Журнал обчислювальної та прикладної математики | 2012, № 3 (109) с. 11–22 | Journal of Numerical & Applied Mathematics

UDC 519.6

RIGOROUS VANISHING SOLUTIONS OF A NONLINEAR HAMMERSTEIN INTEGRAL EQUATION RELATED TO PROBLEMS WITH FREE PHASE

Olena Bulatsyk, Ihor Tupychak, Yuriy Topolyuk

Резюме. Розглядається нелінійне інтегральне рівняння Гаммерштейна, яке виникає в задачах з вільною фазою. Досліджується новий клас дійсних та комплексних розв'язків цього рівняння. Розв'язки подаються у явному вигляді зі скінченим числом невідомих комплексних параметрів, що є нулями спеціально побудованого полінома, і скінченим числом дійсних параметрів – нулів цих розв'язків у їх області визначення. Для знаходження цих параметрів строго отримана нова коректна система трансцендентних рівнянь. Розв'язки цієї системи досліджуються чисельно. Аналізуються точки галуження цих розв'язків відносно дійсного параметра задачі.

Abstract. A nonlinear Hammerstein integral equation that arises in problems with free phase is considered. A new class of real and complex solutions of this equation is investigated. Solutions are represented in an explicit form with a finite number of unknown complex parameters being zeros of a specially built polynomial, and a finite number of real parameters – zeros of these solutions in their domain of definition. A new correctly determined form of earlier obtained transcendental equations system is found. The solutions of this system are numerically investigated. Their branching points are analyzed with respect to a real parameter of the problem.

1. Introduction

Problems with free phase, covering, in particular, the phase problem, attracted the attention of researchers for a long time [1]– [4]. A wide class of these problems complete the phase optimization problems, main idea of which consists in the mean square approximation of a given non-negative function by modulus of the functions being the result of action of a bounded operator on compactly supported complex functions [5]–[9]. These variational problems are reduced in a usual way to nonlinear integral equations of Hammerstein type as the Lagrangue-Euler equations for respective functionals. As a nonlinearity, these equations involve the phase factor (argument of unknown complex function) in the integrant.

One of the ways of solving such type of equations was an approach suggested and developed in [10], [11], and described in details in [12]. In this approach the solutions are represented in an explicit form with a finite number of unknown complex parameters. A system of transcendental equations for calculation of

†*Key words.* Nonlinear integral equation of Hammerstein type, finite-parametric solutions, branching of solutions, phase problem, vanishing solution.

these parameters was obtained. The approach was extended to a general class of Hammerstein equations of the considered type.

As it was noted in [12], above approach does not cover all solutions of the equations, it considers only nonvanishing solutions. The solutions having zeros in their definition domain were particularly considered in [13], [14]. Such problems arise, in particular, when the desired function to be approximated has zeros in its support domain. Among the physical problems of such type we should, in particular, mention the antenna synthesis problem by the given multi-lobe amplitude directivity pattern [15].

Similar problems were investigated in [13]. There were announced the ideas of analytical presentation of the solution for solving the problems. The main theoretical results are given in [14]. Partial numerical results are also considered there. Numerical results of investigation of real solutions of partial system of equations obtained in [14] are conducted in [16].
It should be also mentional the work [17], [18] on the approximation of functions defined on the real axis by the classes of entire functions and more universal approach in works [19], [20] which are close to the ideology of problems in our article.

In the article we consider more general one-dimensional case. Some results given here were announced in [21]. Real solutions, having zeros in the domain of finiteness of given non-negative function were considered in [16]. Some results concerning the solution branching with respect to the real parameter which is included in the kernel of nonlinear equation of Hammerstein type were described. It turns out that the sets of real and complex solutions are not separated. There are such values of the real kernel parameters, at which the complex solution branches into the real one. Branching of the initial complex solutions (this class of solutions will be described later) into other complex solutions is numerically investigated in this article. The starting point of the article is a system of complex transcendental equations, correctness of which is proved theoretically. Numerical results for several particular cases are presented and analyzed.

2. Problem formulation

Consider the Hammerstein integral equation

$$f(\xi) = \int_a^b K(\xi, \xi')F(\xi')e^{i \arg f(\xi')} d\xi'. \tag{1}$$

with the kernel

$$K(\xi, \xi') = \frac{s(\xi)q(\xi') - s(\xi')q(\xi)}{\tau(\xi) - \tau(\xi')}, \tag{2}$$

where $s(\xi)$, $q(\xi)$, $\tau(\xi)$ are real continuous functions, such that the systems of functions, $\{\tau^n(\xi)s(\xi)\}$, $\{\tau^n(\xi)q(\xi)\}$, $(n = 0, 1, \dots)$ are linearly independent, $F(\xi) \in L_2(a, b)$ is a given non-negative function. It is assumed that the solutions of the equation may have real zeros in the interval $[a, b]$.

3. The theoretical results

Let us represent the solution of equation (1) in the form

$$f(\xi) = \gamma \hat{f}(\xi) P_N(\tau) \prod_{j=1}^{M} (\xi - p_j)\,, \tag{3}$$

where $\hat{f}(\xi)$ is a real positive function in $\xi \in [a,b]$; γ is a complex constant with $|\gamma| = 1$; $a \le p_j \le b$ are real zeros of function $f(\xi) : f(p_j) = 0$, M is a positive integer number;

$$P_N(\tau) = \prod_{k=1}^{N} (1 - \eta_{Nk}\tau)$$

is polynomial of the degree N with complex, pairwise nonconjugated zeros η_{Nk}^{-1} :

$$\eta_{Nk} - \bar{\eta}_{Nm} \ne 0, \;\; k, m = 1, 2, ..., N. \tag{4}$$

Without loss of generality, we can set $\gamma = 1$. From (3) we obtain

$$e^{i \arg f(\xi)} = \frac{P_N(\tau)}{|P_N(\tau)|} \prod_{j=1}^{M} \operatorname{sgn}(\xi - p_j)\,. \tag{5}$$

The function $\hat{f}(\xi)$ can be uniquelly defined from the equality

$$\hat{f}(\xi) \left| \prod_{j=1}^{M} (\xi - p_j) \right| = \frac{1}{|P_N(\tau)|} \left| \int_a^b K(\xi,\xi') F(\xi') \frac{P_N(\tau')}{|P_N(\tau')|} \prod_{j=1}^{M} \operatorname{sgn}\left(\xi' - p_j\right) d\xi' \right| \tag{6}$$

which follows from (1).

Theorem 1. *Function $f(\xi)$ of the form (3) is a solution of equation (1) if and only if the real parameters p_j, $j = 1, ..., M$ and complex η_{Nk}, $k = 1, ..., N$, with the condition (4) satisfy the system of transcendental equations:*

$$T_{Nn}(p_j, \eta_{N1}, \eta_{N2}, ..., \eta_{NN}) = 0, \;\; n = 1, 2, ..., N, \;\; j = 1, 2, ..., M, \tag{7}$$

$$\Phi_{Nn}(p_j, \eta_{N1}, \eta_{N2}, ..., \eta_{NN}) = 0, \;\; n = 1, 2, ..., N, \;\; j = 1, 2, ..., M, \tag{8}$$

$$\Psi_{Nn}(p_j, \eta_{N1}, \eta_{N2}, \ldots, \eta_{NN}) = 0, \; n = 1, 2, ..., N, \;\; j = 1, 2, ..., M, \tag{9}$$

where

$$T_{Nn} = \int_a^b K(p_j,\xi') F(\xi') \frac{\operatorname{Re}\left[\bar{P}_N(\tau(p_j)) P_N(\tau')\right]}{|P_N(\tau')|} \prod_{j=1}^{M} \operatorname{sgn}\left(\xi' - p_j\right) d\xi',$$

$$\Phi_{Nn} = \int_a^b \tau^{n-1} s(\xi) \frac{F(\xi)}{|P_N(\tau)|} \prod_{j=1}^{M} \operatorname{sgn}(\xi - p_j)\, d\xi, \tag{10}$$

$$\Psi_{Nn} = \int\limits_a^b \tau^{n-1} q(\xi) \frac{F(\xi)}{|P_N(\tau)|} \prod_{j=1}^{M} \operatorname{sgn}(\xi - p_j)\, d\xi. \tag{11}$$

Proof. Necessity. Let function $f(\xi)$ represented as (3) with $\gamma = 1$ be the solution of equation (1). Substituting expressions (3), (5) into (1) and multiplying both sides by $\bar{P}_N(\xi)$ result in

$$\begin{gathered} \hat{f}(\xi)\, |P_N(\tau)|^2 \prod_{j=1}^{M} (\xi - p_j) = \\ = \bar{P}_N(\tau) \int\limits_a^b K(\xi, \xi') F(\xi') \frac{P_N(\tau')}{|P_N(\tau')|} \prod_{j=1}^{M} \operatorname{sgn}\left(\xi' - p_j\right) d\xi'. \end{gathered} \tag{12}$$

After extracting the imaginary part from (12), we obtain

$$\int\limits_a^b \left(\tau - \tau'\right) K(\xi, \xi') \frac{F(\xi')}{|P_N(\tau')|} \prod_{j=1}^{M} \operatorname{sgn}\left(\xi' - p_j\right) R_{N-1}(\tau, \tau') d\xi' = 0, \tag{13}$$

where

$$R_{N-1}(\tau, \tau') = \frac{2i \operatorname{Im}[P_N(\tau') \bar{P}_N(\tau)]}{\tau - \tau'} = \sum_{k,m=1}^{N} a_{km} \tau^{k-1} (\tau')^{m-1} \tag{14}$$

is a polynomial of two variables with matrix coefficients $A = \{a_{km}\}$. Substitution (14), (2) into (13) and interchanging the variables ξ and ξ', give

$$\begin{gathered} \sum_{k,m=1}^{N} a_{km} \left[q(\xi') \int\limits_a^b \tau^{k-1} s(\xi) \frac{F(\xi)}{|P_N(\tau)|} \prod_{j=1}^{M} \operatorname{sgn}(\xi - p_j)\, d\xi - \right. \\ \left. - s(\xi') \int\limits_a^b \tau^{k-1} q(\xi) \frac{F(\xi)}{|P_N(\tau)|} \prod_{j=1}^{M} \operatorname{sgn}(\xi - p_j)\, d\xi \right] (\tau')^{m-1} \equiv 0. \end{gathered} \tag{15}$$

Since the functions $\{\tau^k s(\xi)\}$, $\{\tau^k q(\xi)\}$, $k = 0, 1, \ldots, N-1$, are linearly independent, expression (15) results in the following systems:

$$\sum_{k=1}^{N} a_{km} \Phi_{Nn} = 0, \;\; n = 1, 2, ..., N, \tag{16}$$

$$\sum_{k=1}^{N} a_{km} \Psi_{Nn} = 0, \;\; n = 1, 2, ..., N, \tag{17}$$

where Φ_{Nn}, Ψ_{Nn} are defined in (10), (11). They can be considered as independent systems of linear algebraic equations for the unknown Φ_{Nn}, Ψ_{Nn}, with the same matrix of coefficients A.

Determinant of matrix A has been found in [11]:

$$\det A = (-1)^{[N/2]} \prod_{k,m=1}^{N} (\bar{\eta}_{Nm} - \eta_{Nk}),$$

where the square brackets mean the integer part of the number. From (4) we get $\det A \neq 0$, and the equation systems (16), (17) have only zero solutions. This means that transcendental equations (8), (9) are satisfied.

Let the solution of equation (1) satisfy the condition $f(p_j) = 0$, $j = 1, ..., M$. Then, according to (1),

$$\int_a^b K(p_j, \xi')F(\xi')e^{i \arg f(\xi')} d\xi' = 0. \tag{18}$$

Multiplying both sides of (18) by $\bar{P}_N(\xi)$ and using (5) result in

$$\int_a^b K(p_j, \xi')F(\xi')\frac{\bar{P}_N(\tau(p_j))P_N(\tau')}{|P_N(\tau')|} \prod_{j=1}^{M} \operatorname{sgn}(\xi' - p_j)\, d\xi' = 0. \tag{19}$$

After extracting the real part from (19), we obtain the system of equations (7). Imaginary part (19) gives the following system:

$$\begin{gathered}\int_a^b (\tau(p_j) - \tau')\, K(p_j, \xi')\frac{F(\xi')}{|P_N(\tau')|} \times \\ \times \prod_{j=1}^{M} \operatorname{sgn}(\xi' - p_j)\, R_{N-1}(\tau(p_j), \tau') d\xi' = 0.\end{gathered} \tag{20}$$

System of equations (20) coincides with the system (13) in case of $\xi = p_j$ and $\tau = \tau(p_j)$.

Sufficiency. Let the system of transcendental equations (7), (8), (9) be satisfied for some integer N, complex numbers η_{Nk}, $k = 1, 2, ..., N$, which satisfy the condition (4) and real numbers p_j, $j = 1, ..., M$. We show that the function of the form (3) is a solution of equation (1) and p_j, $j = 1, ..., M$, are real zeros of the solution.

After reducing system (8), (9) to equalities (16), (17) and substituting (10), (11) into them we get the equality (15). Then, using (13), we have

$$\operatorname{Im}\left[\bar{P}_N(\tau)\int_a^b K(\xi, \xi')\frac{F(\xi')}{|P_N(\tau')|}\prod_{j=1}^{M} \operatorname{sgn}(\xi' - p_j)\, P_N(\tau')d\xi'\right] = 0.$$

Add the real function $\hat{f}(\xi)\,|P_N(\tau)|^2 \prod_{j=1}^{M} (\xi - p_j)$ under the symbol of imaginary part:

$$\begin{gathered}\operatorname{Im}\left[\hat{f}(\xi)\,|P_N(\tau)|^2\prod_{j=1}^{M}(\xi-p_j)+\right.\\ \left.+\bar{P}_N(\tau)\int\limits_a^b K(\xi,\xi')\frac{F(\xi')}{|P_N(\tau')|}\prod_{j=1}^{M}\operatorname{sgn}\left(\xi'-p_j\right)P_N(\tau')d\xi'\right]=0.\end{gathered}\tag{21}$$

Dividing both sides of equality (21) by the positive function $|P_N(\tau)|$ and taking into account the equality

$$\hat{f}(\xi)\,|P_N(\tau)|=\frac{|f(\xi)|}{\left|\prod\limits_{j=1}^{M}(\xi-p_j)\right|},\tag{22}$$

we get

$$\begin{gathered}\operatorname{Im}\left[|f(\xi)|\prod_{j=1}^{M}\operatorname{sgn}(\xi-p_j)+\right.\\ \left.+\frac{\bar{P}_N(\tau)}{|P_N(\tau)|}\int\limits_a^b K(\xi,\xi')\frac{F(\xi')}{|P_N(\tau')|}\prod_{j=1}^{M}\operatorname{sgn}\left(\xi'-p_j\right)P_N(\tau')d\xi'\right]=0.\end{gathered}\tag{23}$$

On the other hand, (6) gives

$$\begin{gathered}\operatorname{Re}\left[|f(\xi)|\prod_{j=1}^{M}\operatorname{sgn}(\xi-p_j)+\right.\\ \left.+\frac{\bar{P}_N(\tau)}{|P_N(\tau)|}\int\limits_a^b K(\xi,\xi')\frac{F(\xi')}{|P_N(\tau')|}\prod_{j=1}^{M}\operatorname{sgn}\left(\xi'-p_j\right)P_N(\tau')d\xi'\right]=0.\end{gathered}\tag{24}$$

Equalities (23) and (24) mean that the expression in brackets equals to zero and, according to (5), function (3) is a solution of integral equation (1). □

Let then the equation system (7) be satisfied. Using (24), (22), (3) with $\xi=p_j$, we get

$$\operatorname{Re}\left[f\left(p_j\right)\right]=0.$$

Similarly, (23), (13), (22), (3) give

$$\operatorname{Im}\left[f\left(p_j\right)\right]=0.$$

Thus, the parameters p_j, $j=1,...,M$ are zeros of the function f of form (3). The theorem is proved.

4. The numerical results

The Newton method was used for solving system (7), (8), (9). Integrals from the left part of the equation system were calculated by the Simpson method. The integration interval was divided into segments by points so, that the integrand was smooth there. Iterative process of Newton method can be described by the following formula:

$$\vec{x}^{(m+1)} = \vec{x}^{(m)} - \left(\Im'\left(\vec{x}^{(m)}; c\right)\right)^{-1} \Im\left(\vec{x}^{(m)}; c\right), \quad m = 0, 1,$$

where $\vec{x}^{(m)} = \left\{\eta_n'^{(m)}, \eta_n''^{(m)}, p_l^{(m)}\right\}$ is an approximation of zeros of the system (7), (8), (9) on the m-th step, $\eta_n^{(m)} = \eta_n'^{(m)} + i\eta_n''^{(m)}$, i is the imaginary unit, $\Im$ is a vector of the left parts of the equation system (7), (8), (9), $\Im'\left(\vec{x}^{(m)}; c\right)$ is the Jacobian matrix of this system in the point $\vec{x}^{(m)}$; c is a real positive parameter; m is an iteration number. The end-point condition of the iterative process in the Newton method is

$$\max_{n=\overline{1,N}} \left|\eta_n'^{(m+1)} - \eta_n'^{(m)}\right| + \max_{n=\overline{1,N}} \left|\eta_n''^{(m+1)} - \eta_n''^{(m)}\right| + \\ + \max_{l=\overline{1,M}} \left|p_l^{(m+1)} - p_l^{(m)}\right| < \varepsilon.$$

The structure of the Jacobian matrix in general case is

$$\Im'\left(\eta_n', \eta_n'', p_l\right) = \begin{pmatrix} \left\{\dfrac{\partial \Phi_{Nj}}{\partial \eta_{Nk}'}\right\}_{j,k=1}^{N} & \left\{\dfrac{\partial \Phi_{Nj}}{\partial \eta_{Nk}''}\right\}_{j,k=1}^{N} & \left\{\dfrac{\partial \Phi_{Nj}}{\partial p_k}\right\}_{j,k=1}^{N,M} \\ \left\{\dfrac{\partial \Psi_{Nj}}{\partial \eta_{Nk}'}\right\}_{j,k=1}^{N} & \left\{\dfrac{\partial \Psi_{Nj}}{\partial \eta_{Nk}''}\right\}_{j,k=1}^{N} & \left\{\dfrac{\partial \Psi_{Nj}}{\partial p_k}\right\}_{j,k=1}^{N,M} \\ \left\{\dfrac{\partial T_{Nj}}{\partial \eta_{Nk}'}\right\}_{j,k=1}^{M,N} & \left\{\dfrac{\partial T_{Nj}}{\partial \eta_{Nk}''}\right\}_{j,k=1}^{M,N} & \left\{\dfrac{\partial T_{Nj}}{\partial p_k}\right\}_{j,k=1}^{M,M} \end{pmatrix}.$$

Equation system (7), (8), (9) was investigated numerically for the case $s(\xi) = \sin c\xi$, $q(\xi) = \cos c\xi$, $\tau = \xi$, $a = 1$, $b = -1$, $c > 0$. In this case equation (1) has the form

$$f(\xi) = \int_{-1}^{1} \frac{\sin c(\xi - \xi')}{\xi - \xi'} F(\xi') \exp(i \arg f(\xi')) d\xi'. \tag{25}$$

We consider two types of given non-negative functions $F(\xi) : F_1(\xi) = |\xi - t|$ and $F_2(\xi) = \sin(\pi \cdot |\xi - t|/(1 + |t|))$, $t \in (-1, 1)$, having one zero in the integration domain.

Real solutions. We consider first the real solutions that correspond to $N = 0$. According to (3), each real solution of equation (1) is represented in the form:

$$f\left(\xi'\right)=\int_{-1}^{1}F\left(\xi\right)\frac{\sin c\left(\xi-\xi'\right)}{\xi-\xi'}\prod_{j=1}^{M}\operatorname{sgn}\left(\xi-p_j\right)d\xi. \quad (26)$$

It follows from (7), that the real parameters p_j of are found from the equation system

$$\int_{-1}^{1}F\left(\xi\right)\frac{\sin c\left(\xi-p_l\right)}{\xi-p_l}\prod_{k=1}^{M}\operatorname{sgn}\left(\xi-p_k\right)d\xi=0,\quad l=1,...,M. \quad (27)$$

The cases M=1 and M=2 are investigated numerically. In these cases the real solution of equation (1) has one or two zeros in the interval [-1,1], respectively.

In case M=1, system (27) becomes the transcendental equation with respect to $p=p_1$:

$$\int_{-1}^{1}F\left(\xi\right)\frac{\sin c(\xi-p)}{|\xi-p|}d\xi=0. \quad (28)$$

We solve this equation by the chord method. Integrals in the left hand sides of the equation are calculated by Simpson method. To apply this method, the integration interval [-1;1] is divided into parts by zeros of the functions F, f so that the integrand is smooth in each of these parts. Then, in particular, unknown parameter p occurs in the integration limits.

Solutions of equation (28) are shown in Fig. 1. Different solutions depending on parameter c are marked by p_{1j}, where j means the solution number.

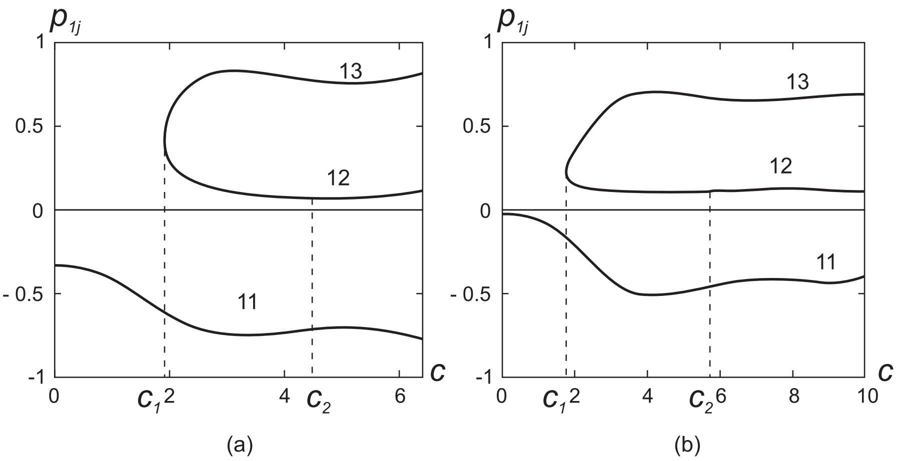

FIG. 1. Solutions of equation (28): (a) $F_1(\xi)=|\xi-t|$, (b) $F_2(\xi)=\sin(\pi\cdot|\xi-t|/(1+|t|))$; $t=0.1$

As follows from Fig. 1, number of solutions varies depending on the real parameter. For $c<c_1$ only one solution p_{11} exists. At the point $c=c_1$ two

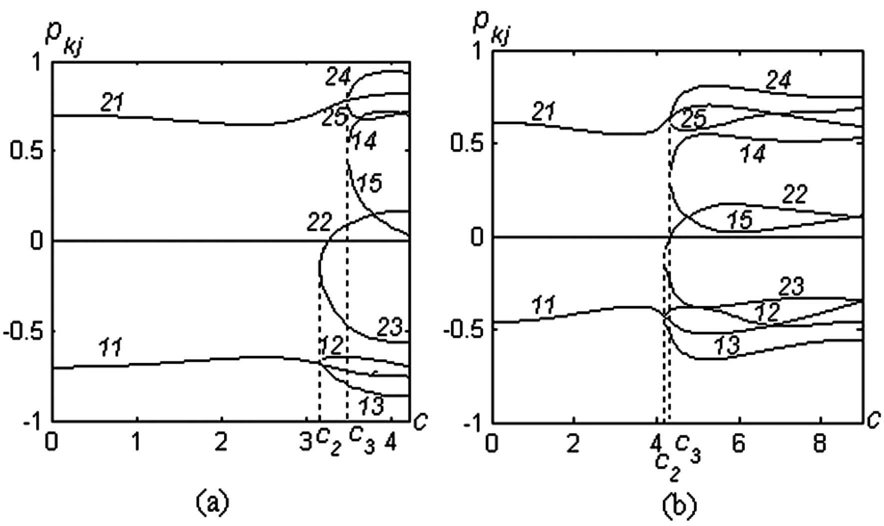

FIG. 2. Solutions of equation system (27) at $M = 2$: (a) $F_1(\xi) = |\xi - t|$, (b) $F_2(\xi) = \sin(\pi \cdot |\xi - t|/(1 + |t|))$; $t = 0.1$

more solutions (p_{12} and p_{13}) appear. These results show that the point c_1 is an isolated bifurcation point, namely a point of appearance of new solutions. For $c>c_1$ we already have three solutions of equation (28).

The more complicated situation arises in the case when two zeros of real solution of equation (1) exist in the interval $[-1; 1]$ ($M = 2$). In this case the equation system (27) is solved by the Newton method. As before, the interval $[-1; 1]$ is divided into parts so that the integrand is smooth in each of these parts.

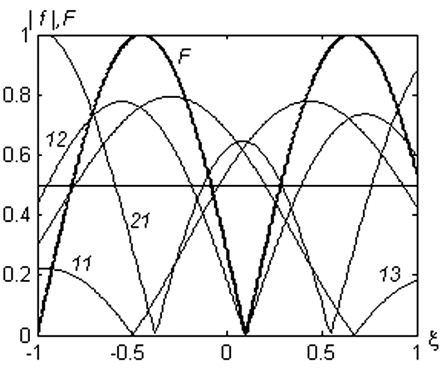

FIG. 3. Different solutions to (25) for given function $F_2(\xi) = \sin(\pi \cdot |\xi - t|/(1 + |t|))$ at $c = 3.5$, $t = 0.1$

The solutions with two zeros p_1, p_2 exist for arbitrary value of $c>0$. The curves corresponding to these solutions are marked with symbols $1j$ and $2j$, respectively. Similarly to the previous case, there is a pair of new solutions with parameters $\{p_{12}, p_{22}\}$ and $\{p_{13}, p_{23}\}$ at the point $c = c_2$. One more pair of solutions $\{p_{14}, p_{24}\}$ and $\{p_{15}, p_{25}\}$ appears at the point $c = c_3$. Thus, when $c > c_3$, there are 5 different solutions of system (27) each of which has two real zeros in the interval: $\{p_{11}, p_{21}\}$, $\{p_{12}, p_{22}\}$, $\{p_{13}, p_{23}\}$, $\{p_{14}, p_{24}\}$, $\{p_{15}, p_{25}\}$.

Moduli $|f_i(\xi)|$ of all found solutions (26) with $M = 1$ and $M = 2$ of the initial equation (25) in case $F_2(\xi) = \sin(\pi \cdot |\xi - t|/(1 + |t|))$, $c = 3.5$ are shown in Fig. 3. It turns out that the closest in modulus to the given function $F_2(\xi)$, is f_{12} having one real zero at the point $\xi = p_{12}$.

Complex solutions. Complex solutions of the system of transcendental equations (7), (8), (9) in the considered particular case were numerically investigated for $N = 1$, $M = 1$. These solutions have one real and one complex parameter. The equation system for this case is of the form

$$\int_{-1}^{1} F(\xi) \frac{\sin c\,(\xi - p_1)}{\xi - p_1} \frac{\operatorname{Re}\left[\bar{P}_1(p_1) P_1(\xi)\right]}{|P_1(\xi)|} \operatorname{sgn}(\xi - p_1)\, d\xi = 0, \tag{29}$$

$$\int_{-1}^{1} \sin c\xi \frac{F(\xi)}{|P_1(\xi)|} \operatorname{sgn}(\xi - p_1)\, d\xi = 0, \qquad \int_{-1}^{1} \cos c\xi \frac{F(\xi)}{|P_1(\xi)|} \operatorname{sgn}(\xi - p_1)\, d\xi = 0,$$

where $P_1(\xi) = 1 - \eta_1 \xi$. This system was solved by the Newton method.

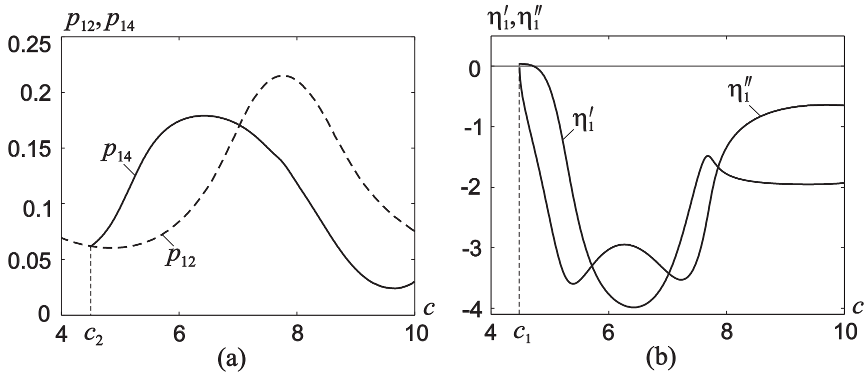

FIG. 4. Parameters of complex solution with $M = 1$, $N = 1$ to equation (29) for $F_1(\xi) = |\xi - 0.1|$

We investigate the real solution f_{12} (with one parameter p_{12}) of initial equation (25). This solution appears at the point $c = c_1$ (see Fig. 1, curve 12). At the point $c = c_2$ two complex conjugated solutions are branched off from f_{12}. At $c > c_2$ they have one real parameter p_{14} and one of two complex parameters $\eta_{1,2} = \eta_1' \pm i\eta_1''$.

Numerical results are shown in Fig. 4 and Fig. 5 for the given functions $F_1(\xi)$ and $F_2(\xi)$, respectively. The results demonstrate that sets of real and complex solutions of equation (1) are not isolated and real solutions branch into the

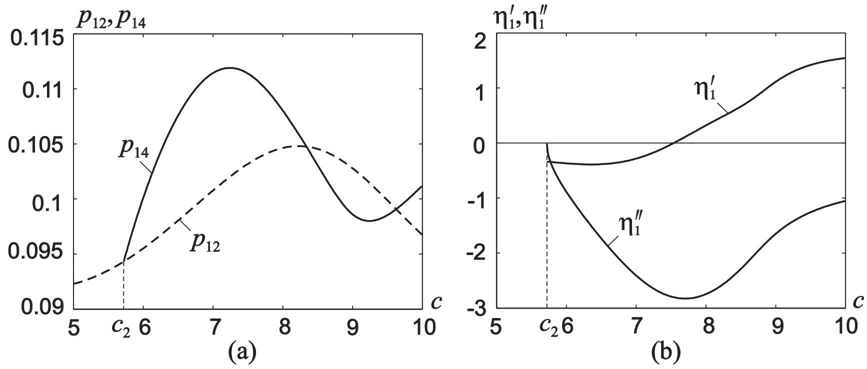

FIG. 5. Parameters of complex solution with $M = 1$, $N = 1$ to equation (29) for $F_2(\xi) = \sin(\pi \cdot |\xi - 0.1|/1.1)$

complex ones (the example of transforming the complex nonvanishing solutions into the real vanishing ones for this type of given function $F(\xi)$ see in [21]).

5. Conclusions

A nonlinear Hammerstein integral equation arisen in problems with free phase has been considered. A new class of real and complex solutions of this equation has been investigated. Solutions have been represented in an explicit form with a finite number of unknown complex parameters being zeros of a complex polynomial, and a finite number of real parameters – zeros of these solutions in their domain of definition. A new correctly determined form of earlier obtained transcendental equations system has been found. The solutions of this system have been numerically investigated for a particular case. The branching points of these solutions with the respect to a real parameter of the problem have been analyzed.

Bibliography

1. Goncharsky A. V. Mathematical models and the phase problem in inverse problem of X-ray diffraction / A. V. Goncharsky, A. A. Stepanov // Mathem. Model.– 1994.– Vol. 6.– P. 117-127 (in Russian).
2. Yezhov P. V. Correlation method for pattern recognition / P.V. Yezhov, A.V. Kuzmenko, V.A. Komarov // Semiconductor Physics, Quantum Electronics & Optoelectronics.– 2002.– Vol. 5.– P. 331-433.
3. Hwi Kim Iterative Fourier transform algorithm withregularization for the optimal design of diffractive optical elements / Kim Hwi, Yang Byungchoon, Lee Byoungho // J. Opt. Soc. Am. A.– 2004.– Vol. 21.– P. 2353-2365.
4. Kuzmenko A. V. Weighting method of the Fourier-kinoform synthesis / A. V. Kuzmenko // Semiconductor Physics, Quantum Electronics & Optoelectronics.– 2008.– Vol. 11.– P. 303-306.
5. Ferwerda H. A. Problem of the wave front phase reconstruction according to amplitude distribution and coherent functions / H.A. Ferwerda In: Inverse Scattering Problems in Optics. Ed. H.P. Baltes.– Berlin: Springer, 1978.
6. Kuznetsova T. I. On the phase problem in optics / T. I. Kuznetsova // Sov. Phys. Usp.– 1988.– Vol. 31.– P. 364-371.

7. Boikova A. T. Solution of phase retrieval problem by a maximum entropy method / A. T. Boikova // Radiophysics and Quantum Electronics.– 1996.– Vol. 39.– P. 321-327.
8. Pol'skikh S. D. The phase problem: analysis of local extrema and image reconstruction algorithms / S. D. Pol'skikh // Journal of Communications Technology and Electronics.– 2008.– Vol. 53.– P. 223-237.
9. Samoilenko M. V. Reconstruction of the amplitude–phase distribution of the field of the received signal in the aperture of a phased antenna array from the measured power / M. V. Samoilenko // Journal of Communications Technology and Electronics.– 2009.– Vol. 54.– P. 1058-1063.
10. Voitovich N. N. Mean square approximation of compactly supported functions with free phase by functions with bounded spectrum / N. N. Voitovich, Yu. P. Topolyuk, O. M. Gis // Dopovidi NAN Ukrainy.– 1999.– Vol. 10.– P. 3-7.
11. Voitovich N. N. Approximation of compactly supported functions with free phase by functions with bounded spectrum / N. N. Voitovich, Yu. P. Topolyuk, O. O. Reshnyak // Fields Institute Communications. AMS.– 2000.– Vol. 25.– P. 531-541.
12. Bulatsyk O. O. Phase Optimization Problems. Applications in Wave Field Theory / O. O. Bulatsyk, B. Z. Katsenelenbaum, Yu. P. Topolyuk, N. N. Voitovich.– Weinheim: WILEY-VCH, 2010.
13. Gis O. M. Closed vanishing solution of the nonlinear problems of the antenna synthesis / O. M. Gis, Yu. P. Topolyuk // Direct and Inverse Problems of Electromagnetic and Acoustic Wave Theory (DIPED-99).– Proc. Int. Seminar / Workshop, Lviv.– 1999.– P. 75-79.
14. Gis O. M. The mean square approximation of nonnegative finite functions by the modulus of functions with the finite spectrum (the case of vanishing solutions of approximation on support the finite function) / O. M. Gis, Yu. P. Topolyuk // Mathem. meth. fiz.-mesh. pola.– 2001.– Vol. 44.– P. 48-54.
15. Andriychuk M. I. Antenna Synthesis according to the Amplitude Radiation Pattern. Numerical Methods and Algorithms / M. I. Andriychuk, N. N. Voitovich, P. A. Savenko, V. P. Tkachuk.– Kiev: Naukova Dumka, 1993 (in Russian).
16. Tupychak I. V. Real vanishing solutions to nonlinear equation related to modified phase problem / I. V. Tupychak, N. N. Voitovich // Direct and Inverse Problems of Electromagnetic and Acoustic Wave Theory (DIPED-2006).– Proc. Int. Seminar / Workshop, Tbilisi.– 2001.– P. 169-173.
17. Stepanets A. I. Classes of functions defined on the real axis, and their approximations by entire functions. I / A. I. Stepanets // Ukr. Math. J.– 1990.– Vol. 42.– P. 102-112 (in Russian).
18. Stepanets A. I. Classes of functions defined on the real axis, and their approximations by entire functions. II / A. I. Stepanets // Ukr. Math. J.– 1990.– Vol. 42. – P. 210-222 (in Russian).
19. Stepanets A. I. Extremal problems of approximation theory in linear spaces / A. I. Stepanets // Ukr. Math. J.– 2003.– Vol. 55.– P. 1662-1698 (in Russian).
20. Stepanets A. I. Extremal problems for integrals of nonnegative functions / A. I. Stepanets // Izvestya RAN, Ser. Math.– 2010.– Vol. 74.– P. 169-224 (Russian).
21. Bulatsyk O. O. Complex vanishing solutions to nonlinear equation related to modified phase problem / O. O. Bulatsyk, I. V. Tupychak, Yu. P. Topolyuk // Direct and Inverse Problems of Electromagnetic and Acoustic Wave Theory (DIPED-2010).– Proc. Int. Seminar / Workshop, Tbilisi.– 2010.– P. 183-186.

Olena Bulatsyk, Ihor Tupychak, Yuriy Topolyuk,
Pidstrygach Institute for Applide Problems of Mechanics and Mathematics, 3b Naukova str., Lviv, 79061, Ukraine

Received 20.04.2012