Журнал обчисл. прикл. матем.
Вип. 2 (105), 2011, стор. 124-135

J. Numer. Appl. Math.
No. 2 (105), 2011, pp. 124-135

EXPONENTIALLY CONVERGENT METHOD FOR THE M-POINT NONLOCAL PROBLEM FOR AN ELLIPTIC DIFFERENTIAL EQUATION IN BANACH SPACE

UDC 519.6

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Анотація. Розглядається m-точкова нелокальна задача для еліптичного диференціального рівняння з операторним коефіцієнтом у банаховому просторі. Розроблено і обґрунтовано експоненціально збіжний алгоритм для чисельного розв'язку цієї задачі в припущенні, що операторний коефіцієнт є сильно позитивний і виконані умови існування та єдиності розв'язку. Цей алгоритм заснований на зображенні операторної функції за допомогою інтеграла Данфорда-Коші по гіперболі, що охоплює спектр оператора, та застосуванні квадратурної формули, що містить невелику сум резольвент. Ефективність запропонованого методу демонструється на модельній задачі.

Abstract. The m-point nonlocal problem for elliptic differential equation with an operator coefficient in Banach space is considered. An exponentially convergent algorithm is proposed and justified for the numerical solution of this problem in assumption that an operator coefficient A is strongly positive and some existence and uniqueness conditions are fulfilled. This algorithm is based on the representations of operator functions by a Dunford-Cauchy integral along a hyperbola, enveloping the spectrum of A, and on the proper quadratures involving short sums of resolvents. The efficiency of the proposed algorithm is demonstrated by numerical example.

1. Introduction

The nonlocal problem for differential equation is one of the important topics in the theory of differential equations. Various problems in this field have been studied by many mathematicians. A lot of questions, however, are still opened. Such nonlocal problems arise in the theory of physics of plasma [20], nuclear physics [15], mathematical chemistry [16], control problems [2] and waveguides design [12]. Modern aircraft constructions are based on the assemblies made from sandwich shells and plates. Such constructions can be described using nonlocal problems for partial differential equations. Nonlocal elliptic boundary-value problems also have important applications to the theory of multi-dimensional diffusion processes [5, 26].

Discretization methods for differential equations in Banach and Hilbert spaces were intensively studied in the last decade (see e.g. [6–9, 17, 18, 21, 22, 24, 25] and the references therein). Methods from [7–9, 17, 18, 22, 24, 25] possess an exponential convergence rate, i.e. the error estimate in an appropriate norm is of the type $\mathcal{O}(e^{-N^{\alpha}})$, $\alpha > 0$ with respect to a discretization parameter $N \to \infty$. For a given tolerance ε such discretizations provide optimal or nearly optimal computational complexity [7]. One of the possible ways to obtain exponentially convergent approximations to abstract differential equations is based on a representation of the solution through the Dunford-Cauchy integral along a parametrized path enveloping the spectrum of the operator coefficient. Choosing a proper quadrature for this integral we obtain a short sum of resolvents. Since the treatment of such resolvents is usually the most time consuming part of any approximation this leads to a low-cost naturally parallelizeable algorithms. Parameters of the algorithms from [9, 18, 22] were optimized in [27, 28] to improve the convergence rate.

Key words. Nonlocal problem, differential equation with an operator coefficient in Banach space, exponentially convergent algorithms.

In this paper we consider the following nonlocal m-point problem:

$$\begin{aligned} &\frac{d^2u}{dx^2} - Au = 0, \quad x \in [0, X] \\ &u(0) = 0, \\ &u(1) = \sum_{k=1}^{m} \alpha_k u(\xi_k) + u_1, \end{aligned} \tag{1.1}$$

where $\alpha_k \in \mathbb{R}$, $k = \overline{1,m}$, $0 < \xi_1, \xi_2, \cdots < \xi_m < 1$, $f(t)$ is a given vector-valued function with values in a Banach space X, u_0, $u_1 \in X$. The operator A with the domain $D(A)$ in a Banach space X is assumed to be densely defined strongly positive (sectorial) operator, i.e. its spectrum $\Sigma(A)$ lies in a sector of the right half-plane with the vertex at the origin, while its resolvent decays inversely proportional to $|z|$ at the infinity (see estimate (2.7) below).

Problem (1.1) is the base problem when one try to solve more general problem

$$\begin{aligned} &\frac{d^2u}{dx^2} - Au = -f(x), \quad x \in [0, X] \\ &u(0) = u_0, \\ &u(1) = \sum_{k=1}^{m} \alpha_k u(\xi_k) + u_1, \end{aligned} \tag{1.2}$$

with the same assumption on initial data. Let us show that problem (1.2) can be reduced to the problem of the type (1.1). We present u in (1.2) as $u(x) = v(x) + w(x)$, where $v(x)$ is the solution of the following problem:

$$\begin{aligned} &\frac{d^2v}{dx^2} - Av = -f(x), \quad x \in [0, X] \\ &v(0) = u_0, \\ &v(1) = u_1. \end{aligned} \tag{1.3}$$

On that, $w(x)$ satisfies equation

$$\frac{d^2w}{dx^2} - Aw = 0, \quad x \in [0, X],$$

and condition

$$w(0) = 0.$$

For the point $x = 1$ we obtain

$$w(1) = u(1) - v(1) = \sum_{k=1}^{m} \alpha_k u(\xi_k) = \sum_{k=1}^{m} \alpha_k w(\xi_k) + \sum_{k=1}^{m} \alpha_k v(\xi_k).$$

Thus, $w(x)$ is a solution of the problem

$$\begin{aligned} &\frac{d^2w}{dx^2} - Aw = 0, \quad x \in [0, X] \\ &w(0) = 0, \\ &w(1) = \sum_{k=1}^{m} \alpha_k w(\xi_k) + w_1, \end{aligned} \tag{1.4}$$

where $w_1 = \sum_{k=1}^{m} \alpha_k v(\xi_k)$.

Remark 1.1 Solution of the problem (1.3) can be presented as

$$v(x) = \sinh(\sqrt{A}(1-x))\sinh^{-1}(\sqrt{A})u_0 + \sinh(\sqrt{A}x)\sinh^{-1}(\sqrt{A})u_1 + \int_0^1 G(x,s;A)f(s)ds,$$

where $G(x,s;A)$ is Green's function

$$G(x,s;A) = [\sqrt{A}\sinh\sqrt{A}]^{-1}\begin{cases}\sinh(x\sqrt{A})\sinh((1-s)\sqrt{A}) & x \le s,\\ \sinh(s\sqrt{A})\sinh((1-x)\sqrt{A}) & x \ge s\end{cases}.$$

Exponentially convergent approximation to this solution was obtained in [11].

The aim of this paper is to construct an exponentially convergent approximation to the problem for an elliptic differential equation with m-point nonlocal condition in abstract setting (1.1). The paper is organized as follows. In Section 2. we discuss the existence and uniqueness of the solution as well as its representation through input data. A numerical algorithm for the problem (1.1) is proposed and justified in section 3.. The main result of this section is theorem 3.1 about the exponential convergence rate of the proposed discretization. The next section 4. is devoted numerical example which confirms theoretical results from the previous section.

2. Existence and representation of the solution

The solution of (1.1) can be formally represented as follows

$$u(x) = \sinh(\sqrt{A}x)\sinh^{-1}(\sqrt{A})\left[\sum_{k=1}^{m}\alpha_k u(\xi_k) + u_1\right]. \tag{2.1}$$

If we set $d = \sum_{k=1}^{m}\alpha_k u(\xi_k)$, then we obtain from (2.1)

$$d = \sum_{k=1}^{m}\alpha_k \sinh(\sqrt{A}\xi_k)\sinh^{-1}(\sqrt{A})d + \sum_{k=1}^{m}\alpha_k \sinh(\sqrt{A}\xi_k)\sinh^{-1}(\sqrt{A})u_1,$$

which is equivalent to

$$d = \left[I - \sum_{k=1}^{m}\alpha_k \sinh(\sqrt{A}\xi_k)\sinh^{-1}(\sqrt{A})\right]^{-1}\sum_{k=1}^{m}\alpha_k \sinh(\sqrt{A}\xi_k)\sinh^{-1}(\sqrt{A})u_1$$

$$= \left[I - \sum_{k=1}^{m}\alpha_k \sinh(\sqrt{A}\xi_k)\sinh^{-1}(\sqrt{A})\right]^{-1} u_1 - u_1 \equiv B^{-1}u_1 - u_1,$$

provided that operator B^{-1} is properly defined.

Therefore, we obtain from (2.1) a representation of the solution of the problem (1.1) as

$$u(x) = \sinh(\sqrt{A}x)\sinh^{-1}(\sqrt{A})B^{-1}u_1. \tag{2.2}$$

For the two dimensional elliptic problem (that is partial case of (1.1)) in [13] the authors proved that

$$\sum_{k=1}^{m}\frac{|\alpha_k| + \alpha_k}{2} \le 1 \tag{2.3}$$

is sufficient condition for existence unique solution (1.1). Different condition for existence and uniqueness of the solution was proposed in [3]:

$$\sum_{k=1}^{m}|\alpha_k|\sqrt{\xi_k} \le 1. \tag{2.4}$$

The condition

$$\sum_{k=1}^{m} |\alpha_k| \le \sigma < 1 \tag{2.5}$$

is used in many publications to construct numerical methods for the nonlocal elliptic problems (see e.g. [1, 19]). We have to mention that in all these cases operator A is strongly self-adjoint and positively definite operator. In this paper we consider more general case when operator A is strongly positive.

Let the operator A in (1.1) be a densely defined strongly positive (sectorial) operator in a Banach space X with the domain $D(A)$, i.e. its spectrum $\Sigma(A)$ lies in the sector

$$\Sigma = \left\{ z = \rho_0 + re^{i\theta} : \quad r \in [0, \infty),\ \rho_0 > 0\ \ |\theta| < \varphi < \frac{\pi}{2} \right\}. \tag{2.6}$$

Additionally, the following estimate for the resolvent holds true

$$\|R_A(z)\| = \left\|(zI - A)^{-1}\right\| \le \frac{M}{1 + |z|} \tag{2.7}$$

outside the sector and on its boundary Γ_Σ. The numbers ρ_0, φ are called the spectral characteristics of A.

The hyperbola

$$\Gamma_0 = \{z(\zeta) = \rho_0 \cosh\zeta - ib_0 \sinh\zeta :\ \zeta \in (-\infty, \infty),\ b_0 = \rho_0 \tan\varphi\} \tag{2.8}$$

in turn is referred as a spectral hyperbola. It has a vertex at $(\rho_0, 0)$ and asymptotes which are parallel to the rays of the spectral angle Σ.

A convenient representation of operator functions is the one through the Dunford-Cauchy integral (see e.g. [4, 14]) where the integration path plays an important role. Using the Dunford-Cauchy integral representation and (2.2) the solution of the problem (1.1) can be represented as

$$u(x) = \frac{1}{2\pi i} \int_{\Gamma_I} \frac{\sinh(\sqrt{z}x)}{\sinh(\sqrt{z}) \left[1 - \sum_{k=1}^{m} \alpha_k \frac{\sinh(\sqrt{z}\xi_k)}{\sinh(\sqrt{z})} \right]} R_A(z) u_1 dz,$$

if

$$\frac{\sinh(\sqrt{z}x)}{\sinh(\sqrt{z}) \left[1 - \sum_{k=1}^{m} \alpha_k \frac{\sinh(\sqrt{z}\xi_k)}{\sinh(\sqrt{z})} \right]}$$

is analytic function inside the integration hyperbola Γ_I that envelopes Γ_0. To obtain numerically stable algorithm we shall slightly modify this integral by changing the resolvent $R_A(z)$ to $R_A^1(z)$ that doesn't change the value of integral when $u_1 \in D(A^\alpha)$, $\alpha > 0$ (for the details see [10, 11]).

$$R_A^1(z) = (zI - A)^{-1} - \frac{I}{z}.$$

Therefore, one can obtain the following representation for the solution of the problem (1.1):

$$u(x) = \frac{1}{2\pi i} \int_{\Gamma_I} \frac{\sinh(\sqrt{z}x)}{\sinh(\sqrt{z}) \left[1 - \sum_{k=1}^{m} \alpha_k \frac{\sinh(\sqrt{z}\xi_k)}{\sinh(\sqrt{z})} \right]} R_A^1(z) u_1 dz. \tag{2.9}$$

We choose the following hyperbola

$$\Gamma_I = \{z(\zeta) = a_I \cosh\zeta - ib_I \sinh\zeta :\ \zeta \in (-\infty, \infty)\}, \tag{2.10}$$

as an integration contour which envelopes the spectrum of A, where the values of a_I, b_I are to be defined later. Using this hyperbola, we obtain from (2.9)

$$u(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \mathcal{F}(x, \zeta) d\zeta, \tag{2.11}$$

with

$$\mathcal{F}(x,\zeta) = F_A(x,\zeta)u_1,$$

$$F_A(x,\zeta) = \frac{\sinh(\sqrt{z(\zeta)}x)z'(\zeta)}{\sinh(\sqrt{z(\zeta)})\left[1 - \sum_{k=1}^{m} \alpha_k \frac{\sinh(\sqrt{z(\zeta)}\xi_k)}{\sinh(\sqrt{z(\zeta)})}\right]} \times$$
$$\times \left[(z(\zeta)I - A)^{-1} - \frac{I}{z(\zeta)}\right],$$

and

$$z'(\zeta) = a_I \sinh\zeta - ib_I \cosh\zeta.$$

The next step toward a numerical algorithm is an approximation of (2.11) by the efficient quadrature formula. For this purpose we need to estimate the width of a strip around the real axis where the function $\mathcal{F}(x,\zeta)$ permits analytical extension (with respect to ζ). The integration hyperbola Γ_I will be translated into the parametric set of hyperbolas with respect to ν after changing ζ to $\zeta + i\nu$

$$\Gamma(\nu) = \{z(\zeta,\nu) = a_I \cosh(\zeta + i\nu) - ib_I \sinh(\zeta + i\nu) :\ \zeta \in (-\infty,\infty)\}$$
$$= \{z(\zeta,\nu) = a(\nu)\cosh\zeta - ib(\nu)\sinh\zeta :\ \zeta \in (-\infty,\infty)\},$$

with

$$a(\nu) = a_I\cos\nu + b_I\sin\nu = \sqrt{a_I^2 + b_I^2}\sin(\nu + \phi/2),$$
$$b(\nu) = b_I\cos\nu - a_I\sin\nu = \sqrt{a_I^2 + b_I^2}\cos(\nu + \phi/2),$$
$$\cos\frac{\phi}{2} = \frac{b_I}{\sqrt{a_I^2 + b_I^2}},\ \sin\frac{\phi}{2} = \frac{a_I}{\sqrt{a_I^2 + b_I^2}}.$$

The analyticity of the function $\mathcal{F}(t,\zeta + i\nu)$, in the strip

$$D_{d_1} = \{(\zeta,\nu) : \zeta \in (-\infty,\infty), |\nu| < d_1/2\},$$

with some d_1 could be violated if the resolvent or the part related to the nonlocal condition (B^{-1}) become unbounded. To avoid this we have to choose d_1 in a way such that for $\nu \in (-d_1/2, d_1/2)$ the hyperbola $\Gamma(\nu)$ remains in the right half-plane of the complex plane. For $\nu = -d_1/2$ the corresponding hyperbola is going through the point $(\rho_1, 0)$, for some $0 \le \rho_1 < \rho_0$. For $\nu = d_1/2$ it coincides with the spectral hyperbola and therefore for all $\nu \in (-d_1/2, d_1/2)$ the set $\Gamma(\nu)$ does not intersect the spectral sector. This fact justifies the choice the hyperbola $\Gamma(0) = \Gamma_I$ as the integration path.

Such requirements for $\Gamma(\nu)$ imply the following system of equations

$$\begin{cases} a_I\cos(d_1/2) + b_I\sin(d_1/2) = \rho_0, \\ b_I\cos(d_1/2) - a_I\sin(d_1/2) = b_0 = \rho_0\tan\varphi, \\ a_I\cos(-d_1/2) + b_I\sin(-d_1/2) = \rho_1, \end{cases}$$

it leads us to the next system

$$\begin{cases} a_I = \rho_0\cos(d_1/2) - b_0\sin(d_1/2), \\ b_I = \rho_0\sin(d_1/2) + b_0\cos(d_1/2), \\ 2a_I\cos(d_1/2) = \rho_0 + \rho_1. \end{cases}$$

Eliminating a_I from the first and the third equations we obtain

$$\rho_0\cos d_1 - b_0\sin d_1 = \rho_1,$$

$$\cos(d_1 + \varphi) = \frac{\rho_1}{\sqrt{\rho_0^2 + b_0^2}},$$

i. e.

$$d_1 = \arccos\left(\frac{\rho_1}{\sqrt{\rho_0^2+b_0^2}}\right) - \varphi, \tag{2.12}$$

with $\cos\varphi = \frac{\rho_0}{\sqrt{\rho_0^2+b_0^2}}$, $\sin\varphi = \frac{b_0}{\sqrt{\rho_0^2+b_0^2}}$. Thus, for a_I, b_I we receive

$$\begin{aligned} a_I &= \sqrt{\rho_0^2+b_0^2}\cos\left(\frac{d_1}{2}+\varphi\right)\\ &= \rho_0\frac{\cos\left(\frac{d_1}{2}+\varphi\right)}{\cos\varphi} = \rho_0\frac{\cos\left(\arccos\left(\frac{\rho_1}{\sqrt{\rho_0^2+b_0^2}}\right)/2+\varphi/2\right)}{\cos\varphi},\\ b_I &= \sqrt{\rho_0^2+b_0^2}\sin\left(\frac{d_1}{2}+\varphi\right)\\ &= \rho_0\frac{\cos\left(\frac{d_1}{2}+\varphi\right)}{\cos\varphi} = \rho_0\frac{\cos\left(\arccos\left(\frac{\rho_1}{\sqrt{\rho_0^2+b_0^2}}\right)/2+\varphi/2\right)}{\cos\varphi}. \end{aligned} \tag{2.13}$$

For a_I and b_I defined as above the resolvent of the operator A is analytic in the strip D_{d_1} with respect to $w = \zeta + i\nu$ for any $x \geq 0$. Note, that for $\rho_1 = 0$ we have $d_1 = \pi/2 - \varphi$ as in [10].

Taking into account (2.13) we can similarly write equations for $a(\nu)$, $b(\nu)$ on the whole interval $-\frac{d_1}{2} \leq \nu \leq \frac{d_1}{2}$

$$\begin{aligned} a(\nu) &= a_I\cos\nu + b_I\sin\nu = \sqrt{\rho_0^2+b_0^2}\cos\left(\frac{d_1}{2}+\varphi\right)\cos(\nu)\\ &+ \sqrt{\rho_0^2+b_0^2}\sin\left(\frac{d_1}{2}+\varphi\right)\sin(\nu) = \sqrt{\rho_0^2+b_0^2}\cos\left(\frac{d_1}{2}+\varphi-\nu\right),\\ b(\nu) &= b_I\cos\nu - a_I\sin\nu = \sqrt{\rho_0^2+b_0^2}\sin\left(\frac{d_1}{2}+\varphi\right)\cos(\nu)\\ &- \sqrt{\rho_0^2+b_0^2}\cos\left(\frac{d_1}{2}+\varphi\right)\sin(\nu) = \sqrt{\rho_0^2+b_0^2}\sin\left(\frac{d_1}{2}+\varphi-\nu\right), \end{aligned}$$

$$\rho_1 \leq a(\nu) \leq \rho_0, \quad b_0 \leq b(\nu) \leq \sqrt{b_0^2+\rho_0^2-\rho_1^2},$$

with d_1, defined by (2.12).

Now, let us establish conditions on α_k, $k = \overline{1,m}$ when expression

$$\left[1 - \sum_{k=1}^{m}\alpha_k\frac{\sinh(\sqrt{z}\xi_k)}{\sinh(\sqrt{z})}\right]$$

related to nonlocal condition dose not become zero inside the integration hyperbola Γ_I. We need some auxiliary estimates for that.

$$\begin{aligned} |e^{-2x\sqrt{z}}| &= \left|\exp\left\{-2x\left(\sqrt{\left(a_I\cosh\zeta+\sqrt{a_I^2\cosh^2\zeta+b_I^2\sinh^2\zeta}\right)/2}\right.\right.\right.\\ &\left.\left.\left.+i\sqrt{\left(-a_I\cosh\zeta+\sqrt{a_I^2\cosh^2\zeta+b_I^2\sinh^2\zeta}\right)/2}\right)\right\}\right|\\ &= \exp\left\{-2x\left(\sqrt{\left(a_I\cosh\zeta+\sqrt{a_I^2\cosh^2\zeta+b_I^2\sinh^2\zeta}\right)/2}\right)\right\}\\ &< e^{-2x\sqrt{a_I\cosh\zeta}} \qquad \forall x \in (0,1],\\ &|1-e^{-2x\sqrt{z}}| \leq 1 + |e^{-2x\sqrt{z}}| \leq 2,\\ &|1-e^{-2x\sqrt{z}}| \geq 1 - |e^{-2x\sqrt{z}}| \geq 1 - e^{-2x\sqrt{a_I\cosh\zeta}}, \end{aligned}$$

$$|e^{(x-1)\sqrt{z}} - e^{-(x+1)\sqrt{z}}| = |e^{(x-1)\sqrt{z}}(1 - e^{-2x\sqrt{z}})| \le 2e^{(x-1)\sqrt{a_I \cosh\zeta}}$$

$$\left|\frac{\sinh(\sqrt{z}\xi_k)}{\sinh(\sqrt{z})}\right| = \left|\frac{e^{\xi_k\sqrt{z}} - e^{-\xi_k\sqrt{z}}}{e^{\sqrt{z}} - e^{-\sqrt{z}}}\right| = \left|\frac{e^{(\xi_k-1)\sqrt{z}} - e^{-(\xi_k+1)\sqrt{z}}}{1 - e^{-2\sqrt{z}}}\right|$$

$$\le \frac{\left|e^{(\xi_k-1)\sqrt{z}}\left(1 - e^{-2\xi_k\sqrt{z}}\right)\right|}{1 - e^{-2\sqrt{a_I\cosh\zeta}}} \le \frac{e^{(\xi_k-1)\sqrt{a_I\cosh\zeta}}\left(1 + e^{-2\xi_k\sqrt{a_I\cosh\zeta}}\right)}{1 - e^{-2\sqrt{a_I\cosh\zeta}}}$$

$$= \frac{\cosh\left(\xi_k\sqrt{a_I\cosh\zeta}\right)}{\sinh\sqrt{a_I\cosh\zeta}} \le \frac{\cosh\left(\xi_k\sqrt{a_I}\right)}{\sinh\sqrt{a_I}}.$$

Therefore

$$\left|\left[1 - \sum_{k=1}^{m}\alpha_k\frac{\sinh(\sqrt{z(\zeta)}\xi_k)}{\sinh(\sqrt{z(\zeta)})}\right]^{-1}\right| \le \left[1 - \left|\sum_{k=1}^{m}\alpha_k\frac{\sinh(\sqrt{z(\zeta)}\xi_k)}{\sinh(\sqrt{z(\zeta)})}\right|\right]^{-1}$$

$$\le \left[1 - \sum_{k=1}^{m}|\alpha_k|\left|\frac{\sinh(\sqrt{z(\zeta)}\xi_k)}{\sinh(\sqrt{z(\zeta)})}\right|\right]^{-1} \le \left[1 - \sum_{k=1}^{m}|\alpha_k|\frac{\cosh\left(\xi_k\sqrt{a_I}\right)}{\sinh\sqrt{a_I}}\right]^{-1},$$

where a_I is defined in (2.13).

So, if the condition

$$\sum_{k=1}^{m}|\alpha_k|\frac{\cosh\left(\xi_k\sqrt{a_I}\right)}{\sinh\sqrt{a_I}} < 1 \tag{2.14}$$

is fulfilled, than

$$\left|\left[1 - \sum_{k=1}^{m}\alpha_k\frac{\sinh(\sqrt{z(\zeta)}\xi_k)}{\sinh(\sqrt{z(\zeta)})}\right]^{-1}\right| \le q < \infty. \tag{2.15}$$

This fact can be formulated as the following lemma.

Lemma 2.1 *Let A be a densely defined strongly positive operator. If the condition* (2.14) *is valid than there exists a unique solution of the problem* (1.1) *which can be represented by* (2.9).

Further, let us establish conditions for the existence of the solution to (1.1) in the case when the operator A is self-adjoint positive definite. To achieve that we have to choose d_1 in a way that for $\nu \in (-d_1/2, d_1/2)$ the hyperbola $\Gamma(\nu)$ remains in the right half-plane of complex plane. For $\nu = -d_1/2$ the corresponding hyperbola turns into the line parallel to the imaginary axis. For $\nu = d_1/2$ it coincides with the ray that lies on the real axis having a vertex at ρ_0. These requirements imply the following system of equations

$$\begin{cases} a_I\cos(d_1/2) + b_I\sin(d_1/2) = \rho_0, \\ b_I\cos(d_1/2) - a_I\sin(d_1/2) = 0, \\ a_I\cos(-d_1/2) + b_I\sin(-d_1/2) = 0, \end{cases}$$

which has the solution

$$a_I = b_I = \frac{\rho_0}{\sqrt{2}},$$
$$d_1 = \frac{\pi}{2}$$

The condition (2.14), then, becomes

$$\sum_{k=1}^{m}|\alpha_k|\frac{\cosh\left(\xi_k\sqrt{\frac{\rho_0}{\sqrt{2}}}\right)}{\sinh\left(\sqrt{\frac{\rho_0}{\sqrt{2}}}\right)} < 1. \tag{2.16}$$

Therefore, in the case of self-adjoint positive operator A we obtain the sufficient condition of existence solution to (1.1) in the form (2.16).

3. Numerical algorithm

Supposing $u_1 \in D(A^\alpha)$, $0<\alpha<1$ it was shown in [11] that

$$\begin{aligned}&\left\|\frac{\sinh(\sqrt{z(\zeta)}x)}{\sinh(\sqrt{z(\zeta)})}z'(\zeta)\left[(z(\zeta)I-A)^{-1}-\frac{1}{z(\zeta)}I\right]u_1\right\|\\&\le(1+M)K\frac{b_I}{1-\mathrm{e}^{-2\sqrt{a_I}}}\left(\frac{2}{a_I}\right)^{1+\alpha}\mathrm{e}^{(x-1)\sqrt{a_I\cosh\xi}-\alpha|\xi|}\|A^\alpha u_1\|,\\&\xi\in\mathbb{R},\quad x\in(0,1],\end{aligned}$$

where K is a constant that depends on α.

The part responsible for the nonlocal condition in (2.11) is estimated by (2.15). Thus, we obtain the following estimate for $\mathcal{F}(t,\xi)$:

$$\begin{aligned}\|\mathcal{F}(x,\xi)\|\le&(1+M)qK\frac{b_I}{1-\mathrm{e}^{-2\sqrt{a_I}}}\left(\frac{2}{a_I}\right)^{1+\alpha}\mathrm{e}^{(x-1)\sqrt{a_I\cosh\xi}-\alpha|\xi|}\|A^\alpha u_1\|,\\&\xi\in\mathbb{R},\quad x\in(0,1].\end{aligned}\tag{3.1}$$

Similarly to [23], we introduce the space $\mathbf{H}^p(D_d)$, $1\le p\le\infty$ of all vector-valued functions $\mathcal{F}$ analytic in the strip

$$D_d=\{z\in\mathbb{C}:-\infty<\mathrm{Re}\ z<\infty,|\Im z|<d\},$$

equipped by the norm

$$\|\mathcal{F}\|_{\mathbf{H}^p(D_d)}=\begin{cases}\lim_{\epsilon\to0}(\int_{\partial D_d(\epsilon)}\|\mathcal{F}(z)\|^p|dz|)^{1/p} & \text{if } 1\le p<\infty,\\ \lim_{\epsilon\to0}\sup_{z\in\partial D_d(\epsilon)}\|\mathcal{F}(z)\| & \text{if } p=\infty,\end{cases}$$

where

$$D_d(\epsilon)=\{z\in\mathbb{C}:|\mathrm{Re}(z)|<1/\epsilon,|\mathrm{Im}(z)|<d(1-\epsilon)\}$$

and $\partial D_d(\epsilon)$ is the boundary of $D_d(\epsilon)$.

We can obtain estimate similar to (2.15) for the part responsible for the nonlocal condition in the form

$$\left|\left[1-\sum_{k=1}^m\alpha_k\frac{\sinh(\sqrt{z(\zeta,\nu)}\xi_k)}{\sinh(\sqrt{z(\zeta,\nu)})}\right]^{-1}\right|\le Q<\infty,\tag{3.2}$$

for $w\in D_{d_1}$. Now, for $\mathcal{F}(x,w)$ we obtain

$$\begin{aligned}\|\mathcal{F}(x,w)\|&\le(1+M)QK\frac{\mathrm{e}^{(x-1)\sqrt{a(\nu)\cosh\zeta}}}{1-\mathrm{e}^{-2\sqrt{a(\nu)\cosh\zeta}}}\\&\times\frac{\sqrt{a^2(\nu)\sinh^2\xi+b^2(\nu)\cosh^2\xi}}{(a^2(\nu)\cosh^2\xi+b^2(\nu)\sinh^2\xi)^{(1+\alpha)/2}}\|A^\alpha u_1\|\\&\le c\frac{b(\nu)}{a(\nu)}\left(\frac{2}{a(\nu)}\right)^\alpha\mathrm{e}^{(x-1)\sqrt{a(\nu)\cosh\xi}-\alpha|\xi|}\|A^\alpha u_1\|\\&\le c\tan\left(\frac{d_1}{2}+\varphi-\nu\right)\left(\frac{2\cos\varphi}{\rho_0\cos\left(\frac{d_1}{2}+\varphi-\nu\right)}\right)^\alpha\mathrm{e}^{-\alpha|\xi|}\|A^\alpha u_1\|,\\&\forall w\in D_d.\end{aligned}$$

Taking into account that the integrals over the vertical sides of the rectangle $D_{d_1}(\epsilon)$ vanish as $\epsilon\to0$, the above estimate for $\|\mathcal{F}(t,w)\|$ implies

$$\begin{aligned}&\|\mathcal{F}(x,\cdot)\|_{\mathbf{H}^1(D_{d_1})}\le\|A^\alpha u_1\|[C_-(\varphi,\alpha)\\&+C_+(\varphi,\alpha)]\int_{-\infty}^\infty\mathrm{e}^{-\alpha|\xi|}d\xi=C(\varphi,\alpha)\|A^\alpha u_1\|\end{aligned}\tag{3.3}$$

with

$$C(\varphi,\alpha)=\frac{2}{\alpha}[C_+(\varphi,\alpha)+C_-(\varphi,\alpha)],$$

$$C_\pm(\varphi,\alpha)=c\tan\left(\frac{d_1}{2}+\varphi\pm\frac{d_1}{2}\right)\left(\frac{2\cos\varphi}{\rho_0\cos\left(\frac{d_1}{2}+\varphi\pm\frac{d_1}{2}\right)}\right)^\alpha.$$

Note, that the influence of both the smoothness parameter of u_1 (given by α) and of the spectral characteristics of the operator A (given by φ and ρ_0) are accounted by the fact, that the constant $C(\varphi,\alpha)$ from (2.13) tends to ∞ if $\alpha\to 0$, $\varphi\to\pi/2$ or $\rho_1\to 0$ (in this case due to (2.12) $d_1\to\frac{\pi}{2}-\varphi$).

We approximate integral (2.11) by the following Sinc-quadrature formula [10, 11, 23–25]:

$$u_N(x)=\frac{h}{2\pi i}\sum_{k=-N}^{N}\mathcal{F}(x,z(kh)), \tag{3.4}$$

with the error

$$\begin{aligned}\|\eta_N(\mathcal{F},h)\|&=\|u(x)-u_N(x)\|\\ \le\left\|u(x)-\frac{h}{2\pi i}\sum_{k=-\infty}^{\infty}\mathcal{F}(x,z(kh))\right\|&+\left\|\frac{h}{2\pi i}\sum_{|k|>N}\mathcal{F}(x,z(kh))\right\|\\ &\le\frac{1}{2\pi}\frac{e^{-\pi d_1/h}}{2\sinh(\pi d_1/h)}\|\mathcal{F}\|_{\mathbf{H}^1(D_{d_1})}\\ &+\frac{C(\varphi,\alpha)h\|A^\alpha u_1\|}{2\pi}\sum_{k=N+1}^{\infty}e^{(x-1)\sqrt{a_I\cosh(kh)}-\alpha kh}.\end{aligned}$$

We obtain the following inequality

$$\begin{aligned}h\sum_{k=N+1}^{\infty}e^{(x-1)\sqrt{a_I\cosh(kh)}-\alpha kh}&\le h\sum_{k=N+1}^{\infty}e^{(x-1)\sqrt{a_I}\cosh(kh/2)-\alpha kh}\\ &\le\frac{1}{\alpha}\exp[(x-1)\sqrt{a_I}\cosh((N+1)h/2)-\alpha(N+1)h],\end{aligned}$$

using the elementary estimate

$$\sqrt{\cosh\xi}=\sqrt{2\cosh^2(\xi/2)-1}\ge\sqrt{\cosh^2(\xi/2)}=\cosh(\xi/2).$$

Therefore, we have

$$\begin{aligned}\|\eta_N(\mathcal{F},h)\|\le&\frac{c\|A^\alpha u_0\|}{\alpha}\left\{\frac{e^{-\pi d_1/h}}{\sinh(\pi d_1/h)}\right.\\ &\left.+\exp[(x-1)\sqrt{a_I}\cosh((N+1)h/2)-\alpha(N+1)h]\right\},\end{aligned} \tag{3.5}$$

where the constant c does not depend on h, N, x.

Equalizing the both exponentials for $x=1$ gives us

$$\frac{\pi d_1}{h}=\alpha(N+1)h.$$

Which will give us the formula for the step size h

$$h=\sqrt{\frac{\pi d_1}{\alpha(N+1)}} \tag{3.6}$$

with the following error estimate

$$\|\eta_N(\mathcal{F},h)\|\le\frac{c}{\alpha}\exp\left(-\sqrt{\pi d_1\alpha(N+1)}\right)\|A^\alpha u_1\| \tag{3.7}$$

here the constant c is independent on x, N. In the case $x < 1$ the first summand in the argument of $\exp[(x-1)\sqrt{a_I}\cosh((N+1)h/2) - \alpha(N+1)h]$ in (3.5) contributes mainly to the error. Setting for such case $h = c_1 \ln N/N$ with some positive constant c_1 we obtain for a fixed x the following estimate:

$$\|\eta_N(\mathcal{F}, h)\| \leq c\left[\mathrm{e}^{-\pi d_1 N/(c_1 \ln N)} + \mathrm{e}^{-c_1(x-1)\sqrt{a_I}N/2 - c_1\alpha \ln N}\right]\|A^\alpha u_1\|. \tag{3.8}$$

Thus, we have proven the following theorem.

Theorem 3.1 *Let A be a densely defined strongly positive operator, $u_1 \in D(A^\alpha)$, $\alpha \in (0,1)$ and condition* (2.14) *is valid. Then Sinc-quadrature* (3.4) *represents an approximate solution of the nonlocal problem* (1.1). *It provides the convergence of exponential order uniformly with respect to $x \in [0,1]$ presented by the estimate* (3.7) *for the step size h defined in* (3.6). *The approximation has the convergence rate* (3.8) *for the case when $x < 1$ and $h = c_1 \ln N/N$.*

Remark 3.2 The integration curve Γ_I is symmetric with respect to the real axis. Therefore $z(-kh) = \overline{z(kh)}$ and $z'(-kh) = -\overline{z'(kh)}$. Approximation (3.4) can be rewritten in the form

$$u_N(x) = \frac{h}{2\pi i}\mathcal{F}(x, z(0)) + Re\left[\sum_{k=1}^{N} h\frac{\mathcal{F}(x, z(kh))}{\pi i}\right],$$

which essentially reduce the number of resolvent calculations by factor of two.

4. Numerical example

Example 4.1 Let us consider the problem (1.1) with the operator A defined by

$$\begin{aligned} &D(A) = \{v(y) \in H^2(0,1) : \ v(0) = v(1) = 0\}, \\ &Av = -v''(y) \ \forall v \in D(A), \end{aligned} \tag{4.1}$$

and nonlocal condition of Bicadze–Samarskii type:

$$u(1) = \frac{1}{3}u(0.5) + u_1,$$

with

$$u_1 = \frac{3\sinh(\pi) - \sinh(0.5\pi)}{3\sinh(\pi)}\sin(\pi y).$$

N	ε_N
4	0.016443141543169
8	0.006303329836203
16	0.000051494072587
32	$9.644507360032 * 10^{-8}$
64	$7.479410192751 * 10^{-19}$
128	$1.777372592378 * 10^{-25}$
256	$1.526391795652 * 10^{-35}$

Tabl. 4.1. The error for $x = 0.25,\ y = 0.5$

This problem can be rewritten in the 2-D form

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0, \qquad x, y \in (0,1)$$

$$u(0,y) = 0, \quad u(x,0) = u(x,1) = 0,$$

$$u(1,y) = \frac{1}{3}u(0.5,y) + \frac{3\sinh(\pi) - \sinh(0.5\pi)}{3\sinh(\pi)}\sin(\pi y).$$

The exact solution of the problem is $u(x,y) = \sinh(\pi x)\sin(\pi y)$. We have performed calculations Maple. The errors at $x = 0.25,\ y = 0.5$ are presented in Table 4.1 for different N. The table clearly exhibits an exponential decay of error according to the theoretical estimate.

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Received 16.05.2011