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TWO-STEP METHOD FOR SOLVING NONLINEAR EQUATIONS WITH NONDIFFERENTIABLE OPERATOR

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Резюме. Запропоновано двокроковий метод для розв'язування нелінійних рівнянь з недиференційовним оператором, побудований на базі двох методів з порядком збіжності $1+\sqrt{2}$. Вивчено локальну та напівлокальну збіжність запропонованого методу та встановлено порядок збіжності. Проведено числове дослідження на тестових задачах та зроблено порівняння отриманих результатів.

Abstract. In this paper we propose a two-step method for solving nonlinear equations with a nondifferentiable operator. Its method is based on two methods of order of convergence $1 + \sqrt{2}$. We study a local and a semilocal convergence of the proposed method and set an order of convergence. We apply our results to the numerical solution of a nonlinear equation and systems of nonlinear equations.

1. Introduction

We consider the equation

$$H(x) \equiv F(x) + G(x) = 0, \tag{1}$$

where F and G are nonlinear operators, defined on a convex subset D of a Banach space X with values in a Banach space Y. F is a Fréchet-differentiable operator, G is a continuous operator.

There are kinds of methods to find a solution of (1). In [1] Argyros studied the two-point iterative process

$$x_{n+1} = x_n - A_n^{-1}(F(x_n) + G(x_n)), \quad n = 0, 1, \dots, \tag{2}$$

where $A_n = A(x_{n-1}, x_n)$ is a bounded linear operator. There was provided a local and a semilocal convergence analysis for the method (2) and some cases where $A_n = F'(x_n)$, $A_n = F'(x_n) + G(x_{n-1}; x_n)$ were considered. Here $G(x; y)$ is a first order divided difference of the operator G at the points x and y. The convergence analysis for the case where $A_n = F'(x_n)$ was given by Zabrejko and Nguen [11]. In the paper [3] the convergence analysis results for modification of the method (2) for some cases of A_n were presented. There are studies in which there are considered difference methods, i.e., the secant method, the parametric secant method [5, 6] and the method based on the method of linear interpolation and the secant method [7]. In [4] Chen studied a Broyden-like method for solving (1). In [9] we researched a semilocal convergence of the

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†*Key words.* Nondifferentiable operator, convergence order, local and semilocal convergence.

method (2) for $A_n = F'(x_n) + G(2x_n - x_{n-1}; x_{n-1})$. The Newton's method cannot be applied, as differentiability of operator H is required.

In this work we propose a two-step method which is based on the methods with the order of convergence $1 + \sqrt{2}$ [8, 10],

$$\begin{aligned} x_{n+1} &= x_n - \Big[F'\Big(\frac{x_n + y_n}{2}\Big) + G(x_n; y_n)\Big]^{-1}(F(x_n) + G(x_n)), \\ y_{n+1} &= x_{n+1} - \Big[F'\Big(\frac{x_n + y_n}{2}\Big) + G(x_n; y_n)\Big]^{-1}(F(x_{n+1}) + G(x_{n+1})), \\ &n = 0, 1, \ldots . \end{aligned} \tag{3}$$

Although the numbers of evaluations of the function values increases by one at each step for the proposed method (3), the convergence order is higher than for the one-step methods.

2. Convergence analysis

Definition 1. Let F be a nonlinear operator defined on a subset D of a linear space X with values in a linear space Y and let x, y be two points of D. A linear operator from X into Y, denoted as $G(x; y)$, which satisfies the condition

$$G(x; y)(x - y) = G(x) - G(y).$$

is called a divided difference of G at the points x and y.

Theorem 1. *Let F and G be nonlinear operators, defined on an open convex subset D of a Banach space X with values in a Banach space Y. F is a twice Fréchet-differentiable operator, G is a continuous operator. Let us suppose that equation (1) has a solution $x^* \in D$, G has a first order divided difference in D and there exist $[A(x, y)]^{-1} = \Big[F'\Big(\frac{x + y}{2}\Big) + G(x; y)\Big]^{-1}$ for all $x \neq y$ and $\|[A(x, y)]^{-1}\| \leq B$. Let in D the following conditions fulfill*

$$\|F'(x) - F'(y)\| \leq 2p_1\|x - y\|, \tag{4}$$

$$\|F''(x) - F''(y)\| \leq p_2\|x - y\|^{\alpha},\ \alpha \in (0, 1], \tag{5}$$

$$\|G(x; y) - G(u; v)\| \leq q_1(\|x - u\| + \|y - v\|). \tag{6}$$

Suppose that $U = \{x :\ \|x - x^*\| < r_*\} \subset D$, where r_* is the smallest positive zero of equations

$$\begin{aligned} q(r) &= 1, \\ 3B(p_1 + q_1)r q(r) &= 1, \end{aligned} \tag{7}$$

$$q(r) = B\Big[(p_1 + q_1)r + \frac{p_2}{4(\alpha + 1)(\alpha + 2)} r^{1+\alpha}\Big].$$

Then the sequences $\{x_n\}_{n\geq 0}$, $\{y_n\}_{n\geq 0}$ generated by the iterative process (3) are well defined for all x_0, $y_0 \in U$, remain in U and converge to the solution x^*. Moreover, the following inequalities hold for all $n \geq 0$

$$\begin{aligned} \|x_{n+1} - x^*\| \leq & B\Big[(p_1 + q_1)\|y_n - x^*\| + \\ & + \frac{p_2}{4(\alpha + 1)(\alpha + 2)}\|x_n - x^*\|^{1+\alpha}\Big]\|x_n - x^*\|, \end{aligned} \tag{8}$$

$$\begin{aligned}\|y_{n+1}-x^*\| \le & B(p_1+q_1)\Big[\|y_n-x^*\|+ \\ & +\|x_n-x^*\|+\|x_{n+1}-x^*\|\Big]\|x_{n+1}-x^*\|.\end{aligned} \tag{9}$$

Proof. Since the following equality holds for all $x, h \in D$ [10]

$$F(x+h)=F(x)+F'(x)h+\int\limits_0^1 (1-t)F''(x+th)hhdt,$$

then

$$\begin{aligned} & F(x_n)-F(x^*)-F'\Big(\frac{x_n+x^*}{2}\Big)(x_n-x^*)= \\ &= F(x_n)-F\Big(\frac{x_n+x^*}{2}\Big)-F'\Big(\frac{x_n+x^*}{2}\Big)\frac{x_n-x^*}{2}- \\ &-\Big[F(x^*)-F\Big(\frac{x_n+x^*}{2}\Big)-F'\Big(\frac{x_n+x^*}{2}\Big)\frac{x^*-x_n}{2}\Big]= \\ &=\int\limits_0^1(1-t)F''\Big(\frac{x_n+x^*}{2}+t\frac{x_n-x^*}{2}\Big)\frac{x_n-x^*}{2}\frac{x_n-x^*}{2}dt- \\ &-\int\limits_0^1(1-t)F''\Big(\frac{x_n+x^*}{2}+t\frac{x^*-x_n}{2}\Big)\frac{x_n-x^*}{2}\frac{x_n-x^*}{2}dt. \end{aligned} \tag{10}$$

Using the condition (5) and the equality (10), we obtain

$$\begin{aligned} & \Big\|F(x_n)-F(x^*)-F'\Big(\frac{x_n+x^*}{2}\Big)(x_n-x^*)\Big\| \le \\ & \le \frac{p_2\|x_n-x^*\|^{2+\alpha}}{4}\int\limits_0^1(1-t)t^\alpha dt=\frac{p_2\|x_n-x^*\|^{2+\alpha}}{4(\alpha+1)(\alpha+2)}. \end{aligned} \tag{11}$$

Let us choose $x_0 \in U$ and show that the sequences given in (3) are well defined. We denote $A_n = F'\Big(\frac{x_n+y_n}{2}\Big)+G(x_n;y_n)$. If $x_n,\ y_n \in U$, then from the definition of the first order divided difference and (4), (6), (11), we obtain

$$\begin{aligned} \|x_{n+1}-x^*\| &= \|x_n-x^*-A_n^{-1}(F(x_n)+G(x_n)-F(x^*)-G(x^*))\| \le \\ & \le \|A_n^{-1}\|\Big\|F(x_n)-F(x^*)-F'\Big(\frac{x_n+x^*}{2}\Big)(x_n-x^*)\Big\|+ \\ & +\|A_n^{-1}\|\Big\|F'\Big(\frac{x_n+x^*}{2}\Big)-F'\Big(\frac{x_n+y_n}{2}\Big)\Big\|\|x_n-x^*\|+ \\ & +\|A_n^{-1}\|\|G(x_n;x^*)-G(x_n;y_n)\|\|x_n-x^*\| \le \\ & \le B\Big[(p_1+q_1)\|y_n-x^*\|+\frac{p_2}{4(\alpha+1)(\alpha+2)}\|x_n-x^*\|^{1+\alpha}\Big]\|x_n-x^*\| \end{aligned}$$

and

$$\begin{aligned} \|y_{n+1}-x^*\| &= \|x_{n+1}-x^*-A_n^{-1}(F(x_{n+1})+G(x_{n+1})-F(x^*)-G(x^*))\| \le \\ & \le \|A_n^{-1}\|\Big\|\int\limits_0^1\Big\{F'(x^*+t(x_{n+1}-x^*))-F'\Big(\frac{x_n+y_n}{2}\Big)\Big\}dt\Big\|\|x_{n+1}-x^*\|+ \end{aligned}$$

$$+\|A_n^{-1}\|\|G(x_{n+1};x^*)-G(x_n;y_n)\|\|x_{n+1}-x^*\| \le$$

$$\le B(p_1+q_1)[\|y_n-x^*\|+\|x_n-x^*\|+\|x_{n+1}-x^*\|]\|x_{n+1}-x^*\|.$$

We prove that inequalities (8) and (9) are fulfilled. Taking $n=0$ above, we obtain

$$\|x_1-x^*\| < B\Big[(p_1+q_1)r_* + \frac{p_2}{4(\alpha+1)(\alpha+2)}r_*^{1+\alpha}\Big]\|x_0-x^*\| \le \|x_0-x^*\| < r_*$$

and

$$\|y_1-x^*\| <3B^2(p_1+q_1)\Big[(p_1+q_1)r_* + \frac{p_2}{4(\alpha+1)(\alpha+2)}r_*^{1+\alpha}\Big]r_*\|x_0-x^*\| \le$$
$$\le\|x_0-x^*\| < r_*.$$

Therefore, $x_1,\ y_1 \in U$. If $\|x_n-x^*\|<r_*$ and $\|y_n-x^*\|<r_*$ then from (7) – (9), it follows

$$\|x_{n+1}-x^*\| <B\Big[(p_1+q_1)r_* + \frac{p_2}{4(\alpha+1)(\alpha+2)}r_*^{1+\alpha}\Big]\|x_n-x^*\| \le$$
$$\le\|x_n-x^*\| < \ldots < r_*,$$

$$\|y_{n+1}-x^*\| <3B^2(p_1+q_1)\Big[(p_1+q_1)r_*+$$
$$+\frac{p_2}{4(\alpha+1)(\alpha+2)}r_*^{1+\alpha}\Big]r_*\|x_n-x^*\| \le$$
$$\le\|x_n-x^*\| < \ldots < r_*.$$

So, iterative process (3) is well defined, the sequences $\{x_n\}_{n\ge0}$, $\{y_n\}_{n\ge0}$ belong to U. From the last inequalities and estimates (8) and (9) we can see that $\{x_n\}_{n\ge0}$ and $\{y_n\}_{n\ge0}$ converge to x^*. □

Corollary 2. *Let us suppose that the hypotheses of Theorem 1 hold. Then the iterative process (3) converges to a solution x^* of the equation (1) with the order of convergence $1+\sqrt{1+\alpha}$.*

Proof. We denote

$$a_n=\|x_n-x^*\|,\ b_n=\|y_n-x^*\|,\ C_1=B(p_1+q_1),\ C_2=\frac{Bp_2}{4(\alpha+1)(\alpha+2)}.$$

By (8) and (9), we get

$$a_{n+1}\le C_1a_nb_n+C_2a_n^{2+\alpha},$$

$$b_{n+1}\le C_1(a_{n+1}+a_n+b_n)a_{n+1}\le C_1(2a_n+b_n)a_{n+1}\le$$
$$\le C_1(2a_n+C_1(2a_0+b_0)a_n)a_{n+1}=C_1(2+C_1(2a_0+b_0))a_na_{n+1},$$

Then for large n and $a_{n-1}<1$, from previous inequalities, we obtain

$$a_{n+1}\le C_1a_nb_n+C_2a_n^2a_{n-1}^{\alpha}\le$$
$$\le C_1^2(2+C_1(2a_0+b_0))a_n^2a_{n-1}+C_2a_n^2a_{n-1}^{\alpha}\le \qquad (12)$$
$$\le[C_1^2(2+C_1(2a_0+b_0))+C_2]a_n^2a_{n-1}^{\alpha}.$$

From (12) we can write down an equation of the convergence order of the iterative process (3): $t^2 - 2t - \alpha = 0$. The order of convergence is the unique positive solution $t^* = 1 + \sqrt{1+\alpha}$. If $\alpha = 1$, we get that the iterative process (3) converges to the solution of the equation (1) with the order $1 + \sqrt{2}$. □

Theorem 2. *Let F and G be nonlinear operators, defined on an open convex subset D of a Banach space X with values in a Banach space Y. F is a Fréchet-differentiable operator, G is a continuous operator. We assume that $U_0 = \{x : \|x - x_0\| \le r_0\}$ is contained in D, the linear operator $A_0 = F'\Big(\frac{x_0+y_0}{2}\Big) + G(x_0;y_0)$, where $x_0,\ y_0 \in D$, is invertible and the Lipschitz conditions are fulfilled*

$$\|A_0^{-1}(F'(x) - F'(y))\| \le 2p_0\|x-y\|, \tag{13}$$

$$\|A_0^{-1}(G(x;y) - G(u;v))\| \le q_0(\|x-u\| + \|y-v\|). \tag{14}$$

Let's a, c $(c > a)$, r_0 be non-negative numbers such that

$$\|x_0 - x_{-1}\| \le a, \quad \|A_0^{-1}(F(x_0) + G(x_0))\| \le c, \tag{15}$$

$$r_0 \ge c/(1-\gamma), \quad (p_0+q_0)(2r_0 - a) < 1,$$

$$\gamma = \frac{(p_0+q_0)(r_0-a) + 0.5p_0r_0}{1-(p_0+q_0)(2r_0-a)}, \quad 0 \le \gamma < 1.$$

Then the following inequalities hold for all $n \ge 0$

$$\|x_n - x_{n+1}\| \le t_n - t_{n+1}, \quad \|y_n - x_{n+1}\| \le s_n - t_{n+1}, \tag{16}$$

$$\|x_n - x^*\| \le t_n - t^*, \quad \|y_n - x^*\| \le s_n - t^*, \tag{17}$$

where

$$t_0 = r_0, \quad s_0 = r_0 - a, \quad t_1 = r_0 - c,$$

$$t_{n+1} - t_{n+2} = \frac{(p_0+q_0)(s_n - t_{n+1}) + 0.5p_0(t_n - t_{n+1})}{1-(p_0+q_0)[(t_0 - t_{n+1}) + (s_0 - s_{n+1})]}(t_n - t_{n+1}), \tag{18}$$

$$t_{n+1} - s_{n+1} = \frac{(p_0+q_0)(s_n - t_{n+1}) + 0.5p_0(t_n - t_{n+1})}{1-(p_0+q_0)[(t_0 - t_n) + (s_0 - s_n)]}(t_n - t_{n+1}), \tag{19}$$

$\{t_n\}_{n\ge0}$, $\{s_n\}_{n\ge0}$ are non-negative, decreasing sequences that converge to certain t^* such that $r_0 - c/(1-\gamma) \le t^* < t_0$; sequences $\{x_n\}_{n\ge0}$, $\{y_n\}_{n\ge0}$ generated by the iterative process (3) are well defined, remain in U_0 that converge to a solution x^* of equation (1).

Proof. Firstly, we prove, by mathematical induction, that the following inequalities hold for all $k \ge 0$

$$t_{k+1} \ge s_{k+1} \ge t_{k+2} \ge r_0 - \frac{c}{1-\gamma} \ge 0, \tag{20}$$

$$t_{k+1} - t_{k+2} \le \gamma(t_k - t_{k+1}), \quad t_{k+1} - s_{k+1} \le \gamma(t_k - t_{k+1}). \tag{21}$$

From (18), (19) for $k = 0$ we obtain

$$t_1 - t_2 = \frac{(p_0+q_0)(s_0 - t_1) + 0.5p_0(t_0 - t_1)}{1-(p_0+q_0)[(t_0 - t_1) + (s_0 - s_1)]}(t_0 - t_1) \le \gamma(t_0 - t_1),$$

$$t_1 - s_1 = [(p_0+q_0)(s_0 - t_1) + 0.5p_0(t_0 - t_1)](t_0 - t_1) \le \gamma(t_0 - t_1),$$

$$t_2 \geq r_0 - c - \frac{(p_0+q_0)s_0 + 0.5p_0t_0}{1-(p_0+q_0)[t_0+s_0]}c =$$
$$= r_0 - (1+\gamma)c = r_0 - \frac{(1-\gamma^2)c}{1-\gamma} \geq r_0 - \frac{c}{1-\gamma} \geq 0,$$
$$t_1 \geq t_2,\ s_1 \geq t_2,\ t_1 \geq s_1 \geq t_2 \geq r_0 - \frac{c}{1-\gamma} \geq 0.$$

Let us suppose that inequalities (20) and (21) hold for $k = 0, 1, \dots, n-1$. Then for $k = n$ we obtain

$$t_{n+1} - t_{n+2} = \frac{(p_0+q_0)(s_n - t_{n+1}) + 0.5p_0(t_n - t_{n+1})}{1-(p_0+q_0)[(t_0-t_{n+1}) + (s_0 - s_{n+1})]}(t_n - t_{n+1}) \leq$$
$$\leq \frac{(p_0+q_0)s_n + 0.5p_0t_n}{1-(p_0+q_0)[t_0+s_0]}(t_n - t_{n+1}) \leq \gamma(t_n - t_{n+1}),$$

$$t_{n+1} - s_{n+1} = \frac{(p_0+q_0)(s_n - t_{n+1}) + 0.5p_0(t_n - t_{n+1})}{1-(p_0+q_0)[(t_0-t_n) + (s_0 - s_n)]}(t_n - t_{n+1}) \leq$$
$$\leq \frac{(p_0+q_0)s_n + 0.5p_0t_n}{1-(p_0+q_0)[t_0+s_0]}(t_n - t_{n+1}) \leq \gamma(t_n - t_{n+1})$$

and

$$t_{n+1} \geq s_{n+1} \geq t_{n+2} \geq t_{n+1} - \gamma(t_n - t_{n+1}) \geq$$
$$\geq r_0 - \frac{1-\gamma^{n+2}}{1-\gamma}c \geq r_0 - \frac{c}{1-\gamma} \geq 0.$$

So, we prove, that sequences $\{t_n\}_{n\geq 0}$ and $\{s_n\}_{n\geq 0}$ are non-negative, decreasing sequences and converge to t^* such that $t^* \geq 0$.

Let us prove, by mathematical induction, that the iterative process (3) is well defined and inequalities (16) hold for all $n \geq 0$.

Using (15) and $t_0 - t_1 = c$, we prove that (16) hold for $n = 0$.

Let denote $A_n = F'\Big(\frac{x_n+y_n}{2}\Big) + G(x_n; y_n)$. Using Lipschitz conditions (13) and (14), we have

$$\|I - A_0^{-1}A_{n+1}\| = \|A_0^{-1}[A_0 - A_{n+1}]\| \leq$$
$$\leq \Big\|A_0^{-1}\Big[F'\Big(\frac{x_0+y_0}{2}\Big) - F'\Big(\frac{x_{n+1}+y_{n+1}}{2}\Big)\Big]\Big\| +$$
$$+ \|A_0^{-1}[G(x_0; y_0) - G(x_{n+1}; y_{n+1})]\| \leq$$
$$\leq 2p_0\Big(\frac{\|x_0 - x_{n+1}\|}{2} + \frac{\|y_0 - y_{n+1}\|}{2}\Big) + q_0(\|x_0 - x_{n+1}\| + \|y_0 - y_{n+1}\|) \leq$$
$$\leq (p_0+q_0)(\|x_0 - x_{n+1}\| + \|y_0 - y_{n+1}\|) \leq$$
$$\leq (p_0+q_0)(t_0 - t_{n+1} + s_0 - s_{n+1}) \leq$$
$$\leq (p_0+q_0)(t_0+s_0) = (p_0+q_0)(2r_0 - a) < 1.$$

By Banach lema on invertible operator, it follows that A_{n+1} is invertible and

$$\|A_{n+1}^{-1}A_0\| \le \Big[1-(p_0+q_0)(\|x_0-x_{n+1}\|+\|y_0-y_{n+1}\|)\Big]^{-1}.$$

Let us prove that iterative process (3) is well defined for $k=n+1$. From the definition of the first order divided difference and (13), (14), we obtain

$$\|A_0^{-1}(F(x_{n+1})+G(x_{n+1}))\| =$$
$$= \|A_0^{-1}\left[F(x_{n+1})+G(x_{n+1})-F(x_n)-G(x_n)-A_n(x_{n+1}-x_n)\right]\| \le$$
$$\le \Big\|A_0^{-1}\Big[\int_0^1 \Big\{F'(x_{n+1}+t(x_n-x_{n+1}))-F'\Big(\frac{x_n+y_n}{2}\Big)\Big\}dt\Big]\Big\|\|x_n-x_{n+1}\|+$$
$$+\left\|A_0^{-1}\left[G(x_n;y_n)-G(x_n;x_{n+1})\right]\right\|\|x_n-x_{n+1}\| \le$$
$$\le 2p_0\Big[\|x_n-x_{n+1}\|\int_0^1\Big|t-\frac{1}{2}\Big|dt+\frac{\|y_n-x_{n+1}\|}{2}\Big]\|x_n-x_{n+1}\|+$$
$$+q_0\|y_n-x_{n+1}\|\|x_n-x_{n+1}\| =$$
$$= (p_0+q_0)\|y_n-x_{n+1}\|\|x_n-x_{n+1}\|+0.5p_0\|x_n-x_{n+1}\|^2.$$

Hence, using (16), we have

$$\|x_{n+1}-x_{n+2}\| = \|A_{n+1}^{-1}(F(x_{n+1})+G(x_{n+1}))\| \le$$
$$\le \|A_{n+1}^{-1}A_0\|\|A_0^{-1}(F(x_{n+1})+G(x_{n+1}))\|\|x_n-x_{n+1}\| \le$$
$$\le \frac{(p_0+q_0)\|y_n-x_{n+1}\|+0.5p_0\|x_n-x_{n+1}\|}{1-(p_0+q_0)(\|x_0-x_{n+1}\|+\|y_0-y_{n+1}\|)}\|x_n-x_{n+1}\| \le$$
$$\le \frac{(p_0+q_0)(s_n-t_{n+1})+0.5p_0(t_n-t_{n+1})}{1-(p_0+q_0)[(t_0-t_{n+1})+(s_0-s_{n+1})]}(t_n-t_{n+1}) = t_{n+1}-t_{n+2},$$
$$\|x_{n+2}-y_{n+2}\| = \|A_{n+1}^{-1}(F(x_{n+2})+G(x_{n+2}))\| \le$$
$$\le \|A_{n+1}^{-1}A_0\|\|A_0^{-1}(F(x_{n+2})+G(x_{n+2}))\|\|x_n-x_{n+1}\| \le$$
$$\le \frac{(p_0+q_0)\|y_{n+1}-x_{n+2}\|+0.5p_0\|x_{n+1}-x_{n+2}\|}{1-(p_0+q_0)(\|x_0-x_{n+1}\|+\|y_0-y_{n+1}\|)}\|x_{n+1}-x_{n+2}\| \le$$
$$\le \frac{(p_0+q_0)(s_{n+1}-t_{n+2})+0.5p_0(t_{n+1}-t_{n+2})}{1-(p_0+q_0)[(t_0-t_{n+1})+(s_0-s_{n+1})]}(t_{n+1}-t_{n+2}) = s_{n+2}-t_{n+2}.$$

So, iterative process (3) is well defined and (15) holds for all $n \ge 0$. From this it follows

$$\|x_n-x_k\| \le t_n-t_k,\ \|y_n-x_k\| \le s_n-t_k, \|y_n-y_k\| \le s_n-s_k, \quad 0 \le n \le k, \quad (22)$$

i.e., $\{x_n\}_{n\ge 0}$ and $\{y_n\}_{n\ge 0}$ are fundamental sequences in a Banach space X. From (22) for $k \to \infty$ it follows inequalities (17). Let's show that x^* is solution of equation (1). Indeed,

$$\|A_0^{-1}(F(x_{n+1})+G(x_{n+1}))\| \le$$
$$\le (p_0+q_0)\|y_n-x_{n+1}\|\|x_n-x_{n+1}\|+0.5p_0\|x_n-x_{n+1}\|^2 \to 0, \quad n \to \infty.$$

So, $H(x^*) = 0$. □

Remark 5. *If we choose $F(x) = 0$, $p_1 = 0$, $p_2 = 0$ then the estimates (8) and (9) reduce to similar ones in* [8] *for the case $\alpha = 1$.*

Remark 6. *If the divided difference of the operator G satisfies the condition (6), i.e. the operator $G(x; y)$ is Lipschitz continuous, then G is Fréchet-differentiable.*

3. Numerical experiments

For the numerical investigation we choose the equation and the systems of equations considered in [1, 4, 5, 6, 7].

Example 1.

$$e^{x-0.5} - 1.05 + 0.2x|x-1| = 0,$$
$$\boldsymbol{x}^* = 0.5.$$

Example 2.

$$3x^2y - y^2 - 1 + |x-1| = 0,$$
$$x^4 + xy^3 - 1 + |y| = 0,$$
$$(\boldsymbol{x}^*; \boldsymbol{y}^*) \approx (0.894655; 0.327827).$$

Example 3.

$$x^2 - y + 1 + \frac{1}{9}|x-1| = 0,$$
$$y^2 + x - 7 + \frac{1}{9}|y| = 0,$$
$$(\boldsymbol{x}^*; \boldsymbol{y}^*) \approx (1.15936; 2.36182).$$

Example 4.

$$z^2(1-y) - xy + |y - z^2| = 0,$$
$$z^2(x^3 - x) - y^2 + |3y^2 - z^2 + 1| = 0,$$
$$6xy^3 + y^2z^2 - xy^2z + |x + z - y| = 0,$$
$$(\boldsymbol{x}^*; \boldsymbol{y}^*; \boldsymbol{z}^*) = (-1; 2; 3).$$

Let $X = Y = \mathbb{R}^m$, $m = 1, 2, 3$. In this case the first order divided difference $G(x; y)$ is a matrix of dimension $m \times m$. Its elements are calculated as [8]

$$G(x; y)_{i,j} = \frac{G_i(x^1, \ldots, x^j, y^{j+1}, \ldots, y^m) - G_i(x^1, \ldots, x^{j-1}, y^j, \ldots, y^m)}{x^j - y^j},$$
$$i, j = \overline{1, m}.$$

In calculations we use the norm $\|x\|_\infty = \max\limits_{1 \le i \le m} |x^i|$. In the following Tables there are results obtained by methods (3) and (2) in particular, for such cases

$$x_{n+1} = x_n - [F'(x_n)]^{-1}(F(x_n) + G(x_n)), \quad n = 0, 1, \ldots, \tag{23}$$

$$x_{n+1} = x_n - [F'(x_n) + G(x_{n-1}; x_n)]^{-1}(F(x_n) + G(x_n)), \quad n = 0, 1, \dots, \tag{24}$$

$$x_{n+1} = x_n - [H(x_{n-1}; x_n)]^{-1}(F(x_n) + G(x_n)), \quad n = 0, 1, \dots. \tag{25}$$

TABL. 1. Numbers of iterations for solving equations with initial points $x_0 = 1 \cdot d$, $x_{-1} = y_0 = 2 \cdot d$ – for Example 1, $x_0 = (1, 0)d$, $x_{-1} = y_0 = (5, 5)d$ – for Example 2

d	ε	**Example 1**				**Example 2**			
		(23)	(24)	(25)	(3)	(23)	(24)	(25)	(3)
1	10^{-5}	5	5	6	5	11	4	5	5
	10^{-15}	6	7	8	6	33	6	9	6
10	10^{-5}	14	15	20	13	19	13	18	12
	10^{-15}	15	17	22	14	41	15	21	13
100	10^{-5}	104	105	–	88	27	21	30	19
	10^{-15}	105	107	–	89	49	23	32	20

The calculations were conducted in MATLAB 7.1. Iterations were stopped after conditions $\|x_{n+1} - x_n\|_\infty \le \varepsilon$ and $\|H(x_{n+1})\|_\infty \le \varepsilon$ were satisfied. Sign "'-"' means, that in this case the solution was not possible to be found. We examined the convergence of the considered method for such variants of choice of the additional initial approximation y_0: for Example 1 – $x_{-1} = y_0 = 2 \cdot d$, for Examples 2, 3 y_0 was chosen as x_{-1} in the works [1, 5, 6, 7] and $x^i_{-1} = y^i_0 = x^i_0 + 10^{-4}$, $i = 1, 2, 3$ – for Example 4.

The obtained results show that the methods (24) and (3) differ a little for the initial points that are close to the solution. But the method (3) converge faster than (2) for the initial points with $d = 100$. In this case $\|x_0 - x^*\|$ takes the largest value. The method (23) has the lowest speed of convergence.

TABL. 2. Numbers of iterations for solving equations with initial points $x_0 = (1, 1)d$, $x_{-1} = y_0 = (0.9, 1.1)d$ – for Example 3, $x_0 = (-2, 3, 5)d$, $x^i_{-1} = y^i_0 = x^i_0 + 10^{-4}$ – for Example 4

d	ε	**Example 3**				**Example 4**			
		(23)	(24)	(25)	(3)	(23)	(24)	(25)	(3)
1	10^{-5}	6	5	6	5	85	7	10	7
	10^{-15}	13	7	9	6	266	10	12	8
10	10^{-5}	8	7	9	6	102	10	25	14
	10^{-15}	15	9	11	7	284	20	27	16
100	10^{-5}	11	11	14	9	110	28	39	23
	10^{-15}	18	12	16	10	292	30	41	24

In Table 3 the numerical results are presented for the example 1 with $\varepsilon = 10^{-10}$, where n is the iteration number, x_n is the approximate value for x^*, TABL. 3. Numerical results for the Example 1: $x_0 = 1$, $y_0 = 2$

n	x_n	$\lvert x_n - x_{n-1}\rvert$	$\lvert H(x_n)\rvert$
0	1		0.5987212707001
1	0.8079964212227	0.1920035787772	0.3417237602029
2	0.5200746907444	0.28792173047835	0.02019694382837
3	0.5000182789519	0.02005641179247	$1.827905217970 \cdot 10^{-5}$
4	0.5000000000006	$1.827895124595 \cdot 10^{-5}$	$6.967343368913 \cdot 10^{-13}$
5	0.5	$6.967759702547 \cdot 10^{-13}$	$4.163336342344 \cdot 10^{-17}$

$|x_n - x_{n-1}|$ is the norm of correction and $|H(x_n)|$ is the norm of deviation on every step of the iterative process (3).

Now we verify whether the hypothesis of Theorem 2 are satisfied. The research are carried out for the example 1. Since $m = 1$ than $\|\cdot\|_\infty = |\cdot|$. In [9] we showed that the following estimates hold for all $x,\ y \in [0;1]$

$$|A_0^{-1}(F'(x) - F'(y))| \le |A_0^{-1}||(F'(x) - F'(y))| \le \frac{e^{0.5}}{|A_0|}|x - y|,$$

$$|A_0^{-1}(G(x,y) - G(u,v))| \le |A_0^{-1}||(G(x,y) - G(u,v))| \le$$
$$\le \frac{1}{5|A_0|}(|x-u| + |y-v|).$$

Hence $p_0 = \dfrac{e^{0.5}}{2|A_0|}$ and $q_0 = \dfrac{1}{5|A_0|}$. Let us choose $x_0 = 0.43$, $y_0 = 0.47$. Then we get

$$\frac{1}{|A_0|} = 1.049985813745361, \quad p_0 = 0.8655669725276801,$$
$$q_0 = 0.2099971627490723, \quad c = 0.07201451611773883, \quad a = 0.04.$$

Let us choose $r_0 = 0.1$. Then, according to formulas (18) and (19), we get

$$t_0 = 0.1000000000000000, \quad s_0 = 0.06000000000000000,$$
$$t_1 = 0.0798548388226117, \quad s_1 = 0.02326130579394141,$$
$$t_2 = 0.0226355142098747, \quad s_2 = 0.02261740817032270,$$
$$t_3 = 0.02261727501017343, \ldots, \ t^* \approx 0.02261727484294557,$$
$$0.01720355125317807 < t^* < 0.1, \quad \gamma = 0.1302221628134378 < 1.$$

The solution x^* is obtained in 3 iterations with $\varepsilon = 10^{-5}$.

TABL. 4. Numerical results for the Example 1

n	$\lvert x_{n-1} - x_n\rvert$	$t_{n-1} - t_n$	$\lvert y_{n-1} - x_n\rvert$	$s_{n-1} - t_n$
1	$7.0617898 \cdot 10^{-2}$	$7.2014516 \cdot 10^{-2}$	$3.0617898 \cdot 10^{-2}$	$3.2014516 \cdot 10^{-2}$
2	$6.1790108 \cdot 10^{-4}$	$5.3499697 \cdot 10^{-3}$	$1.8418431 \cdot 10^{-5}$	$6.2579158 \cdot 10^{-4}$
3	$3.4257955 \cdot 10^{-9}$	$1.8239200 \cdot 10^{-5}$	$6.1617378 \cdot 10^{-13}$	$1.3316015 \cdot 10^{-7}$

Thus for the given values hypothesis of the Theorem 2 are satisfied (See Tabl. 4). According to this theorem, the iterative process (2) is well-defined, remains in U_0 and converges to the solution $x^* \in U_0$.

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