson // Acta Numerica.– 1995.– P. 1-54.
8. Kvasnytsya G. Comparison of simple a posteriori error estimators of finite element method for elastostatic problems / G. Kvasnytsya, H. Shynkarenko // Visnyk Lviv Univ. Ser. Appl. Math. Comp. Sci.– 2003.– No 7.– P. 162-174. (in Ukrainian).
9. Kvasnytsya G. Finite element method adaptive approximations for elastostatic problems / G. Kvasnytsya, H. Shynkarenko // Visnyk Lviv Univ. Ser. Appl. Math. Comp. Sci.– 2003.– No 5.– P. 95-106. (in Ukrainian).
10. Kozarevska Yu. S. The analysis of the similarity criteria and of the sensitivity of substance migration problems solutions to its coefficients perturbation / Yu. S. Kozarevska, H. A. Shynkarenko // Visnyk Lviv Univ. Ser. Appl. Math. Comp. Sci.– 2000.– No 2.– P. 116-125. (in Ukrainian).
11. Shynkarenko H. The reguralization of the numerical solutions for substance migration problems: h-adaptive finite element method. Part 1 / Yu. Kozarevska, H. Shynkarenko // Visnyk Lviv Univ. Ser. Appl. Math. Comp. Sci.– 2002.– No 5.– P. 153-164. (in Ukrainian).
12. Verfurth R. A Review of A Posteriori Estimation and Adaptive Mesh-Refinement Techniques, Advances in Numerical Mathematics / R. Verfurth.– New York, Stuttgart: Wiley-Teubner, 1996.

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