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ON ONE CLASS OF INTERPOLATING FORMULAS FOR FUNCTIONS OF MATRIX VARIABLES

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Анотація. Розглянуто тригонометричні і експоненціальні інтерполяційні матричні многочлени різної структури. Отримано тригонометричний варіант формули Лагранжа–Сильвестра, а також формули Ерміта–Біркгофа для випадку матриць. Розглянуто многочлени, в які входять псевдообернені матриці. Побудовано формули з використанням диференціалів Гато і для них знайдено вигляд поліномів, відносно яких формули інваріантні.

Abstract. The trigonometric and exponential interpolation matrix polynomials of various structure are considered. The trigonometric variant of the Lagrange–Silvester formula and also the Hermite–Birkhoff formulas for the case of matrixes are obtained. Polynomials into which pseudoinverse matrixes enter are considered. Formulas with use of Gateaux differentials are constructed and the form of polynomials, for which these formulas are invariant, is found.

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1. Introduction

In the interpolation theory of functions of matrix variables it seems that the problem of algebraic interpolation is mostly completely considered. Interpolation algebraic matrix polynomials of different types (Lagrange's, Newton's, Hermite's, Hermite–Birkhoff's and others) are constructed.

Several of these formulas for functions of one matrix variable were obtained and analyzed in monograph [1], and also in papers [2]- [7]. Side with interpolation of functions on sets of matrices with usual multiplication the problem of interpolation is considered also on sets of matrices with another rules of multiplication: Jordan's and Kronecker's multiplication, multiplication by Hadamard and Frobenius [4, 7].

In paper [8] interpolation polynomials for matrix multivariable functions were constructed on sets of stationary and functional matrices. The interpolation problem of multivariable functionals is discussed in work [9].

In the theory of interpolation of functions of scalar arguments interpolation polynomials construct with respect to arbitrary Chebyshev systems of functions and its particular cases: trigonometric, exponential, fractional-rational and other classes of Chebyshev systems. Such interpolation formulas are also useful in different areas of mathematics and its applications.

In the present paper we obtained trigonometric interpolation matrix polynomials of Lagrange's and Hermite's types and interpolation polynomials with respect to matrix exponents.

2. Trigonometric interpolation matrix polynomials

In this section interpolation matrix polynomials of mentioned two types for 2π-periodic entire functions are constructed. Both cases of stationary, and functional matrices are considered here.

Theorem 2.1 *For entire 2π-periodic function $F(z)$ and matrix A with different eigenvalues $\lambda_0, \lambda_1, ..., \lambda_n$ for which*

$$\cos\lambda_k \neq \cos\lambda_\nu$$

Key words. Interpolation, matrix functions, interpolation matrix polynomials, formulas of Lagrange and Hermite type, pseudoinverse matrix.

when $k \neq \nu$ the following equality occurs:

$$F(A) = \tag{2.1}$$

$$= \sum_{k=0}^{n} \frac{(\cos A - \cos\lambda_0 I)\cdots(\cos A - \cos\lambda_{k-1} I)(\cos A - \cos\lambda_{k+1} I)\cdots(\cos A - \cos\lambda_n I)}{(\cos\lambda_k - \cos\lambda_0)\cdots(\cos\lambda_k - \cos\lambda_{k-1})(\cos\lambda_k - \cos\lambda_{k+1})\cdots(\cos\lambda_k - \cos\lambda_n)}\times$$

$$\times\left[\frac{F(\lambda_k)+F(-\lambda_k)}{2} I + \frac{F(\lambda_k)-F(-\lambda_k)}{2\sin\lambda_k}\sin A\right],$$

where I is an identity matrix.

Proof. Let $F(z)$ be entire 2π-periodic function $F(z+2\pi)=F(z),\ z\in\mathbb{C}$. Fourier series

$$F(z) = \alpha_0 + \sum_{k=1}^{\infty}(\alpha_k\cos kz + \beta_k \sin kz)$$

for these functions converges uniformly in any bounded closed domain of region $0 \le \operatorname{Re} z \le 2\pi$ of complex plane. Thus that $\cos kz$ and $\sin kz$ are expressed in terms of $\cos^\nu z$ ($\nu = 1,\ 2,\ \ldots,\ k$) and $\sin z$ according formulas

$$\cos kz = \sum_{\nu=0}^{k} a_{k\nu}\cos^\nu z,\quad \sin kz = \sin z\sum_{\nu=0}^{k-1} b_{k\nu}\cos^\nu z,$$

where $a_{k\nu},\ b_{k\nu}$ are corresponding numbers, then $F(z)$ can be presented also in the form of series

$$F(z) = \sum_{k=0}^{\infty} a_k\cos^k z + \sin z\sum_{k=0}^{\infty} b_k \cos^k z = F_1(z) + \sin z F_2(z).$$

After applying change $\cos z = \xi$ functions $F_1(z)$ and $F_2(z)$ are reduced to analytical in circle $|\xi| \le r$ $(0<r<\infty)$ functions with respect to variable ξ. If $\lambda_0,\ \lambda_1,\ \ldots,\ \lambda_n$ are different eigenvalues of matrix A, then matrix $\cos A$ will have eigenvalues $\cos\lambda_k$ ($k = 0,\ 1,\ \ldots,\ n$), and using further well-known Lagrange-Silvester's formula (see [9] p. 108) we obtain

$$F_1(A) = \tag{2.2}$$

$$= \sum_{k=0}^{n} \frac{(\cos A - \cos\lambda_0 I)\cdots(\cos A - \cos\lambda_{k-1} I)(\cos A - \cos\lambda_{k+1} I)\cdots(\cos A - \cos\lambda_n I)}{(\cos\lambda_k - \cos\lambda_0)\cdots(\cos\lambda_k - \cos\lambda_{k-1})(\cos\lambda_k - \cos\lambda_{k+1})\cdots(\cos\lambda_k - \cos\lambda_n)}\times$$

$$\times F_1(\lambda_k).$$

$$F_2(A) = \tag{2.3}$$

$$= \sum_{k=0}^{n} \frac{(\cos A - \cos\lambda_0 I)\cdots(\cos A - \cos\lambda_{k-1} I)(\cos A - \cos\lambda_{k+1} I)\cdots(\cos A - \cos\lambda_n I)}{(\cos\lambda_k - \cos\lambda_0)\cdots(\cos\lambda_k - \cos\lambda_{k-1})(\cos\lambda_k - \cos\lambda_{k+1})\cdots(\cos\lambda_k - \cos\lambda_n)}\times$$

$$\times F_2(\lambda_k).$$

Then taking account of relations $F_1(z) = \dfrac{F(z)+F(-z)}{2}$, $F_2(z) = \dfrac{F(z)-F(-z)}{2\sin z}$ and equalities (2.2), (2.3) we obtained formula (2.1). The theorem 2.1 is proved. □

Thus, for any 2π-periodic entire function $F(z)$ matrix A coincides with trigonometric matrix polynomial $T_{n+1}(A)$ of the degree $n+1$, which has form

$$T_{n+1}(A) = \widetilde{a}_0 I + \sum_{k=1}^{n}(\widetilde{a}_k\cos kA + \widetilde{b}_k\sin kA) + \widetilde{b}_{n+1}\sin(n+1)A.$$

Formula (2.1) is trigonometric analog of interpolation Lagrange–Silvester's formula.

For the case of second order matrices (i.e. when $n=1$) formula (2.1) takes form:

$$F(A) = \frac{\cos A - \cos\lambda_1 I}{\cos\lambda_0 - \cos\lambda_1}\left[\frac{F(\lambda_0)+F(-\lambda_0)}{2} I + \frac{F(\lambda_0)-F(-\lambda_0)}{2\sin\lambda_0}\sin A\right] + \tag{2.4}$$

$$+\frac{\cos A - \cos\lambda_0 I}{\cos\lambda_1 - \cos\lambda_0}\left[\frac{F(\lambda_1)+F(-\lambda_1)}{2} I + \frac{F(\lambda_1)-F(-\lambda_1)}{2\sin\lambda_1}\sin A\right].$$

Example 2.2 Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a given matrix, and $\lambda_0,\ \lambda_1\ (\lambda_0 \neq \lambda_1)$ are its eigenvalues. Let us calculate matrices $F(A) = e^{\cos A}$ and $F(A) = e^{\sin A}$.

According to formula (2.4) we have for $F(A) = e^{\cos A}$

$$e^{\cos A} = \frac{e^{\cos\lambda_0} - e^{\cos\lambda_1}}{\cos\lambda_0 - \cos\lambda_1}\cos A + \frac{e^{\cos\lambda_1}\cos\lambda_0 - e^{\cos\lambda_0}\cos\lambda_1}{\cos\lambda_0 - \cos\lambda_1} I. \tag{2.5}$$

Analogically for the function $F(A) = e^{\sin A}$ we get

$$e^{\sin A} = \left[\frac{e^{\sin\lambda_0} - e^{-\sin\lambda_0}}{4\sin\lambda_0(\cos\lambda_0 - \cos\lambda_1)} + \frac{e^{\sin\lambda_1} - e^{-\sin\lambda_1}}{4\sin\lambda_1(\cos\lambda_1 - \cos\lambda_0)}\right]\sin 2A + \tag{2.6}$$

$$+\frac{e^{\sin\lambda_0} + e^{-\sin\lambda_0} - e^{\sin\lambda_1} - e^{-\sin\lambda_1}}{2(\cos\lambda_0 - \cos\lambda_1)}\cos A -$$

$$-\left[\frac{\cos\lambda_1(e^{\sin\lambda_0} - e^{-\sin\lambda_0})}{2\sin\lambda_0(\cos\lambda_0 - \cos\lambda_1)} + \frac{\cos\lambda_0(e^{\sin\lambda_1} - e^{-\sin\lambda_1})}{2\sin\lambda_1(\cos\lambda_1 - \cos\lambda_0)}\right]\sin A +$$

$$+\frac{\cos\lambda_0(e^{\sin\lambda_1} + e^{-\sin\lambda_1}) - \cos\lambda_1(e^{\sin\lambda_0} + e^{-\sin\lambda_0})}{2(\cos\lambda_0 - \cos\lambda_1)} I.$$

In particular, for the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ we have $\lambda_0 = 0,\ \lambda_1 = 1$. By the formula (2.5) for the function $F(A) = e^{\cos A}$ we obtain

$$e^{\cos A} = \frac{e - e^{\cos 1}}{1 - \cos 1}\cos A + \frac{e^{\cos 1} - e\cos 1}{1 - \cos 1} I. \tag{2.7}$$

Using direct series expansion of functions $\cos z$ and $\sin z$, it can be shown that for this matrix

$$\cos A = I - (1 - \cos 1)A,\ \ \sin A = A\sin 1. \tag{2.8}$$

Substituting expressions (2.8) for $\cos A$ and $\sin A$ in (2.5) and (2.6) we get

$$e^{\cos A} = eI + (e^{\cos 1} - e)A.$$

Analogously for the function $F(A) = e^{\sin A}$ we have

$$e^{\sin A} = \left[\frac{e^{\sin 1} - e^{-\sin 1}}{2} - \sin 1\right]A^2 + \left[\sin 1 - 1 + \frac{e^{\sin 1} + e^{-\sin 1}}{2}\right]A + I.$$

Since $A = A^2$ then finally we obtain equality

$$e^{\sin A} = \left(e^{\sin 1} - 1\right)A + I.$$

Further, we consider some other formulas of trigonometric interpolation, in particular, formulas of the form

$$T_n(A) = \sum_{k=0}^{2n} \Psi_k(A)\Psi_k^{-1}(A_k)F(A_k), \tag{2.9}$$

where

$$\Psi_k(A) = \sin\frac{A - A_0}{2}\cdots\sin\frac{A - A_{k-1}}{2}\sin\frac{A - A_{k+1}}{2}\cdots\sin\frac{A - A_{2n}}{2}.$$

This formula exists if matrices $\sin\dfrac{A_\nu - A_k}{2}$ $(\nu \neq k)$ are invertible.

Consider now interpolation matrix polynomial, in the case, when the existence of inverse matrices $\sin^{-1}\dfrac{A_\nu - A_k}{2}$ is not required, moreover matrix A and nodes of interpolation A_k can be even rectangular.

Let S_{lr} and S_{rl} be $l \times r$– and $r \times l$– matrices $(r \geq l)$ of the following structures:

$$S_{lr} = [I_l | O_{l,r-l}] \ \text{and}\ S_{rl} = \begin{bmatrix} I_l \\ O_{r-l,l} \end{bmatrix},$$

where I_l is the identity square matrix of the dimension l, where $O_{l,r-l}$ and $O_{r-l,l}$ are null matrices of introduced dimensions. It is obvious that $S_{lr}S_{rl} = E_l$.

Suppose

$$\Psi_k(A) = \prod_{i=0,\ i\neq k}^{2n} \sin\frac{A - A_i}{2} \sin^+\frac{A_k - A_i}{2},$$

where $\sin^+\frac{A_k - A_i}{2}$ is pseudoinverse Moor-Penrose matrix for matrix $\sin\frac{A_k - A_i}{2}$, which always exists and unique [9] for arbitrary matrix, and let r_k and l_k be the ranks of the matrices $\Psi_k(A_k)$ and $F(A_k)$ $(k = 0,\ 1,\ ...,\ 2n)$.

Theorem 2.3 *Let $\Psi_k(A_k) = B_kC_k$ and $F(A_k) = M_kN_k$ be the skeleton representations of the matrices $\Psi_k(A_k)$ and $F(A_k)$ $(k = 0,\ 1,\ ...,\ 2n)$. Then for the matrix polynomial*

$$T_n(A) = \sum_{k=0}^{2n} F(A_k)N_k^+ S_{l_k r_k} B_k^+ \Psi_k(A) C_k^+ S_{r_k l_k} M_k^+ F(A_k)$$

at the condition, that $l_k \leq r_k$ $(k = 0,\ 1,\ ...,\ 2n)$, the equations

$$T_n(A_\nu) = F(A_\nu)\ (\nu = 0,\ 1,\ ...,\ 2n)$$

hold true.

Proof. Note, that

$$\Psi_k(A_\nu) = \delta_{k\nu}\Psi_k(A_k) = \delta_{k\nu}B_kC_k,$$

where $\delta_{k\nu}$ is Kronecker symbol. It follows from the properties of the skeleton representations of the matrices $\Psi_\nu(A_\nu)$ and $F(A_\nu)$ that

$$B_\nu^+B_\nu = C_\nu C_\nu^+ = I_{r_\nu}, \quad \text{and} \quad F^+(A_\nu) = N_\nu^+ M_\nu^+.$$

Therefore,

$$T_n(A_\nu) = F(A_\nu)N_\nu^+ S_{l_\nu r_\nu} I_{r_\nu} S_{r_\nu l_\nu} M_\nu^+ F(A_\nu) = F(A_\nu)F^+(A_\nu)F(A_\nu) = F(A_\nu).$$

The theorem 2.3 is proved. □

Further we consider interpolation trigonometric polynomials of Hermite's type. Formulas of this form besides the values of interpolated function contain also the values of its derivatives in all or only isolated nodes of interpolation. It should be noted that construction of generalized variants of these formulas requires the coincidence between the values of differential operators of interpolation polynomials and the values of differential operators of interpolated function in given nodes. A number of such formulas for the functions of scalar argument is obtained in [10]. In particular, it was constructed the trigonometric polynomial of the degree $n + 1$ in the form

$$T_{n+1}(t) = H_n(t) + \frac{2^{2n}}{(2n+1)!}\frac{\Omega_{n+1}(t)}{\cos\left((n+1)t_j - t_0 - \frac{1}{2}\sum_{k=1}^{2n} t_k\right)} L_{2n+1}(f;\ t_j), \tag{2.10}$$

where

$$H_n(t) = \frac{1}{2}\sum_{k=0}^{2n} \frac{\psi_n(t)}{\sin\frac{t - t_k}{2}\psi_n'(t_k)} f(t_k), \quad \psi_n(t) = \sin\frac{t - t_0}{2}\sin\frac{t - t_1}{2}\cdots\sin\frac{t - t_{2n}}{2},$$

$$\Omega_{n+1}(t) = 2\cos\frac{t - t_0}{2}\psi_n(t)$$

and

$$\cos\left((n+1)t_j - t_0 - \frac{1}{2}\sum_{k=1}^{2n} t_k\right) \neq 0,$$

for which the equality $T_{n+1}(t_i) = f(t_i)$ holds true in all nodes of interpolation t_i $(0 \le t_i < 2\pi,\ i = 0,\ 1,\ ...,\ 2n)$ and also in one of fixed nodes t_j the equality $L_{2n+1}(T_{n+1};\ t_j) = L_{2n+1}(f;\ t_j)$ takes place.

We consider the matrix version of formula (2.10). Suppose, that X is the set of square matrices, and $F(z)$ is entire 2π-periodic function, $z \in \mathbb{C}$. Let $A_k \in X$ $(k = 0,\ 1,\ ...,\ 2n)$ be the set of different matrix nodes. The values $F(A_i)$ of the function $F(A)$ are given in these nodes. Besides at one of the nodes A_j the value of the operator $L_{2n+1}(F;\ A_j) \equiv L_{2n+1}F(A_j)$ is given, where L_{2n+1} is the differential matrix operator of the form

$$L_{2n+1}F(A) = D(D^2 + 1^2) \cdots (D^2 + n^2)F(z)|_{z=A},\quad D = \frac{d}{dz}.$$

Further we specify the trigonometric polynomial $T_{n+1}(A)$ of the degree $n + 1$ for which the conditions

$$T_{n+1}(A_i) = F(A_i)\ \ (i = 0,\ 1,\ ...,\ 2n);\quad L_{2n+1}(T_{n+1};\ A_j) = L_{2n+1}(F;\ A_j) \tag{2.11}$$

hold true.

Theorem 2.4 *The trigonometric polynomial*

$$T_{n+1}(F;\ A) = H_n(A) + \tag{2.12}$$

$$+\frac{2^{2n}}{(2n+1)!}\cos^{-1}\left((n+1)A_j - A_0 - \frac{1}{2}\sum_{k=1}^{2n} A_k\right)\Omega_{n+1}(A)L_{2n+1}(F;\ A_j),$$

where

$$H_n(A) = \sum_{k=0}^{2n} \Psi_k(A)\Psi_k^{-1}(A_k)F(A_k), \tag{2.13}$$

$$\Psi_k(A) = \sin\frac{A - A_0}{2} \cdots \sin\frac{A - A_{k-1}}{2} \sin\frac{A - A_{k+1}}{2} \cdots \sin\frac{A - A_{2n}}{2},$$

$$\Omega_{n+1}(A) = 2\cos\frac{A - A_0}{2} \prod_{k=0}^{2n} \sin\frac{A - A_k}{2},$$

the matrices

$$\sin\frac{A_k - A_\nu}{2}\quad (k \ne \nu)$$

and

$$\cos\left((n+1)A_j - A_0 - \frac{1}{2}\sum_{k=1}^{2n} A_k\right)$$

are invertible, satisfies the first group of conditions (2.11). *If matrices* A_k $(k = 0, 1, ..., 2n)$ *are mutually permutable, then polynomial* (2.12) *satisfies also the second condition of* (2.11).

Proof. It is obvious that $H_n(A_i) = F(A_i)$ $(i = 0,\ 1,\ ...,\ 2n)$. As multiplier $\sin\dfrac{A - A_i}{2}$ enters in product $\Omega_{n+1}(A)$, then $\Omega_{n+1}(A_i) = 0$ $(i = 0,\ 1,\ ...,\ 2n)$. Thus, the first group of conditions (2.11) holds true.

Under the condition that matrix A is permutable with matrices A_k $(k = 0,\ 1,\ ...,\ 2n)$, matrix polynomials $\Psi_k(A)$ $(k = 0,\ 1,\ ...,\ 2n)$ and consequently polynomial $H_n(A)$ are representable in the form

$$H_n(A) = \widetilde{A}_0 + \sum_{k=1}^{n}\left[\widetilde{A}_k \cos kA + \widetilde{B}_k \sin kA\right],$$

where $\widetilde{A}_0$, $\widetilde{A}_k$, $\widetilde{B}_k$ $(k = 1,\ 2,\ ...,\ n)$ are corresponding matrices, which are not dependent on variable A.

Since

$$L_{2n+1}(\cos kA) = L_{2n+1}(\sin kA) = 0\ \ (k = 0,\ 1,\ ...,\ 2n), \tag{2.14}$$

that also $L_{2n+1}(H_n(A)) = 0$.

At the same conditions of commutativity of matrices A and A_k it can be shown by mathematical induction method that the equality

$$\Omega_{n+1}(A) = \tag{2.15}$$
$$= \frac{(-1)^n}{2^{2n}}\left[-\sin\frac{1}{2}\left(A_0+\sum_{i=0}^{2n}A_i\right)\cos(n+1)A+\cos\frac{1}{2}\left(A_0+\sum_{i=0}^{2n}A_i\right)\sin(n+1)A\right]+$$
$$+\widetilde{A}_{n0}+\sum_{k=1}^{n}\left[\widetilde{A}_{nk}\cos kA+\widetilde{B}_{nk}\sin kA\right].$$

Since

$$L_{2n+1}[cos(n+1)A] = (-1)^{n+1}(2n+1)!\sin(n+1)A,$$

and

$$L_{2n+1}[sin(n+1)A] = (-1)^{n}(2n+1)!\cos(n+1)A,$$

then we obtain

$$L_{2n+1}\left[\Omega_{n+1}(A)\right] = \tag{2.16}$$
$$= \frac{(2n+1)!}{2^{2n}}\left[\sin\frac{1}{2}\left(A_0+\sum_{i=0}^{2n}A_i\right)\sin(n+1)A+\cos\frac{1}{2}\left(A_0+\sum_{i=0}^{2n}A_i\right)\cos(n+1)A\right]=$$
$$=\frac{(2n+1)!}{2^{2n}}\cos\left((n+1)A-A_0-\frac{1}{2}\sum_{i=1}^{2n}A_i\right).$$

Finally, substituting node A_j instead of A in (2.16), we obtain, that the second equality in (2.11) holds true. The theorem 2.4 is proved. □

Further we construct formula, similar to (2.12), in which the operator of the form $L_{2n+1}F(A)$ will be define by means of Gateaux differentials of the function $F(A)$.

Let us introduce the following notations

$$\widetilde{L}_1F(A) = \delta F[A;\ H_0] \equiv \widetilde{D}F(A),$$

$$\widetilde{L}_3F(A) = \widetilde{D}\left(\widetilde{D}^2+H_2H_1\right)F(A) = \delta^3F[A;\ H_2H_1H_0]+H_2H_1\delta F[A;\ H_0],$$

where $\delta F[A;\ H_0]$ is the first order Gateaux differential of F at the point A in the direction H_0, and $\delta^3F[A; H_2H_1H_0]$ is the third order Gateaux differential of $F(A)$ at the same point in the directions $H_0,\ H_1,\ H_2$.

Denote $\varphi_n(A) = \sin nA,\ \psi_n(A) = \cos nA$ and let the matrices A and H be permutable. Then for Gateaux differentials of the functions $\varphi_n(A)$ и $\psi_n(A)$ in the direction H the equalities

$$\delta\varphi_n[A;\ H] = nH\psi_n(A);\quad \delta\psi_n[A;\ H] = -nH\varphi_n(A)$$

hold true.

It is easy to verify, that the functions $F(A) = \varphi_1(A),\ F(A) = \psi_1(A)$ and also arbitrary matrix function, not depending on A, are the solution of equation $\widetilde{L}_3F(A) = 0$.

Analogously for the differential-matrix operator of the general form

$$\widetilde{L}_{2n+1}F(A) = \tag{2.17}$$
$$= \widetilde{D}\left(\widetilde{D}^2+1^2H_2H_1\right)\left(\widetilde{D}^2+2^2H_4H_3\right)\cdots\left(\widetilde{D}^2+n^2H_{2n}H_{2n-1}\right)F(A)$$

the equation $\widetilde{L}_{2n+1}F(A) = 0$ has the solutions $\varphi_k(A) = \sin kA$ and $\psi_k(A) = \cos kA$ ($k = 0,\ 1,\ ...,\ n$).

Suppose, that the matrices A, A_ν and H_k ($\nu,\ k = 0,\ 1,\ ...,\ 2n$) are mutually permutable. Further we construct the trigonometric polynomial $\widetilde{T}_{n+1}(A)$ of the degree $n+1$, for which the conditions

$$\widetilde{T}_{n+1}(A_i) = F(A_i)\ \ (i = 0,\ 1,\ ...,\ 2n);\ \ \widetilde{L}_{2n+1}\left(\widetilde{T}_{n+1};\ A_j\right) = \widetilde{L}_{2n+1}(F;\ A_j) \tag{2.18}$$

would be satisfied.

Theorem 2.5 *The trigonometric polynomial*

$$\widetilde{T}_{n+1}(F;\ A) = H_n(A) + \frac{2^{2n}}{(2n+1)!}\cos^{-1}\left((n+1)A_j - A_0 - \frac{1}{2}\sum_{k=1}^{2n} A_k\right)\times \tag{2.19}$$

$$\times\Omega_{n+1}(A)H_0^{-1}H_1^{-1}\cdots H_{2n}^{-1}\widetilde{L}_{2n+1}(F;\ A_j),$$

where $H_n(A)$ is the matrix polynomial (2.13) with the nodes of interpolation A_k, the same as in previous theorem, and the matrices $H_0,\ H_1,\ \ldots,\ H_{2n}$ are invertible, satisfies the first group of conditions (2.18). If matrices $A,\ A_k$ and H_ν $(k,\ \nu = 0,\ 1,\ \ldots,\ 2n)$ are mutually permutable, then polynomial (2.19) satisfies also the second condition of (2.18).

Proof. At the proof of this theorem the equalities

$$\delta^2[\psi_k(A);\ H_{2k}H_{2k-1}] + k^2 H_{2k}H_{2k-1}\psi_k(A) = 0,$$

$$\delta^2[\varphi_k(A);\ H_{2k}H_{2k-1}] + k^2 H_{2k}H_{2k-1}\varphi_k(A) = 0$$

are used.

Therefore

$$\widetilde{L}_{2n+1}[\psi_k(A)] = \widetilde{L}_{2n+1}[\varphi_k(A)] = 0 \quad (k = 0,\ 1,\ \ldots,\ n),$$

and respectively $\widetilde{L}_{2n+1}[H_n(A)] = 0$. Since

$$\delta^2[\psi_{n+1}(A);\ H_{2k}H_{2k-1}] + k^2\psi_{n+1}(A)H_{2k}H_{2k-1} =$$
$$= -\left[(n+1)^2 - k^2\right]\psi_{n+1}(A)H_{2k}H_{2k-1},$$

then we have

$$\widetilde{L}_{2n+1}[\psi_{n+1}(A)] = (-1)^{n+1}(2n+1)!\varphi_{n+1}(A)H_{2n}\cdots H_1H_0,$$

and since

$$\delta^2[\varphi_{n+1}(A);\ H_{2k}H_{2k-1}] + k^2\varphi_{n+1}(A)H_{2k}H_{2k-1} =$$
$$= -\left[(n+1)^2 - k^2\right]\varphi_{n+1}(A)H_{2k}H_{2k-1},$$

then

$$\widetilde{L}_{2n+1}[\varphi_{n+1}(A)] = (-1)^n(2n+1)!\psi_{n+1}(A)H_{2n}\cdots H_1H_0.$$

And consequently

$$L_{2n+1}\left[\Omega_{n+1}(A)\right] = \tag{2.20}$$

$$= \frac{(2n+1)!}{2^{2n}}\left[\sin\frac{1}{2}\left(A_0 + \sum_{i=0}^{2n} A_i\right)\sin(n+1)A + \cos\frac{1}{2}\left(A_0 + \sum_{i=0}^{2n} A_i\right)\cos(n+1)A\right]\times$$

$$\times H_{2n}\cdots H_1H_0 = \frac{(2n+1)!}{2^{2n}}\cos\left((n+1)A - A_0 - \frac{1}{2}\sum_{i=1}^{2n} A_i\right)H_{2n}\cdots H_1H_0.$$

Substituting node A_j instead of A in (2.20), we obtain, that the second equality in (2.18) holds true. The theorem 2.5 is proved. □

Now we perform one more formula, which contains the first order Gateaux differential of interpolated function.

Theorem 2.6 *Let $A_k = A_k(t)$ $(k = 0,\ 1,\ 2)$ be the nodes of interpolation, and also the matrices $[\cos(A_0 - A_2) - \cos(A_1 - A_2)]^{-1}$, $\sin^{-1}\dfrac{2A_2 - A_0 - A_1}{2}$ exist for arbitrary $t \in T \subseteq \mathbb{R}$. Then for the matrix polynomial*

$$L_2(A) = F(A_0) + \int_0^1 \delta F[A_0 + \tau(A_1 - A_0);\ l_{10}(A)(A_1 - A_0)]d\tau + \delta F[A_2;\ l_{11}(A)], \tag{2.21}$$

where

$$l_{10}(A) = [\cos(A_0 - A_2) - \cos(A_1 - A_2)]^{-1} [\cos(A_0 - A_2) - \cos(A - A_2)], \tag{2.22}$$

$$l_{11}(A) = \sin^{-1} \frac{2A_2 - A_0 - A_1}{2} \left[\cos \frac{A_1 - A_0}{2} - \cos \frac{2A - A_0 - A_1}{2}\right], \tag{2.23}$$

the interpolation conditions

$$L_2(A_i) = F(A_i) \ (i = 0,\ 1); \ \ \delta L_2[A_2;\ H] = \delta F[A_2;\ H], \tag{2.24}$$

hold true, at the condition, that the matrices A, A_k and H are permutable.

If the matrices A, A_k are mutually permutable, then formula (2.21) *is invariant with respect to polynomials of the form*

$$P_1(A) = C_0(s) + \int_T C_1(s,t) \cos A(t) D_1(s,t) dt + \int_T C_2(s,t) \sin A(t) D_2(s,t) dt, \tag{2.25}$$

where $C_0(s)$, $C_1(s,t)$, $C_2(s,t)$, $D_1(s,t)$, $D_2(s,t)$ $(s \in \mathbb{R}^m,\ t \in T \subseteq \mathbb{R})$ are given matrices.

Proof. Since $l_{10}(A_0) = l_{11}(A_0) = 0$, then $L_2(A_0) = F(A_0)$. Because of $l_{10}(A_1) = I$, and $l_{11}(A_1) = 0$, then the condition $L_2(A_1) = F(A_1)$ holds true. Further we verify the truth of the second condition in (2.24).

The Gateaux differential of the polynomial (2.21) at the point $A = A(t)$ in the direction H has the form

$$\delta L_2[A;\ H] = \int_0^1 \delta F\left[A_0 + \tau(A_1 - A_0);\ \delta l_{10}[A;\ H](A_1 - A_0)\right] d\tau + \delta F[A_2;\ \delta l_{11}[A;\ H]],$$

where

$$\delta l_{10}[A;\ H] = [\cos(A_0 - A_2) - \cos(A_1 - A_2)]^{-1} \sin(A - A_2) H,$$

$$\delta l_{11}[A;\ H] = \sin^{-1} \frac{2A_2 - A_0 - A_1}{2} \sin \frac{2A - A_0 - A_1}{2} H.$$

From this it follows that $\delta L_2[A_2;\ H] = \delta F[A_2;\ H]$, i.e. the second condition in (2.24) also takes place.

Now we show that formula (2.21) is exact for the polynomial of the form (2.25).

Suppose that $F_0(A) = B$, $F_1(A) = \cos A$, $F_2(A) = \sin A$, where B is given matrix.

It is obvious, that $L_2(F_0;\ A) = F_0(A)$. Now we calculate for $F(A) = F_1(A)$ the second summand in the quality (2.21). Since

$$\delta \cos\left[A_0 + \tau(A_1 - A_0);\ l_{10}(A)(A_1 - A_0)\right] = -\sin\left(A_0 + \tau(A_1 - A_0)\right) l_{10}(A)(A_1 - A_0),$$

then because of permutability of the matrices A_0 and $\tau(A_1 - A_0)$ it takes the form

$$-\int_0^1 \sin\left(A_0 + \tau(A_1 - A_0)\right)(A_1 - A_0) l_{10}(A) d\tau = [\cos A_1 - \cos A_0]\, l_{10}(A).$$

Since $\delta \cos[A_2;\ l_{11}(A)] = -\sin A_2 l_{11}(A)$, then we have

$$L_2(F_1;\ A) = \cos A_0 + [\cos A_1 - \cos A_0]\, l_{10}(A) - \sin A_2 l_{11}(A) = \cos A = F_1(A).$$

Analogously it can be shown that formula (2.21) is invariant with respect to the function $F_2(A) = \sin A$. Thus we get

$$L_2(F_2;\ A) = \sin A_0 + [\sin A_1 - \sin A_0]\, l_{10}(A) + \cos A_2 l_{11}(A) = \sin A = F_2(A).$$

Considering the structure of the polynomial (2.25), we finally obtain, that $L_2(P_1;\ A) = P_1(A)$. The theorem 2.6 is proved. □

3. Exponential interpolation matrix polynomials

At first we consider the first order Hermite–Birkhoff interpolation formula, constructed on base of the exponential functions $\varphi_k(A) = e^{\lambda_k A}$ $(k = 0,\ 1,\ 2)$, $A = A(t)$.

Let $A_k = A_k(t) \in X$ $(k = 0,\ 1)$ be the matrix nodes. The values $F(A_k)$ of the function $F(A)$ are given in these nodes and at one of the nodes it is given the value $D_2(F;\ A_j)$ of differential operator of the form

$$D_2F(A) = D(D-\lambda_1)F(z)|_{z=A}, \quad D = \frac{d}{dz}, \tag{3.1}$$

where $0 = \lambda_0 < \lambda_1 < \lambda_2$ are given real numbers.

Theorem 3.1 *If the matrix* $\left[e^{\lambda_1 A_0} - e^{\lambda_1 A_1}\right]$ *is invertible, then for the interpolation polynomial*

$$\widetilde{L}_2(A) = L_1(A) + \frac{\Omega_2(A)e^{-\lambda_2 A_j} D_2(F;\ A_j)}{\lambda_2(\lambda_2-\lambda_1)}, \tag{3.2}$$

where

$$L_1(A) = \left[e^{\lambda_1 A} - e^{\lambda_1 A_1}\right]\left[e^{\lambda_1 A_0} - e^{\lambda_1 A_1}\right]^{-1} F(A_0) +$$
$$+ \left[e^{\lambda_1 A} - e^{\lambda_1 A_0}\right]\left[e^{\lambda_1 A_1} - e^{\lambda_1 A_0}\right]^{-1} F(A_1),$$
$$\Omega_2(A) = e^{\lambda_2 A} - e^{\lambda_2 A_0} + \left[e^{\lambda_1 A} - e^{\lambda_1 A_0}\right]\left[e^{\lambda_1 A_1} - e^{\lambda_1 A_0}\right]^{-1}\left[e^{\lambda_2 A_0} - e^{\lambda_2 A_1}\right],$$

the conditions

$$\widetilde{L}_2(A_k) = F(A_k) \ \ (k = 0,\ 1); \quad D_2\left(\widetilde{L}_2;\ A_j\right) = D_2\left(F;\ A_j\right),$$

where $j = 0$ or $j = 1$, hold true.

If the matrices A_0 and A_1 are permutable, then formula (3.2) is exact for the polynomials of the form

$$P_2(A) = B + e^{\lambda_1 A}C + e^{\lambda_2 A}G, \tag{3.3}$$

where $B,\ C,\ G$ are some given matrices.

Proof. Really, coincidence between the polynomial $\widetilde{L}_2(A)$ and the interpolated function $F(A)$ in the nodes A_k follows from this that $L_1(A_k) = F(A_k)$ and $\Omega_2(A_k) = 0$ for $k = 0,\ 1$.

Since $D_2\left(e^{\lambda_k A}\right) = 0$ $(k = 0,\ 1)$, and also $D_2\left(e^{\lambda_2 A}\right) = \lambda_2(\lambda_2-\lambda_1)e^{\lambda_2 A}$, then we obtain that $D_2\left(\widetilde{L}_2;\ A_j\right) = D_2\left(F;\ A_j\right)$.

Now we verify the invariance of formula (3.2) with respect to the polynomials of the form (3.3). Since $D_2(F;\ A) = 0$ for $F(A) = B$ and $F(A) = e^{\lambda_1 A}$, and also $L_1(A) = F(A)$, then for these functions $\widetilde{L}_2(A) \equiv F(A)$. Further, because of $D_2(F;\ A) = \lambda_2(\lambda_2-\lambda_1)e^{\lambda_2 A}$ for the function $F(A) = e^{\lambda_2 A}$, then after simple evaluations we obtain the formula $\widetilde{L}_2(A) = L_1(A) + \Omega_2(A) \equiv F(A)$. Thus, the formula (3.2) is exact for the polynomials of the form (3.3). The theorem 3.1 is proved. □

Further we construct the Hermite-Birkhoff formula for the case, when the functions $\varphi_k(A) = e^{kA}$ $(k = 0,\ 1,\ ...,\ n)$ are chosen as the basis. Let X be a set of permutable matrices, $A_k \in X$ $(k = 0,\ 1,\ ...,\ n)$ is the collection of matrix nodes. The values $F(A_k)$ of the function $F(A)$ are given in these nodes and at one of these nodes it is given the value $D_{n+1}(F;\ A_j)$ of differential operator of the form

$$D_{n+1}F(A) = \widetilde{D}\left(\widetilde{D} - H_1\right)\left(\widetilde{D} - 2H_2\right)\cdots\left(\widetilde{D} - nH_n\right)F(A), \tag{3.4}$$

where $\widetilde{D}F(A) \equiv \delta F[A;\ H]$ is the Gateaux differential of F at the point A in the direction H.

Theorem 3.2 *If the matrices H_k, $\left[e^{A_k} - e^{A_\nu}\right]$ $(k \neq \nu,\ k,\ \nu = 0,\ 1,\ ...,\ n)$ are invertible, then the Hermite–Birkhoff interpolation polynomial*

$$\widetilde{L}_{n+1}(A) = L_n(A) + \frac{\Omega_{n+1}(A)e^{-(n+1)A_j}H_0^{-1}H_1^{-1}\cdots H_n^{-1}D_{n+1}(F;\ A_j)}{(n+1)!}, \tag{3.5}$$

where

$$L_n(A) = \sum_{k=0}^{n} \Phi_k^{-1}(A_k)\Phi_k(A)F(A_k), \tag{3.6}$$

$$\Phi_k(A) = \left[e^A - e^{A_0}\right] \cdots \left[e^A - e^{A_{k-1}}\right] \left[e^A - e^{A_{k+1}}\right] \cdots \left[e^A - e^{A_n}\right],$$

$$\Omega_{n+1}(A) = \left[e^A - e^{A_0}\right] \left[e^A - e^{A_1}\right] \cdots \left[e^A - e^{A_n}\right],$$

satisfies the conditions

$$\widetilde{L}_{n+1}(A_k) = F(A_k) \;\; (k = 0,\ 1,\ ...,\ n); \;\; D_{n+1}\left(\widetilde{L}_{n+1};\ A_j\right) = D_{n+1}\left(F;\ A_j\right). \tag{3.7}$$

Proof. Since $\Phi_k(A_\nu) = 0$ when $(\nu \neq k)$ and $\Omega_{n+1}(A_\nu) = 0$ $(k,\ \nu = 0,\ 1,\ ...,\ n)$, then the first group of the conditions (3.7) holds true. Further, the polynomial (3.5) can be represented in the form

$$\widetilde{L}_{n+1}(A) = \sum_{k=0}^{n} B_k e^{kA} + \frac{e^{(n+1)A}e^{-(n+1)A_j}H_0^{-1}H_1^{-1}\cdots H_n^{-1}D_{n+1}(F;\ A_j)}{(n+1)!},$$

where B_k $(k = 0,\ 1,\ ...,\ n)$ are matrices, not depending on A. Since $D_{n+1}\left(e^{kA}\right) = 0$ $(k = 0,\ 1,\ ...,\ n)$, and also $D_{n+1}\left(e^{(n+1)A}\right) = (n+1)!H_n \cdots H_1 H_0 e^{(n+1)A}$, then we obtain, that the second equality in (3.7) also takes place. The theorem 3.2 is proved. □

The interpolation formula of the form (3.6) is known. Such formula can be always constructed, if the inverse matrices $\left[e^{A_k} - e^{A_\nu}\right]^{-1}$ exist when $(\nu \neq k)$. However, the obtaining of these inverse matrices for the majority of the matrix nodes A_k is enough challenge.

Let as well as earlier S_{lr} and S_{rl} be $l \times r-$ and $r \times l-$ matrices $(r \geq l)$ of the following structures:

$$S_{lr} = [I_l | O_{l,r-l}] \text{ and } S_{rl} = \begin{bmatrix} I_l \\ O_{r-l,l} \end{bmatrix},$$

where I_l is the identity square matrix of the dimension l, where $O_{l,r-l}$ and $O_{r-l,l}$ are null matrices of introduced dimensions. It is obvious that $S_{lr}S_{rl} = E_l$.

Suppose

$$\Phi_k(A) = \prod_{i=0,\ i\neq k}^{n} \left[e^A - e^{A_i}\right]\left[e^{A_k} - e^{A_i}\right]^+,$$

where $\left[e^{A_k} - e^{A_i}\right]^+$ is pseudoinverse Moor-Penrose matrix for matrix $\left[e^{A_k} - e^{A_i}\right]$, and let r_k and l_k be the ranks of the matrices $\Phi_k(A_k)$ and $F(A_k)$ $(k = 0,\ 1,\ ...,\ n)$.

Theorem 3.3 *Let $\Phi_k(A_k) = B_k C_k$ and $F(A_k) = M_k N_k$ be the skeleton representations of the matrices $\Phi_k(A_k)$ and $F(A_k)$ $(k = 0,\ 1,\ ...,\ n)$. Then for the matrix polynomial*

$$L_n(A) = \sum_{k=0}^{n} F(A_k)N_k^+ S_{l_k r_k} B_k^+ \Phi_k(A) C_k^+ S_{r_k l_k} M_k^+ F(A_k)$$

at the condition, that $l_k \leq r_k$ $(k = 0,\ 1,\ ...,\ n)$, the equations

$$L_n(A_\nu) = F(A_\nu) \;\; (\nu = 0,\ 1,\ ...,\ n)$$

hold true.

The proof is similar to the proof of the theorem 2.2

Further we consider one more formula, which contains the first order Gateaux differential of interpolated function.

Theorem 3.4 *Let $A_k = A_k(t)$ $(k = 0,\ 1,\ 2)$ be the nodes of interpolation, and also the matrices $\left[2e^{A_1+A_2} - e^{2A_1} - 2e^{A_0+A_2} + e^{2A_0}\right]^{-1}$, $\left[2e^{2A_2} - e^{A_1+A_2} - e^{A_0+A_2}\right]^{-1}$ exist for arbitrary $t \in T \subseteq \mathbb{R}$. Then the matrix polynomial*

$$L_2(A) = F(A_0) + \int_0^1 \delta F[A_0 + \tau(A_1 - A_0);\ l_{10}(A)(A_1 - A_0)]d\tau + \delta F[A_2;\ l_{11}(A)], \tag{3.8}$$

where

$$l_{10}(A) = \left[2e^{A_1+A_2} - e^{2A_1} - 2e^{A_0+A_2} + e^{2A_0}\right]^{-1} \times \\ \times \left[2e^{A+A_2} - e^{2A} - 2e^{A_0+A_2} + e^{2A_0}\right], \quad (3.9)$$

$$l_{11}(A) = \left[2e^{2A_2} - e^{A_1+A_2} - e^{A_0+A_2}\right]^{-1} \left[e^{2A} - e^{A+A_0} - e^{A+A_1} + e^{A_0+A_1}\right], \quad (3.10)$$

satisfies the interpolation conditions

$$L_2(A_i) = F(A_i)\ (i = 0,\ 1);\quad \delta L_2[A_2;\ H] = \delta F[A_2;\ H], \quad (3.11)$$

at the condition, that the matrices A, A_k are permutable with the matrix H.

If the matrices A, A_k are mutually permutable, then formula (3.8) is invariant with respect to the polynomials of the form

$$P_2(A) = C_0(s) + \int_T C_1(s,t)e^{A(t)}D_1(s,t)dt + \int_T C_2(s,t)e^{2A(t)}D_2(s,t)dt, \quad (3.12)$$

where $C_0(s),\ C_1(s,t),\ C_2(s,t),\ D_1(s,t),\ D_2(s,t)\ \ (s \in \mathbb{R}^m,\ t \in T \subseteq \mathbb{R})$ are given matrices.

Proof. Since $l_{10}(A_0) = l_{11}(A_0) = 0$, then $L_2(A_0) = F(A_0)$. Owing to that $l_{10}(A_1) = I$, and $l_{11}(A_1) = 0$, the condition $L_2(A_1) = F(A_1)$ takes place. Thus, the first group of the interpolation conditions holds true. Now we verify the truth of the second condition in (3.11).

The Gateaux differential of the polynomial (3.8) at the point $A = A(t)$ in the direction H has the form

$$\delta L_2[A;\ H] = \int_0^1 \delta F\left[A_0 + \tau(A_1 - A_0);\ \delta l_{10}[A;\ H](A_1 - A_0)\right] d\tau + \delta F[A_2;\ \delta l_{11}[A;\ H]],$$

where

$$\delta l_{10}[A;\ H] = \left[2e^{A_1+A_2} - e^{2A_1} - 2e^{A_0+A_2} + e^{2A_0}\right]^{-1} \left[2e^{A+A_2} - 2e^{2A}\right] H,$$

$$\delta l_{11}[A;\ H] = \left[2e^{2A_2} - e^{A_1+A_2} - e^{A_0+A_2}\right]^{-1} \left[2e^{2A} - e^{A+A_0} - e^{A+A_1}\right] H.$$

Thus, it follows that $\delta L_2[A_2;\ H] = \delta F[A_2;\ H]$, i.e. the second conditions in (3.11) also takes place.

Further we show that formula (3.8) is invariant with respect to the polynomials of the form (3.12).

Suppose that $F_0(A) = B,\ F_1(A) = e^A,\ F_2(A) = e^{2A}$, where B is some given matrix.

It is obvious, that $L_2(F_0;\ A) = F_0(A)$. Now we calculate for $F(A) = F_1(A)$ the second summand in the quality (3.8). Since

$$\delta F_1\left[A_0 + \tau(A_1 - A_0);\ l_{10}(A)(A_1 - A_0)\right] = e^{A_0+\tau(A_1-A_0)} l_{10}(A)(A_1 - A_0),$$

then because of permutability of the matrices A_0 and $\tau(A_1 - A_0)$ it takes the form

$$\int_0^1 e^{A_0+\tau(A_1-A_0)} l_{10}(A)(A_1 - A_0)d\tau = \left[e^{A_1} - e^{A_0}\right] l_{10}(A).$$

Since $\delta F_1\left[A_2;\ l_{11}(A)\right] = e^{A_2} l_{11}(A)$, then we have

$$L_2(F_1;\ A) = e^{A_0} + \left[e^{A_1} - e^{A_0}\right] l_{10}(A) + e^{A_2} l_{11}(A) = e^A = F_1(A).$$

The invariance of formula (3.8) with respect to the function $F_2(A) = e^{2A}$ is similarly shown. Thus we obtain

$$L_2(F_2;\ A) = e^{2A_0} + \left[e^{2A_1} - e^{2A_0}\right] l_{10}(A) + 2e^{2A_2} l_{11}(A) = e^{2A} = F_2(A).$$

Considering the structure of the polynomial (3.12), we finally obtain, that $L_2(P_2;\ A) = P_2(A)$. The theorem 3.4 is proved. □

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