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FORMULATION AND WELL-POSEDNESS OF THE VARIATIONAL PROBLEM OF VISCOUS HEAT-CONDUCTING FLUID ACOUSTICS

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Резюме. На підставі законів збереження сформульовано лінійну початково-крайову та відповідну їй варіаційну задачу у термінах невідомих вектора зміщень та температури, яка описує процес поширення акустичних хвиль у в'язкій теплопровідній рідині з урахуванням зв'язаності механічного та температурного полів. Окреслено клас регулярності вхідних даних варіаційної задачі, який гарантує єдність та неперервну залежність шуканого розв'язку в енергетичній нормі задачі. На додаток доведено існування розв'язку розглядуваної задачі як границі послідовності напівдискретних (за просторовими змінними) апроксимацій Гальоркіна.

Abstract. On the basis of conservation laws, we formulate linear initial-boundary value problem and corresponding variational problem in terms of displacement vector and temperature, which describes the process of spreading of acoustic waves in viscous heat-conducting fluid taking into account connectivity of mechanical and thermal fields. We determined input data regularity for the variational problem, which guarantee uniqueness and continuous dependence of the solution in the energy norm of the problem. In addition we prove the existence of the solution of the problem as a limit of a sequence of the semi-discrete spatial Galerkin approximations.

1. Introduction

In most applications, when considering acoustic vibrations, the viscosity of fluid is neglected, hence considering it to be 'ideal'[5, 3]. However, there is a considerable number of problems, which are first of all connected to spreading of the high-frequency vibrations and vibrations at frequencies close to resonance, for which neglecting medium viscosity (even for traditionally "ideal" water or air) leads to considerable inaccuracies in solutions [1, 2, 10]. Furthermore, analysis of dissipative loss of energy in such problems, as well as estimation of reciprocal influence of acoustic and thermal processes are impossible without introducing viscosity of the medium to the model. The general principles of building corresponding models of acoustics of viscous heat-conducting fluid ("dissipative acoustics" is a widely-used term) are studied in papers [11, 6, 7, 9, 10].

† *Key words.* Thermohydrodynamycs, dissipative acoustics, initial-boundary value problem, variation problem, balance equation, the semi-discrete Galerkin method, well-posedness of variation formulation.

In paper [9], for numerical analysis of problems of dissipative acoustics with additional assumption of vortex-free flow in fluid, it is proposed to use Raviart-Thomas finite element approximations, and time integration schemes for semi-discretized problem are built by means of Galerkin method. However, the authors [2] proved earlier the correctness of application of classical approximations of finite element method for solving problems of spreading acoustic waves in viscous fluids and fluid-structure systems in terms of unknown displacements without additional assumptions. It is proposed that a similar approach should be used for problems of thermal and hydro acoustics.

This paper is organized as follows. In section 2, with reference to conservation law, we state a fundamental system of non-linear differential equations and phenomenological relations, which describe the motion of viscous heat-conducting Newtonian fluid, and complement them with possible initial and boundary conditions. Although the obtained system of equations is open in relation to density, mass, velocity, temperature, entropy of the fluid, the hypotheses of acoustics and thermodynamics applied in sections 3 and 4 allowed us to formulate a linear initial-boundary value problem of acoustics with closed system of equations of motion and heat conductivity in terms of acoustic displacement vector and temperature. In section 5 we state variational formulation of this problem as the main object of our study and in section 6 we characterize the components of its equations with regard to continuity and ellipticity. Based on these, in section 7 we describe an important instrument for research of the variational problem – a concretized energy equations of dissipative acoustics. A priori estimates, constructed on this basis in sections 8 and 9, make it possible to determine (quite usable) conditions of regularity of input data of the problem, which guarantee uniqueness and stability of its solution. To prove existence of this solution, in section 10 we recourse to space semi-discretization Galerkin method [4], and in section 11 we show that approximations built in such a way converge to such displacement vector and temperature, which satisfy variation equations of the problem of dissipative acoustics.

2. Fundamental equations of thermohydrodynamics of Newtonian fluid

Below we will consider mathematical models which describe motion of a viscous fluid, which in each moment of time $t \in [0,T], 0 < T < +\infty$, occupies connected bounded domain Ω of points $x = (x_1, \dots, x_d)$ of Euclidian space $\mathbb{R}^d$ (in applications $d = 1, 2$ or 3). We denote as Γ the domain boundary Ω, $\Gamma = \partial\Omega$, and assume that it is Lipschitz-continuous. The latter hypothesis guarantees that almost everywhere on Γ we can build a unit vector of outward normal

$$n = (n_1, \dots, n_d), \quad n_i := \cos(n, x_i).$$

It is well known that physical features of fluid are defined by coefficients of bulk viscosity η and shear viscosity $\mu = \mathit{const} > 0$, and its state can be characterized by means of *velocity vector* $v = \{v_i(x,t)\}_{i=1}^d$ of its particles, density of its mass

$$\rho = \rho(x,t) \geq 0$$

and scalar field of *hydrostatic pressure* $p = p(x,t)$. If the above-mentioned characteristics of the fluid are defined, then with the help of Cauchy relations we can find the components of *strain tensor*

$$e_{ij}(\upsilon) := \frac{1}{2}\left(\frac{\partial \upsilon_i}{\partial x_j} + \frac{\partial \upsilon_j}{\partial x_i}\right), \quad i,j = 1, \dots, d, \tag{1}$$

and components of *stress tensor*

$$\sigma_{ij}(\upsilon,p) := -p\delta_{ij} + \tau_{ij}(\upsilon), \quad i,j = 1, \dots, d, \tag{2}$$

where $\tau_{ij}(\upsilon)$ - components *of viscous stress tensor*,

$$\tau_{ij}(\upsilon) := 2\mu\, e_{ij}(\upsilon) + (\eta - \tfrac{2}{3}\mu)\delta_{ij}\nabla.\upsilon, \quad i,j = 1, \dots, d, \tag{3}$$

δ_{ij} -Kronecker's symbol,

$$\delta_{ij} := \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Modeling of fluid flows reduces to initial-boundary value problems for the partial differential equation system, which are based on the *laws of mass conservation*, momentum, energy, etc. [10, 11]. So, for example, the *law of mass conservation* of continuous medium states that given the absence of sources for mass increase, the density $\rho = \rho(x,t)$ and the vector of fluid velocity $\upsilon = \{\upsilon_i(x,t)\}_{i=1}^d$ satisfy the so-called *equation of mass continuity*

$$D_t\rho + \rho\nabla.\upsilon = 0 \quad in \quad \Omega \times (0,T]. \tag{4}$$

At the same time, *laws of momentum conservation* can be presented as a system of Navier-Stokes equations

$$\rho D_t\upsilon_i - \frac{\partial}{\partial x_m}\sigma_{im}(\upsilon,p) = \rho f_i, \quad i = 1, \dots, d, \quad in \quad \Omega \times (0,T], \tag{5}$$

where vector $f = \{f_i(x,t)\}_{i=1}^d$ describes volume forces which act on the considered fluid.

Finally, the *law of energy conservation* leads to *equation of continuity of entropy* $s = s(x,t)$ formulated as

$$\rho\theta D_t s + \nabla.q(\theta) - \tau(\upsilon) : e(\upsilon) = \rho g \quad in \quad \Omega \times (0,T], \tag{6}$$

where $g = g(x,t)$ is intensity of distributed in the fluid volume sources of heat, $q = \{q_i(x,t)\}_{i=1}^d$ is *vector of heat flow*, which is connected in most important cases to the temperature $\theta = \theta(x,t)$ and coefficient of heat conductivity $\chi > 0$ of fluid through *phenomenological Fourier law*

$$q(\theta) = -\chi\nabla\theta \quad in \quad \Omega \times (0,T]. \tag{7}$$

Here and further on we shall use the summation convention from 1 to d with repeated indexes, eliminating the sign of summation itself; e.g. scalar product in space $\mathbb{R}^d$ is written as

$$a.b \equiv a_i b_i := \sum_{i=1}^d a_i b_i \qquad \forall a = \{a_i\}_{i=1}^d,\ b = \{b_i\}_{i=1}^d \in \mathbb{R}^d,$$

and

$$\sigma : e \equiv \sigma_{mi} e_{im} := \sum_{i,m=1}^{d} \sigma_{mi} e_{im} \qquad \forall \sigma = \{\sigma_{ij}\}_{i,j=1}^{d}, \ e = \{e_{ij}\}_{i,j=1}^{d} \in \mathbb{R}^{d\times d}.$$

Finally, in the equations (4)-(7) we utilize widely-used symbols for full and partial derivatives of a scalar function by time variable and its gradient by spatial variable.

$$D_t w := w' + v.\nabla w, \quad w' := \frac{\partial}{\partial t} w(x,t), \quad \nabla w := \left\{ \frac{\partial w}{\partial x_m} \right\}_{m=1}^{d}.$$

Let us complement the system (1)-(7) with appropriate initial and boundary conditions. If on the outer surface of fluid $\Gamma_\sigma \subset \Gamma$ is affected by the applied stress vector $\hat{\sigma} = \{\hat{\sigma}_i(x,t)\}_{i=1}^{d}$, then the law of momentum conservation leads to the following boundary condition for stress:

$$\sigma_{ij}(v,p) n_j = \hat{\sigma}_i \quad i = 1, \ \ldots \ , d, \qquad on \quad \Gamma_\sigma \times [0,T]. \tag{8}$$

Similarly, if a part of the boundary $\Gamma_q \subset \Gamma$ is affected by heat flow, the normal component of which is determined by the function $\hat{q} = \hat{q}(x,t)$, then according to the law of energy conservation, the boundary condition will be

$$n.q(\theta) = \hat{q} \qquad on \quad \Gamma_q \times [0,T]. \tag{9}$$

Finally, if, for example, particles of the remaining fluid surface $\Gamma_v := \Gamma \backslash \Gamma_\sigma$ move in compliance with the known rule at the speed $\hat{v} = \{\hat{v}_i(x,t)\}$, then the boundary condition on this part of the surface should be

$$v = \hat{v} \qquad on \quad \Gamma_v \times [0,T], \ \ \Gamma_v := \Gamma \backslash \Gamma_\sigma. \tag{10}$$

Similarly, if it is known that the part of the surface $\Gamma_\theta := \Gamma \backslash \Gamma_q$ is maintained at the defined temperature, $\hat{\theta} = \hat{\theta}(x,t)$, then the boundary condition assigned to it is

$$\theta = \hat{\theta} \qquad on \quad \Gamma_\theta \times [0,T], \ \ \Gamma_\theta := \Gamma \backslash \Gamma_q. \tag{11}$$

We have to mention that there might be boundary conditions for different classes of applications, as a rule, formulated as linear combinations of condition components (8), (9) and (10), (11) correspondigly.

Finally, considering the specifics of the structure of system relations and equations (1)-(7), namely, the absence of pressure derivatives by time variable in it, we come to a conclusion that during the study of viscous fluid motion it is sufficient to reduce it to studying the initial conditions and values of mass density, velocity vector and temperature

$$\rho|_{t=0} = \rho_0, \quad v|_{t=0} = v_0, \quad \theta|_{t=0} = \theta_0 \quad in \quad \Omega. \tag{12}$$

The obtained nonlinear problem of thermohydrodynamics (1)-(12) contains less equations (d+2) than, the unknowns (d+4), and must be complemented by additional equations based on phenomenological deductions. For this purpose we shall use the hypotheses of acoustic approximation, which will allow us not only to find a closed equation system, but also to linearize it.

3. Linear equation system of dissipative acoustics in terms of acoustic displacement and temperature

Below we assume that, for one reason or another, there are connections between the unknowns $\{\rho, p, s, \theta\}$, which are expressed as

$$p = p(\rho, \theta), \quad s = s(\rho, \theta).$$

It is known that pressure is related to density and temperature by the following thermodynamic connections [10]:

$$\left.\frac{\partial p}{\partial \rho}\right|_{\theta} = \frac{c^2}{\gamma}, \qquad \left.\frac{\partial p}{\partial \theta}\right|_{\rho} = \frac{c^2 \rho \alpha}{\gamma},$$

where c is velocity of sound, α coefficient of thermal expansion, $\gamma = c_p c_v^{-1}$, c_p and c_v specific heat of fluid at constant pressure and volume respectively. Then to accuracy of an additive constant

$$p = p_0 + c^2 \gamma^{-1} [\rho + \rho \alpha \theta].$$

In addition we can linearize the obtained rule in the following way:

$$p(x,t) \cong p_0 + c^2 \gamma^{-1} [\rho(x,t) + \rho_0 \alpha \theta(x,t)], \tag{13}$$

where ρ_0 is mass density distribution of fluid in the state undisturbed by acoustic factors. Here we implicitly assume that the mass density of fluid admits the following decomposition

$$\begin{cases} \rho(x,t) = \rho_0 + \rho_*(x,t) \qquad \forall x \in \Omega \;\; \forall t \in [0,T], \\ \rho_*|_{t=0} = 0 \qquad in \quad \Omega, \\ ||\rho_*|| \ll ||\rho_0||. \end{cases} \tag{14}$$

Now we shall convey the velocity of fluid motion as a sum formulated as

$$\begin{cases} v(x,t) = v_0(x) + v_*(x,t) \qquad \forall x \in \Omega \;\; \forall t \in [0,T], \\ v_*|_{t=0} = 0 \qquad in \quad \Omega, \qquad ||v_*|| \ll ||v_0|| \end{cases} \tag{15}$$

And turn to the continuity equation from (4). Bearing in mind the hypotheses (14) and (15), we shall linearize it in the following way

$$\rho' + v.\nabla \rho + \rho \nabla . v \cong \rho'_* + \rho_0 \nabla . v_* + v_0 . \nabla \rho_* \cong$$
$$\cong \rho'_* + \rho_0 \nabla . v_* = 0 \qquad in \;\; \Omega \times \; (0,T]],$$

And later integrate the obtained approximation into a time interval $(0,t)$, $0 < t \leq T$. As a result, we find out that

$$\begin{aligned} \rho_*(x,t) &= -\rho_0 \nabla . \textstyle\int_0^t v_*(x,\tau) d\tau = \\ &= -\rho_0 \nabla . u(x,t) \quad \forall x \in \Omega \;\; \forall t \in [0,T], \end{aligned} \tag{16}$$

where $u = u(x,t)$ – vector of acoustic displacement of fluid particles

$$u(x,t) := u_0(x) + \int_0^t v_*(x,\tau) d\tau \qquad in \;\; \Omega \times (0,T].$$

Taking into account (13) and (16), we come to a final expression for the linear approximation of acoustic pressure in fluid

$$\begin{aligned} p(x,t) &\cong p_0 + c^2\gamma^{-1}[\rho_*(x,t) + \rho_0\alpha\theta(x,t)] \cong \\ &\cong p_0 + c^2\gamma^{-1}\rho_0[-\nabla.u(x,t) + \alpha\theta(x,t)] \equiv \\ &\equiv p_0 + \pi(u,\theta) \qquad \forall x \in \Omega \;\; \forall t \in [0,T]. \end{aligned} \tag{17}$$

Introducing the vector of acoustic displacements $u = u(x,t)$ also leads to change of notation and structure of stress tensor of fluid, such as

$$\begin{aligned} \sigma_{ij}(\upsilon,p) &= -p\delta_{ij} + \tau_{ij}(\upsilon) \cong \\ &\cong -p_0\delta_{ij} + \pi(u,\theta)\delta_{ij} + \tau_{ij}(u') = \\ &= -p_0\delta_{ij} + \bar{\sigma}_{ij}(u,\theta) \qquad \forall x \in \Omega \;\; \forall t \in [0,T]. \end{aligned}$$

In other words, taking into consideration the relation (17), pressure is excluded when determining the stress tensor, instead we include the dependence of its components from the temperature of fluid. Taking into account the hypotheses of acoustics and linearization of convective constituents, the motion equations (5) undergo some changes, such as

$$\begin{aligned} &\rho[\upsilon_i'(t) + \upsilon_m\frac{\partial \upsilon_i}{\partial x_m}] - \frac{\partial \sigma_{im}(\upsilon,p)}{\partial x_m} - \rho f_i \cong \\ &\cong \rho_0 u_i''(t) + \frac{\partial}{\partial x_i}p_0 - \frac{\partial \bar{\sigma}_{im}(u')}{\partial x_m} - \rho_0 f_i = 0. \end{aligned}$$

It follows that considering the hypotheses of acoustics and the linearization of motion equations, performed above, lead to excluding pressure and density from the unknown, and after this procedure the motion equations acquire the form

$$\begin{aligned} &\rho_0 u_i''(t) - \frac{\partial \bar{\sigma}_{im}(u')}{\partial x_m} = \rho_0 f_i - \frac{\partial}{\partial x_i}p_0, \\ &\bar{\sigma}_{ij}(u) := -\pi(u,\theta)\delta_{ij} + \tau_{im}(u'), \\ &\pi(u,\theta) := c^2\gamma^{-1}\rho_0[-\nabla.u + \alpha\theta]\,, \\ &\tau_{ij}(u') := 2\mu\, e_{ij}(u') + (\eta - \tfrac{2}{3}\mu)\delta_{ij}\nabla\cdot u', \\ &e_{ij}(u) := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad in \quad \Omega \times (0,T]. \end{aligned}$$

Since entropy is related to density and temperature through thermodynamic links expressed as [10]

$$(\frac{\partial s}{\partial \rho})_\theta = -\frac{c^2\alpha}{\rho\gamma}, \quad (\frac{\partial s}{\partial \theta})_\rho = \frac{c_V}{\theta},$$

then

$$\begin{aligned} \frac{\partial s}{\partial t} &= -\frac{c^2\alpha}{\rho\gamma}\frac{\partial \rho}{\partial t} + \frac{c_V}{\theta}\frac{\partial \theta}{\partial t} \cong -\frac{c^2\alpha}{\rho_0\gamma}\frac{\partial \rho}{\partial t} + \frac{c_V}{\theta_0}\frac{\partial \theta}{\partial t} \cong \\ &\cong \frac{c^2\alpha}{\rho_0\gamma}\rho_0\nabla.u' + \frac{c_V}{\theta_0}\frac{\partial \theta}{\partial t} = \frac{c^2\alpha}{\gamma}\nabla.u' + \frac{c_V}{\theta_0}\frac{\partial \theta}{\partial t} \end{aligned}$$

and after substitution of this expression in the equation of conservation of entropy in (6) and its linearization, we will come to the equation of thermal conductivity of viscous fluid

$$\rho\theta D_t s + \nabla \cdot q + \tau(u') : e(u') - \rho g \cong$$
$$\cong \rho_0\theta_0 \left[\frac{c^2\alpha}{\gamma}\nabla.u' + \frac{c_V}{\theta_0}\frac{\partial\theta}{\partial t}\right] - \nabla\cdot[\chi\nabla\theta] - \tau(u') : e(u') - \rho_0 g$$

or

$$\rho_0 c_V \frac{\partial\theta}{\partial t} - \nabla\cdot[\chi\nabla\theta] + c^2\gamma^{-1}\rho_0\theta_0\alpha\nabla.u' = \rho_0 g \;\; in \;\; \Omega\times(0,T].$$

4. Linearized initial-boundary value problem of dissipative acoustics

Summarizing the results of section 3, we come to the following linearized initial-boundary value problem of dissipative acoustics with a closed system of fundamental equations:

Find displacement $u = \{u_i(x,t)\}_{i=1}^d$ *and temperature* $\theta = \theta(x,t)$ *which satisfy the linearized system of equations of dissipative acoustics*

$$\begin{cases} \rho_0 c_V \theta_0^{-1}\dfrac{\partial\theta}{\partial t} - \theta_0^{-1}\nabla.[\chi\nabla\theta] + c^2\gamma^{-1}\rho_0\alpha\nabla.u' = \rho_0\theta_0^{-1} g, \\ \rho_0 u_i''(t) + \dfrac{\partial}{\partial x_i}\pi(u,\theta) - \dfrac{\partial\tau_{im}(u')}{\partial x_m} = \rho_0 f_i - \dfrac{\partial}{\partial x_i} p_0, \\ \pi(u,\theta) := c^2\gamma^{-1}\rho_0[-\nabla.u + \alpha\theta], \\ \tau_{ij}(u') := 2\mu\, e_{ij}(u') + (\eta - \tfrac{2}{3}\mu)\delta_{ij}\nabla.u', \\ e_{ij}(u) := \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right) \qquad in \quad \Omega\times(0,T], \end{cases} \tag{18}$$

boundary conditions

$$\begin{cases} \sigma_{ij}n_j = \hat{\sigma}_i, & on \quad \Gamma_\sigma\times[0,T], \;\; \Gamma_\sigma\subset\Gamma, \\ u = \hat{u}, & on \quad \Gamma_v\times[0,T], \;\; \Gamma_v := \Gamma\backslash\Gamma_\sigma, \\ q.n = \hat{q}, & on \quad \Gamma_q\times[0,T], \;\; \Gamma_q\subset\Gamma, \\ \theta = \hat{\theta} & on \quad \Gamma_\theta\times[0,T], \;\; \Gamma_\theta := \Gamma\backslash\Gamma_q \end{cases} \tag{19}$$

and initial conditions

$$u|_{t=0} = u_0, \quad u'|_{t=0} = v_0, \quad \theta|_{t=0} = \theta_0 \quad in \quad \Omega. \tag{20}$$

5. Variational problem of dissipative acoustics

To build a variational formulation of the initial-boundary value problem (18)-(20), we first (taking into account Dirichlet boundary conditions) introduce the space of admissible displacement vectors

$$V := \{v = \{v_i\}_{i=1}^d \in [H^1(\Omega)]^d : \; v = 0 \; on \; \Gamma_u\}$$

and the space of admissible temperatures

$$G := \{\zeta \in H^1(\Omega) : \; \zeta = 0 \;\; on \;\; \Gamma_\theta\}$$

respectively.

Now we shall multiply the equation of heat conductivity of the problem (18)-(20) by arbitrary function $\zeta \in G$ and integrate the obtained result over the domain Ω using integration by parts

$$\begin{aligned}
&\int_\Omega \rho_0 \theta_0^{-1} g(t)\zeta dx = \\
&= \int_\Omega \{\rho_0 c_V \theta_0^{-1}\theta'(t) - \theta_0^{-1}\nabla\cdot[\chi\nabla\theta(t)] + c^2\gamma^{-1}\rho_0\alpha\nabla.u'(t)\}\zeta dx\rho_0 = \\
&= \int_\Omega \{\rho_0 c_V \theta_0^{-1}\theta'(t)\zeta + \theta_0^{-1}\nabla\zeta.[\chi\nabla\theta(t)] + c^2\gamma^{-1}\rho_0\alpha\nabla.u'(t)\}dx + \\
&+ \int_{\Gamma_q} \theta_0^{-1}\zeta q_m(\theta)n_m d\gamma = \\
&= \int_\Omega [\rho_0 c_V \theta_0^{-1}\theta'(t)\zeta + \theta_0^{-1}\nabla\zeta.[\chi\nabla\theta(t)] + c^2\gamma^{-1}\rho_0\alpha\nabla.u'(t)]dx + \\
&+ \int_{\Gamma_q} \theta_0^{-1}\hat{q}(t)\zeta d\gamma \qquad \forall\zeta \in G.
\end{aligned}$$

Let us introduce bilinear and linear forms

$$\left\{\begin{array}{l}
\chi(\theta,\zeta) := \int_\Omega \theta_0^{-1}\chi\nabla\zeta.\nabla\theta dx \\
s(\theta,\zeta) := \int_\Omega \rho_0 c_V\theta_0^{-1}\theta\zeta dx \qquad \forall\theta,\zeta \in G, \\
b(v,\zeta) := \int_\Omega c^2\gamma^{-1}\rho_0\alpha\zeta(\nabla.v)dx \qquad \forall v \in V \;\; \forall\zeta \in G
\end{array}\right. \tag{21}$$

and

$$< z,\zeta > := \int_\Omega \rho_0\theta_0^{-1} g\zeta dx - \int_{\Gamma_q} \theta_0^{-1}\hat{q}\zeta d\gamma \qquad \forall\zeta \in G$$

and re-write the equation obtained above as

$$s(\theta'(t),\zeta) + \chi(\theta(t),\zeta) + b(u'(t),\zeta) = < z(t),\zeta > \qquad \forall\zeta \in G.$$

Similarly, we shall multiply the equation of motion of the problem (18)-(20) by arbitrary vector $v \in V$ and integrate the obtained result over the domain Ω

$$\begin{aligned}
&\int_\Omega \rho_0\, f(t).v dx = \\
&= \int_\Omega \left\{\rho_0 u_i''(t) + \frac{\partial}{\partial x_m}[\pi[u(t),\theta(t)]\delta_{im} - \tau_{im}(u'(t))]\right\} v_i dx = \\
&= \int_\Omega \rho v.\, u''(t)dx + \int_\Omega c^2\gamma^{-1}\rho_0[\nabla.u(t)](\nabla.v)dx - \\
&\qquad - \int_\Omega c^2\gamma^{-1}\rho_0\alpha\theta(t)\nabla.v dx + \\
&\quad + \int_\Omega \tau(u'(t)) : e(v)dx - \int_{\Gamma_\sigma} v\cdot\hat{\sigma}(t)d\gamma \qquad \forall v \in V.
\end{aligned}$$

Taking the obtained equation into account, we introduce the forms

$$\left\{\begin{array}{l}
m(u,v) := \int_\Omega \rho_0 u.v dx, \\
a(u,v) := \int_\Omega \tau(u) : e(v)dx \equiv \\
\qquad \equiv \int_\Omega [2\mu\, e(u) : e(v) + (\eta - \frac{2}{3}\mu)(\nabla.u)(\nabla.v)]dx, \\
c(u,v) := \int_\Omega c^2\gamma^{-1}\rho_0(\nabla.u)(\nabla.v)dx, \qquad \forall u,v \in V,
\end{array}\right.$$

$$< l,v > := m(f - \rho_0^{-1}\nabla p_0, v) + \int_{\Gamma_\sigma} v.\hat{\sigma}d\gamma \quad \forall v \in V \tag{22}$$

and finally write the variational formulation of the initial-boundary value problem of dissipative acoustics

$$
\begin{cases}
\textit{Find pair } \{u(t),\theta(t)\} \in V \times G \textit{ such that} \\
m(u''(t),v) + a(u'(t),v) + c(u(t),v) - \\
\qquad -b(v,\theta(t)) =< l(t), v >, \\
s(\theta'(t),\zeta) + \chi(\theta(t),\zeta) + b(u'(t),\zeta) = \\
\qquad =< z(t), \zeta > \quad \forall t \in (0,T], \\
m(u'(0)-v_0, v) = 0, \quad a(u(0)-u_0, v) = 0, \qquad \forall v \in V, \\
\qquad s(\theta(0)-\theta_0,\zeta) = 0 \qquad \forall \zeta \in G.
\end{cases}
\tag{23}
$$

Let us remark that bilinear form $b(\ . \ , \ . \ ) : G \times V \to \mathbb{R}$, we determined in (21), binds variational equations of the problem (23) into a system for determining thermal and mechanical fields of acoustic wave. On the other hand, as we shall see later, this bilinear form describes the mechanism of heat-to-work conversion, and, since it is present in both variational equations, a contraria.

6. Properties of components of variational problem of dissipative acoustics

To perform the analysis of properties of bilinear forms and linear functional which constitute the structure of variational problem (23), we shall first introduce the following notation for spaces of scalar and vector functions

$$H := L^2(\Omega), \quad H := H^d, \quad H(div;\Omega) := \{v \in H : \quad \nabla . v \in H\}.$$

Taking into account the additive values of the problem data (22), it is easy to notice that continuous symmetric bilinear forms

$$
\begin{aligned}
m(u,v) &= \textstyle\int_\Omega \rho_0 u.v dx \qquad \forall u,v \in H, \\
s(\theta,\zeta) &= \textstyle\int_\Omega \rho_0 c_v \theta_0^{-1} \theta\zeta dx \qquad \forall \ \theta,\zeta \in H
\end{aligned}
\tag{24}
$$

are scalar products on spaces $\mathbf{H}$ and H and as consequence, form norms on them

$$
\begin{aligned}
|v||_H &:= \sqrt{m(v,v)} \qquad \forall v \in H, \\
||\zeta||_H &:= \sqrt{s(\zeta,\zeta)} \qquad \forall \ \zeta \in H,
\end{aligned}
$$

Equivalent to the norms of spaces $[L^2(\Omega)]^d$ and $L^2(\Omega)$ respectfully.

Similarly, taking into consideration Korn inequality, continuous symmetric bilinear forms

$$
\begin{aligned}
a(u,v) &= \textstyle\int_\Omega [2\mu e_{ij}(u)e_{ij}(v) + (\eta - \frac{2}{3}\mu)(\nabla.u)(\nabla.v)]dx \quad \forall u,v \in V, \\
\chi(\theta,\zeta) &= \textstyle\int_\Omega \theta_0^{-1}(\chi\nabla\theta).(\nabla\zeta)dx \qquad\qquad \forall \ \theta,\zeta \in G
\end{aligned}
\tag{25}
$$

are scalar products on spaces $\mathbf{V}$ and G respectively, and as consequence, form norms on them

$$||v||_V := \sqrt{a(v,v)} \qquad \forall v \in V \quad (equivalent \; || \; \cdot \; ||_{[H^1(\Omega)]^d}),$$
$$||\zeta||_G := \sqrt{\chi(\zeta,\zeta)} \qquad \forall \; \zeta \in G \; \; (equivalent \; || \; \cdot \; ||_{H^1(\Omega)} \;).$$

The properties of bilinear forms of variational problem that we have mentioned here are well known for problems of elastodynamics and heat conductivity which, as a matter of fact, form the core structure of variational problem of dissipative acoustics.

One of the specific properties of the problem of dissipative acoustic is illustrated by a continuous symmetric bilinear form

$$c(u,v) = \int_\Omega c^2 \rho_0 \gamma^{-1} (\nabla . u)(\nabla . v) dx \quad \forall u, v \in V,$$

which is non-negative on the space of admissible displacements $\mathbf{V}$ and creates seminorm in space $H(div;\Omega)$. We shall denote the latter as follows:

$$|v|_V := \sqrt{c(v,v)} \qquad \forall v \in V.$$

And finally, the bilinear form

$$b(v,\zeta) := \int_\Omega c^2 \gamma^{-1} \rho_0 \alpha \zeta (\nabla . v) dx \qquad \forall v \in V \;\; \forall \zeta \in G,$$

which determines the interaction mechanism of thermal and mechanical fields in the process of spreading acoustic waves, is continuous on the space $V \times G$. Linear functionals also possess this property

$$< z, \zeta >= \int_\Omega \rho_0 \theta_0^{-1} g \zeta dx - \int_{\Gamma_q} \theta_0^{-1} \hat{q} \zeta d\gamma \qquad \forall \zeta \in G, \tag{26}$$

$$< l, v >= m(f - \rho_0^{-1} \nabla p_0, v) + \int_{\Gamma_\sigma} v . \hat{\sigma} d\gamma \quad \forall v \in V \tag{27}$$

in case that external sources of mechanics and thermal energy of the problem possess the following properties of regularity

$$g \in H, \;\; \hat{q} \in L^2(\Gamma_q), \; p_0 \in H^1(\Omega),$$
$$f \in H, \;\; \hat{\sigma} \in [L^2(\Gamma_\sigma)]^d.$$

7. Energy equalities of dissipative acoustics

We shall accept for the problem equations (23) for admissible functions $v = u'(t)$ and $\zeta = \theta(t)$ and add the first pair of variational equations. As a result of elimination of summands with the value of bilinear form $b(u'(t), \theta(t))$ (which indicates energy conversion without losses!) and using norms from p.6, we shall obtain energy equations of this problem

$$\frac{1}{2}\frac{d}{dt}[||u'(t)||_H^2 + |u(t)|_V^2 + ||\theta(t)||_H^2] + ||u'(t)||_V^2 + ||\theta(t)||_G^2 =$$
$$=< l(t), u'(t) > + < z(t), \theta(t) > \qquad \forall t \in (0,T]$$

Or after integrating over arbitrary time interval $[0,t]$, $\; 0 \leq t \leq T$,

$$\frac{1}{2}[||u'(t)||^2_H + |u(t)|^2_V + ||\theta(t)||^2_H] + \int_0^t [||u'(t)||^2_V + ||\theta(\tau)||^2_G] d\tau =$$
$$= \frac{1}{2}[||v(0)||^2_H + |u(0)|^2_V + ||\theta(0)||^2_H] + \tag{28}$$
$$+ \int_0^t [< l(\tau), u'(\tau) > + < z(\tau), \theta(\tau) >] d\tau \quad \forall t \in [0, T].$$

We shall write the last equation as

$$K_S[u'(t)] + P_S[u(t)] + P_C[\theta(t)] + \int_0^t \{D_S[u(\tau)] + D_C[\theta(\tau)]\} d\tau =$$
$$= K_S[v_0] + P_S[u_0] + P_C[\theta_0] + Q_S[u'(t)] + Q_C[\theta(t)] \quad \forall t \in [0,\ T],$$

where

$$K_S[u'(t)] := \frac{1}{2}||u'(t)||^2_H, \quad P_S[u(t)] := \frac{1}{2}|u(t)|^2_V,$$
$$D_S[u'(t)] := ||u'(t)||^2_V$$

are instantaneous values of kinetic and potential energy, and its dissipation caused by kinetic motion of fluid, in the function

$$P_C[\theta(t)] := ||\theta(t)||^2_H, \quad D_C[\theta(t)] := ||\theta(t)||^2_G$$

they are instantaneous values of energy and its losses, caused by the existence of heat flow pattern of fluid,

$$Q_S[u'(t)] := \int_0^t < l(\tau), u'(\tau) > d\tau, \quad Q_C[\theta(t)] := \int_0^t < \mu(\tau), \theta(\tau) > d\tau.$$

8. Data regularity of a problem of dissipative acoustics

Let us consider the conditions of data regularity for the variation problem (22), as functions of space and time variables, which can be determined on the basis of equality analysis (28). In particular, to allow the total energy of acoustic field of fluid

$$E[u(t), \theta(t)] := \frac{1}{2}[||u'(t)||^2_H + |u(t)|^2_V + ||\theta(t)||^2_H]$$

take finite values in each moment of time $t \in (0, T]$, it is necessary that the following conditions are held

$$u' \in L^\infty(0, T; H), \ \ u \in L^\infty(0, T; H(div; \Omega)), \ \ \theta \in L^\infty(0, T; H).$$

Similarly, to allow the the losses of acoustic field of fluid

$$D[u(t), \theta(t)] := \int_0^t [||u'(t)||^2_V + ||\theta(\tau)||^2_G] d\tau$$

take finite values in each moment of time $(0, t] \subset (0, T]$, it is necessary that the following conditions are held

$$u' \in L^2(0, T; V), \quad \theta \in L^2(0, T; G).$$

Thus, appropriate solutions of the variational problem of dissipative acoustics should satisfy the following conditions

$$\begin{cases} u' \in L^\infty(0,T;H) \cap L^2(0,T;V), \\ u \in L^\infty(0,T;H(div;\Omega)), \\ \theta \in L^\infty(0,T;H) \cap \theta \in L^2(0,T;G). \end{cases}$$

Now based on the requirement

$$\left| \int_0^t [< l(\tau), u'(\tau) > + < z(\tau), \theta(\tau) >] d\tau \right| < +\infty \quad \forall t \in (0,T]$$

we find sufficient requirements of regularity for energy sources, such as,

$$l \in L^2(0,T;V'), \;\; z \in L^2(0,T;G')$$

Or in more detail, taking into consideration the structures (26) and (27) of these functionals

$$\begin{cases} f \in L^2(0,T;H), & \hat{\sigma} \in L^2(0,T;[L^2(\Gamma_\sigma)]^d), \\ g \in L^2(0,T;H), & \hat{q} \in L^2(0,T;L^2(\Gamma_q)). \end{cases}$$

The latter sum

$$E[u(0), \theta(0)] := \frac{1}{2}[||u'(0)||_H^2 + |u(0)|_V^2 + ||\theta(0)||_H^2]$$

of energy equality (28) shows that the total energy of the acoustic field at the initial moment of time $t = 0$ will have finite values, if the initial data of the problem of dissipative acoustics are selected according to the rules

$$\upsilon_0 \in H, \quad u_0 \in V, \quad \theta_0 \in H.$$

9. Uniqueness and stability of solution of the variational problem of dissipative acoustics

Now we are ready to prove the next theorem

Theorem 1. *Assume that the variational problem of dissipative acoustics* (23)*, whose data satisfy the conditions of regularity*

$$\upsilon_0 \in H, \quad u_0 \in V, \;\; \theta_0 \in H \tag{29}$$

and

$$\begin{cases} f \in L^2(0,T;H), & \hat{\sigma} \in L^2(0,T;[L^2(\Gamma_\sigma)]^d), \\ g \in L^2(0,T;H), & \hat{q} \in L^2(0,T;L^2(\Gamma_q)), \end{cases} \tag{30}$$

has the solution $\psi(t) = \{u(t), \, \theta(t)\}$ *.*

Then the pair $\psi(t) = \{u(t), \, \theta(t)\}$ *will be the unique solution to the problem* (23) *and*

$$\begin{cases} L^\infty(0,T;H(div;\Omega)), \quad u' \in L^\infty(0,T;H) \cap L^2(0,T;V), \\ \theta \in L^\infty(0,T;H) \cap L^2(0,T;G); \end{cases}$$

Moreover, the solution $\psi(t) = \{u(t),\, \theta(t)\}$ is continuously dependent on the problem data (23) *and under these conditions the following a priori estimate is correct*

$$\begin{aligned}&\frac{1}{2}[||u'(t)||_H^2 + |u(t)|_V^2 + ||\theta(t)||_H^2] + \int_0^t [||u'(t)||_V^2 + ||\theta(\tau)||_G^2]d\tau \leq \\ &\leq C\left\{[||v_0||_H^2 + |u_0|_V^2 + ||\theta_0||_H^2] + \int_0^t [||l(\tau)||_{V'}^2 + ||(\tau)||_{G'}^2]d\tau\right\}, \\ &\qquad \forall t \in [0,T].\end{aligned} \tag{31}$$

with constant $C > 0$, the value of which is independent of quantities under consideration.

Proof. Bearing in mind the conditions (30)

$$l \in L^2(0,T;V'),\ \ z \in L^2(0,T;G'),$$

we conclude that the following estimates are correct

$$\begin{aligned}&|< l(\tau), u'(\tau) >| \leq ||l(\tau)||_{V'}||u'(\tau)||_V \leq \frac{1}{2}\|u'(\tau)\|_V^2 + \frac{1}{2}\|l(\tau)\|_{V'}^2, \\ &|< z(\tau), \theta(\tau) >| \leq \frac{1}{2}\|\theta(\tau)\|_G^2 + \frac{1}{2}\|z(\tau)\|_{G'}^2, \qquad \forall t \in [0,T].\end{aligned} \tag{32}$$

From the initial condition of the problem (23)

$$m(u'(0) - v_0, v) = 0, \qquad \forall v \in H$$

After substituting $v = u'(0)$ and $v = v_0$ we obtain that

$$||u'(0)||_H^2 = m(u'(0), v_0) = m(v_0, u'(0)) = m(v_0, v_0) = ||v_0||_H^2. \tag{33}$$

Applying the same principle

$$|u(0)||_V = ||u_0||_V,\ \ ||\theta(0)||_H = ||\theta_0||_H. \tag{34}$$

Next, taking into account the results from p.6, we find $C = const > 0$, such that

$$|v|_V \leq C||v||_V \qquad \forall v \in V$$

and, in particular,

$$|u(0)|_V \leq C||u(0)||_V = C||u_0||_V. \tag{35}$$

Summarizing (32)-(34) and (35) in energy equality (28), we come to an estimate (31).

Based on the same estimate, by contradiction, we demonstrate the uniqueness of the problem solution (23). □

Corollary 1. *Let us assume that the hypotheses of theorem 1 are satisfied in relation to the variation problem of dissipative acoustics* (23).

Then the natural norm for its solution $\psi(t) = \{u(t),\, \theta(t)\}$ *is*

$$||\psi(t)\,||^2 := ||u'(t)||_H^2 + |u(t)|_V^2 + ||\theta(t)||_H^2 +$$
$$+ \int_0^t [||u'(\tau)||_V^2 + ||\theta(\tau)||_G^2] d\tau \qquad \forall t \in [0,T].$$

10. Galerkin semi-discretization of variational problem of dissipative acoustics

Let us assume that $\{V_h\}$ та $\{G_h\}$ are sequences of finite-dimensional spaces, such that

$$\begin{cases} V_h \subset V,\;\; G_h \subset G \qquad \forall h > 0, \\ \mathrm{dim} V_h = N = N(h) \to \infty, \\ \dim G_h = M = M(h) \to \infty, \;\; if \;\; h \to 0, \\ \bigcup\limits_{h>0} V_h \; dense \;\; in \;\; V, \quad \bigcup\limits_{h>0} G_h \, dense \;\; in \;\; G. \end{cases}$$

On this basis we determine the sequence of semi-discrete Galerkin approximations $\{\psi_h\}_{h>0} = \{(u_h, \theta_h)\}_{h>0}$ expressed as solutions of the following variational problems:

$$given\, h > 0;\; find\, pair\, \psi_h(t) = (u_h(t),\, \theta_h(t)) \,\in\, V_h \times G_h \, such\, that$$
$$\begin{cases} m(u_h''(t), v) + a(u_h'(t), v) + c(u_h(t), v) - \\ \qquad -b(\theta_h(t), v) = < l(t), v >, \\ s(\theta_h'(t), \zeta) + k(\theta_h(t), \zeta) + b(\zeta, u_h'(t)) = < z(t), \zeta > \qquad \forall t \in (0,T], \\ m(u_h'(0) - v_0, v) = 0, \;\; a(u_h(0) - u_0, v) = 0 \qquad \forall v \in V_h, \\ \qquad s\,(\theta_h(0) - \theta_0, \zeta) = 0 \qquad \forall \zeta \in G_h. \end{cases} \tag{36}$$

To concretize the structure of problems we have just formulated and the required approximations $(u_h, \theta_h) \in L^2(0,T; V_h \times G_h)$, let us select certain bases $\{\phi_k(x)\}_{k=1}^N$ and $\{\varphi_k(x)\}_{k=1}^M$ of spaces V_h and G_h respectively. First of all, this selection univalently determines the form of each sequence member of semi-discrete approximations as a linear combination

$$u_h(x,t) = \sum_{k=1}^{N} u_k(t)\phi_k(x)\;,$$
$$\theta_h(x,t) = \sum_{k=1}^{M} \vartheta_k(t)\varphi_k(x) \quad \forall (x,t) \in \Omega \times [0,T]$$

with unknown coefficients $U(t) = \{u_k(t)\}_{k=1}^N$ and $\Theta(t) = \{\vartheta_m(t)\}_{m=1}^M$, and secondly, after application of Galerkin procedure, allows obtaining Cauchy problem for finding the above-mentioned coefficients

$$\left\{\begin{array}{l} MU''(t) + AU'(t) + CU(t) - B\ \Theta(t) \ = L(t), \\ \qquad S\,\Theta'(t) + K\Theta(t) + B^T U'(t) = Z(t) \quad \forall t \in (0,T], \\ MU'(0) = Y^0, \quad AU(0) = U^0, \\ \qquad S\,\Theta(0) = \Theta^0. \end{array}\right. \tag{37}$$

Here the components of matrices and vectors of the right side of equation are calculated according to the rules

$$C = \{c(\phi_i, \phi_k)\}_{i,k=1}^{N}\,, \quad B = \{b(\varphi_i, \phi_k)\}_{i,k=1}^{M,N}\,, \quad K = \{k(\varphi_i, \varphi_k)\}_{i,k=1}^{M}\,,$$

$$L(t) = \{< l(t), \phi_i >\}_{i=1}^{N}\,, \quad Z(t) = \{< z(t), \varphi_i >\}_{i=1}^{M} \qquad \forall T \in (0,t],$$

and

$$Y_0 = \{m(v_0, \phi_k)\}_{k=1}^{N}\,, \quad U_0 = \{a(u_0, \phi_k)\}_{k=1}^{N}\,, \ \Theta_0 = \{s(\theta_0, \varphi_i >\}_{i=1}^{M}.$$

Since the rest of the matrices

$$M = \{m(\phi_i, \phi_k)\}_{i,k=1}^{N}\,, \quad A = \{a(\phi_i, \phi_k)\}_{i,k=1}^{N}\,, \quad S = \{s(\varphi_i, \varphi_k)\}_{i,k=1}^{M}$$

are the Gram matrices in systems of linearly independent functions $\{\phi_k(x)\}_{k=1}^{N}$ and $\{\varphi_k(x)\}_{k=1}^{M}$ (in relation to scalar products described in p. 6, see (24) and (25)), it follow that they are positively defined. This fact guarantees the possibility of unique solution of the system of ordinary differential equations of Cauchy problem (37) and also systems of linear algebraic equations of its initial conditions in relation to vectors $U(0)$, $U'(0)$ and $\Theta(0)$. From here it follows that for each constant $h > 0$ the Cauchy problem (37) has a unique solution$\{U(t), \Theta(t)\}$, which allows finding univalently the semi-discrete Galerkin approximation $(u_h, \theta_h) \in L^2(0,T; V_h \times G_h)$ as (36).

Theorem 2. *Let us assume that the data of variational problem of dissipative acoustics* (23) *is characterized by the conditions of regularity*

$$v_0 \in H, \quad u_0 \in V, \ \ \theta_0 \in H$$

and

$$\left\{\begin{array}{l} f \in L^2(0,T;H), \quad \hat{\sigma} \in L^2(0,T;[L^2(\Gamma_\sigma)]^d), \\ g \in L^2(0,T;H), \quad \hat{q} \in L^2(0,T;L^2(\Gamma_q)). \end{array}\right.$$

Then for each value of discretization parameter $h > 0$ the following statements will be true:

(i) the semi discretized problem has a unique solution (36) $\psi_h = \{u_h,\, \theta_h\}$ *and*

$$\left\{\begin{array}{l} u_h \in L^\infty[0,T;H(div;\Omega)], \quad u_h' \in L^\infty(0,T;H) \cap L^2(0,T;V), \\ \theta_h \in L^\infty(0,T;H) \cap L^2(0,T;G); \end{array}\right.$$

(ii) semi-discrete approximation $\psi_h = \{u_h,\, \theta_h\}$ is continuously dependent on the problem data (23)*, more, the following a priori estimate is correct*

$$\frac{1}{2}[||u_h'(t)||_H^2 + |u_h(t)|_V^2 + ||\theta_h(t)||_H^2] + \int_0^t [||u_h'(t)||_V^2 + ||\theta_h(\tau)||_G^2]d\tau \leq$$
$$\leq C\left\{[||v_0||_H^2 + |u_0|_V^2 + ||\theta_0||_H^2] + \int_0^t [||l(\tau)||_{V'}^2 + ||z(\tau)||_{G'}^2]d\tau\right\}$$
$$\forall t \in [0,T] \quad \forall h > 0.$$

with constant $C > 0$*, the value of which is independent of quantities under consideration.*

11. Existence of solution variation problem of dissipative acoustics

Theorem 3. *Let us assume that the data of problem of dissipative acoustics* (23) *are characterized by regularity conditions* (29) *and* (30)*. Then the variational problem* (23) *has a unique solution* $\psi = \{u,\, \theta\}$ *and*

$$\begin{cases} u_h \in L^\infty[0,T;H(div;\Omega)], \quad u_h' \in L^\infty(0,T;H) \cap L^2(0,T;V), \\ \theta_h \in L^\infty(0,T;H) \cap L^2(0,T;G); \end{cases}$$

moreover

$$\frac{1}{2}[||u'(t)||_H^2 + |u(t)|_V^2 + ||\theta(t)||_H^2] + \int_0^t [||u'(t)||_V^2 + ||\theta(\tau)||_G^2]d\tau \leq$$
$$\leq C\left\{[||v_0||_H^2 + |u_0|_V^2 + ||\theta_0||_H^2] + \int_0^t [||l(\tau)||_{V'}^2 + ||z(\tau)||_{G'}^2]d\tau\right\},$$
$$\forall t \in [0,T].$$

with constant $C > 0$*, the value of which is independent of quantities under consideration.*

Proof. Bearing in mind the theorem 1 we need to estimate the existence of solution (23).

As it follows from the theorem 10.1, the sequence of semi-discrete Galerkin approximations $\psi_h = \{u_h,\, \theta_h\}$ (and also $\{u_h'\}$) form at $h \to 0$ bounded sets in the space $L^\infty(0,T;V) \times [L^\infty(0,T;H) \cap L^2(0,T;G)]$ (respectively$L^\infty(0,T;H) \cap L^2(0,T;V)$).

Therefore, among them we can select convergent subsequence $\psi_\Delta = \{u_\Delta,\, \theta_\Delta\}$ and $\{u_\Delta'\}$such that

$$\begin{cases} \psi_\Delta = \{u_\Delta,\, \theta_\Delta\} \underset{\Delta\to 0}{\to} \psi = \{u,\, \theta\} \;\; in \;\; L^2(0,T;V \times G) \;\; \text{weakly}, \\ u_\Delta' \underset{\Delta\to 0}{\to} u' \;\; in \;\; L^2(0,T;V) \;\; \text{weakly}. \end{cases}$$

After that it remains for us to show that the limit $\psi = \{u,\, \theta\}$ obtained in this way from space $L^2(0,T;V \times G)$is the solution of the problem (23); more precise, it is the matter of direct verification to prove that the pair $\psi = \{u, \theta\}$ satisfies the equation of this problem.

For this purpose we select the spaces $V_h \subset V$, $G_h \subset G$ and $W := \{g \in C^1([0,T]) \mid g(T) = 0\}$. Let us assume that as before $\{\phi_k(x)\}_{k=1}^N$ and $\{\varphi_k(x)\}_{k=1}^M$ are bases of the spaces V_h and G_h respectively and

$$v_h(t) = \sum_{i=1}^{n} q_i(t)\phi_i \in V_h \quad \forall q_i \in W, \quad g_h(t) = \sum_{i=1}^{k} \eta_i(T)\varphi_i \in G_h \quad \forall \eta_i \in W.$$

Due to the problem (36) we have

$$\begin{cases} m(u''_\Delta(t), v_h(t)) + a(u'_\Delta(t), v_h(t)) + c(u_\Delta(t), v_h(t)) - \\ \qquad -b(\theta_\Delta(t), v_h(t)) =< l(t), v_h(t) >, \\ s(\theta'_\Delta(t), g_h(t)) + k(\theta_\Delta(t), g_h(t)) + b(g_h(t), u'_\Delta(t)) = \\ \qquad =< \mu(t), g_h(t) > \quad \forall t \in (0,T]. \end{cases}$$

After time integration over the interval $(0,T)$ when applying integration by parts and initial conditions from (36), we obtain

$$\begin{cases} \int_0^T \{-m(u'_\Delta, v'_h) + a(u'_\Delta, v_h) + c(u_\Delta, v_h) - b(\theta_\Delta, v_h) - < l, v_h) >\} d\tau = \\ \qquad = -m(u'_\Delta(0), v_h(0)) = -m(v_0, v_h(0)), \\ \int_0^T -s(\theta_\Delta, g'_h) + k(\theta_\Delta, g_h) + b(g_h, u'_\Delta) - < \mu, g_h >\} d\tau = \\ \qquad = -s(\theta_\Delta(0), g_h(0)) = -s(\theta_0, g_h(0)). \end{cases}$$

In the derived equations we proceed to the limit with $\Delta \to 0$, and then again perform integration by parts, we obtain

$$\begin{cases} \int_0^T \{m(u'', v_h) + a(u', v_h) + c(u, v_h) - b(\theta, v_h) - < l, v_h) >\} d\tau = \\ \qquad = m(u'(0) - v_0, v_h(0)) \qquad \forall v_h \in C^1([0,T]; V_h) \\ \int_0^T s(\theta', g_h) + k(\theta, g_h) + b(g_h, u') - < \mu, g_h >\} d\tau = \\ \qquad = s(\theta(0) - \theta_0, g_h(0)) \qquad \forall g_h \in C^1([0,T]; G_h). \end{cases}$$

Since V_h is dense in space V, and G_h is dense in space G, the final equations is true for each $v \in C^1([0,T]; V)$ and $g \in C^1([0,T]; G)$.

$$\begin{cases} m(u'', v) + a(u', v) + c(u, v) - b(\theta, v) =< l, v >, \\ \qquad s(\theta', g) + k(\theta, g) + b(g, u') =< \mu, g >, \\ m(u'(0) - v_0, v) = 0 \quad \forall v \in V, \ s(\theta(0) - \theta_0, g) = 0 \quad \forall g \in G. \end{cases}$$

Finally, from the initial conditions and considering (36)

$$a(u_0, v) = a(u_\Delta(0), v) \to a(u(0), v) \qquad \forall v \in V.$$

It follows that the pair $\psi = \{u, \theta\}$ is the solution of the variational problem (23). Moreover, for this solution the energy equation (28) and estimate (31) stay true. The uniqueness of solution of variational problem (23) results from (31) and proof by contradiction. □

12. Conclusions

On the basis of the conservation laws, we have formulated fundamental equations, phenomenological relations, initial and boundary conditions that describe the motion of Newtonian viscous heat-conducting fluid in terms of mass density, vector of velocity, pressure, entropy and temperature. By applying for this non-closed model of hydrodynamics the hypotheses of acoustic disturbances of fluid by linearization, we have found the initial boundary value problem and corresponding variational problem only in terms of vector of acoustic displacements and temperature, which describes the process of spreading acoustic waves with consideration of connectivity of mechanical and thermal fields. We have determined the regularity class of input data of variational problem, which guarantee uniqueness and continuous dependence of the required solution in the energy norm of the problem. In addition, the existence of solution of the considered problem has been presented as a limit of sequence of semi-discrete (by spatial variables) Galerkin approximations.

The obtained results form a fully-functional system for successful modeling and analysis of numeric schemes for solving problems of dissipative acoustics. In particular, one of such schemes can be obtained by direct application of the one-step recurrent scheme for time integration of semi-discretized variational problem (36) using classic approximation spaces of the finite element method [8]. The results of modeling and analysis of convergence of such schemes will be presented in the nearest future.

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