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ON THE RECOVERY OF CONTINUOUS FUNCTIONS OF TWO VARIABLES FROM NOISY FOURIER COEFFICIENTS

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Резюме. Розглянуто некоректну задачу відновлення гладких функцій двох змінних по наближено заданим коефіцієнтам Фурьє. Ця задача розглянута для двох модельних класів функцій скінченної гладкості: функцій соболевського типу гладкості та функцій з домінуючою змішаною частинною похідною.

Abstract. We consider the ill-posed problem of the recovery of smooth functions of two variables from noisy Fourier coefficients.This problem is considered for two model classes of function of finite smoothness: functions of Sobolev type of smoothness and functions with dominating mixed partial derivative.

1. Introduction

Let $L_2 = L_2(Q_n)$ be the space of square integrable real-valued functions of n variables on a cube $Q_n = [0,1]^n$. Denote by $C = C(Q_n)$ the space of continuous functions on Q_n.

This paper is dedicated to the problem of summation of Fourier series of continuous functions with inaccurately given coefficients. Note that almost all previously known results on this problem were obtained mainly for classes of functions of one variable ($n = 1$).

Let us briefly consider the history of the problem under investigation. Assume that the system of functions $\{\varphi_k(t)\}_{k=1}^{\infty}$ is orthonormal in $L_2(Q_1)$ with respect to the standard scalar product $\langle \cdot, \cdot \rangle$, and $\sum_{k=1}^{\infty} y_k \cdot \varphi_k(t)$ is a Fourier series of the function $y(t) \in C$. Suppose that instead of Fourier coefficients their approximate values $y_{\delta,k}$ are given : the condition

$$\sum_{k=1}^{\infty} (y_k - y_{\delta,k})^2 \le \delta^2$$

is fulfilled.

It is well known (see, for example [6],[7]) that the problem of summation of Fourier series of a continuous function $y(t)$ with approximately given coefficients $\{y_{\delta,k}\}_{k=1}^{\infty}$ on some orthonormal system $\{\varphi_k(t)\}_{k=1}^{\infty}$ is ill-posed, since deviation of a function $y(t) \in C$ of the amount of its series $\sum_{k=1}^{\infty} y_{\delta,k} \cdot \varphi_k(t)$ in the metric of the space C can be arbitrary large.

Papers of the many authors, see, example [1]-[8] are dedicated to the solution of this problem.

†*Key words.* Orthonormal system, stable summation, Fourier series, regularization method.

For the first time to solve this problem A.N. Tikhonov proposed regularization method [7], having the form

$$T^{\alpha,s}(y_\delta)(t)=\sum_{k=1}^{\infty}\frac{y_{\delta,k}}{1+\alpha\cdot k^{2s}}\cdot\varphi_k(t) \tag{1}$$

where α is a regularization parameter, and s characterizes the smoothness of the function to be recovered. Convergence and stability of the method (1) to small perturbations of Fourier coefficients on any orthonormal system $\{\varphi_k(t)\}_{k=1}^{\infty}$ on the class of continuous functions, satisfying the condition

$$\sum_{k=1}^{\infty}|\langle y,\varphi_k\rangle|^2\cdot\psi_k<\infty$$

were proved in [8]. In the condition $\{\psi_k\}_{k=1}^{\infty}$ is a sequence of positive numbers, the order of which is not less than $k^{2+\varepsilon}$, $\varepsilon>0$.

Later V.A. Il'in and E.G. Poznjak [3] in the case of the trigonometric system and B.Aliev [1] in the case of any orthonormal uniform boundary systems for the special classes of functions have obtained the estimate

$$\left\|y(t)-T^{\alpha,s}(y_\delta)(t)\right\|_C\le C\cdot\left(\sqrt{\alpha}+\frac{\delta}{\alpha}\right). \tag{2}$$

We note that one of the major topics within the theory of ill-posed problems is the optimal choice of the regularization parameter α, or the discretization level n, depending on the level of error δ. From (2) one can see that the optimal choice for α is $\alpha_0=\delta^{2/3}$ for which

$$\left\|y(t)-T^{\alpha,s}(y_\delta)(t)\right\|_C\le C\cdot\delta^{\frac{1}{3}}.$$

Later,in [4] P.Mathe and S.V.Pereverzev have considered a general method of summation which is defined as follows

$$T_n^{\lambda}(y_\delta)(t)=\sum_{k=1}^{n}\lambda_k\cdot y_{\delta,k}\cdot\varphi_k(t) \tag{3}$$

where for a triangular array $\lambda=\{\lambda_k=\lambda_k^n,\quad k=1,2,...n,\quad n\in N\}$ it is assumed that there exists a constant C and some number $\theta>0$, such that the condition

$$|1-\lambda_k|\le C\cdot\left(\frac{k}{n}\right)^{\theta}$$

is satisfied. In this case we say that the method of summation (3) is of degree θ.

Error estimates of the method (3) in [4] were obtained for the class

$$W_2^{\mu}=\left\{y\in L_2(Q_1):\quad \|y\|_{\mu}^2=\sum_{k=1}^{n}k^{2\mu}\cdot|\langle y,\varphi_k\rangle|^2<\infty\right\}$$

in the cases of arbitrary orthonormal systems that satisfy various conditions. In particular, in the case of systems of functions $\{\varphi_k(t)\}_{k=1}^{\infty}$ satisfying condition

$$\|\varphi_k\|_C\asymp k^{\beta},\quad \beta\ge 0 \tag{4}$$

the estimate was obtained

$$\|y - T_n^\lambda(y_\delta)\|_C \le C \cdot \delta^{\frac{\mu-\beta-\frac{1}{2}}{\mu}}. \tag{5}$$

In [5] the last result was generalized to the case of a class of continuous functions W_2^ψ related to a given orthonormal system $\{\varphi_k(t)\}_{k=1}^\infty$, satisfying the condition (4) as follows

$$W_2^\psi = \left\{ y \in L_2(Q_1) : \quad \|y\|_\psi^2 = \sum_{k=1}^\infty \psi^2(k) \cdot |\langle y, \varphi_k \rangle|^2 < \infty \right\}.$$

where $\psi(k)$ is some monotone increasing function. At the same,for the method of summation (3) from [4] on a class of functions W_2^ψ the estimate

$$\left\| y - T_n^\lambda(y_\delta) \right\|_C \le C \cdot \delta \cdot \left[\psi^{-1}\left(\frac{1}{\delta}\right) \right]^{\beta+1/2}.$$

was obtained.

The aim in this paper is to obtain results on this problem for some classes of continuous functions of two variables ($n = 2$). Bellow we will consider two model classes of functions of finite smoothness: functions of Sobolev type class and a class of functions with dominating mixed partial derivative.

2. Generalized class of functions with dominating mixed partial derivative

Let $\{\varphi_k(t)\}_{k=1}^\infty$ be an orthonormal system of functions in $L_2(Q_1)$ for which the condition (4) is fulfilled, and

$$\sum_{i=1}^\infty \sum_{j=1}^\infty y_{ij} \cdot \varphi_i(t) \cdot \varphi_j(\tau), \quad y_{ij} = \langle y, \varphi_i \varphi_j \rangle,$$

is Fourier series of a function $y(t,\tau) \in C(Q_2)$.

Suppose that instead of Fourier coefficients $\{y_{ij}\}_{i,j=1}^\infty$ their inaccurate values are given, i.e. a sequence of numbers $y_\delta := \{y_{\delta,i,j}\}_{i,j=1}^\infty$ is given, such that

$$y_{\delta,i,j} = y_{i,j} + \delta \cdot \xi_{i,j}, \quad i,j = 1,2,... \tag{6}$$

where $\xi = \{\xi_{i,j}\}_{i,j=1}^\infty$ is a noise. It is assumed that $\delta \in (0,1)$ and

$$\|\xi\|_{l_2} = \left(\sum_{i=1}^\infty \sum_{j=1}^\infty |\xi_{i,j}|^2 \right)^{\frac{1}{2}}.$$

Consider the two-dimensional analogue of the summation method (3) from [4] and [5], which has the form

$$T_n^\lambda(y_\delta)(t,\tau) = \sum_{i=1}^n \sum_{j=1}^n \lambda_{i,j} \cdot y_{\delta,i,j} \cdot \varphi_i(t)\varphi_j(\tau). \tag{7}$$

The quality of the method $T_n^\lambda(y_\delta)$ depends on truncation level n and on the properties of the set $\lambda = \{\lambda_{i,j} = \lambda_{i,j}^n : \quad i,j = 1,2,... \quad n \in N\}$. We will assume that there exists a constant C and some $\theta > 0$, such that

$$|1-\lambda_{i,j}| \le C \cdot \left(\frac{ij}{n^2}\right)^{\theta}. \tag{8}$$

In this section we study regularization properties of the summation methods (7) on the class

$$L_2^{\mu} = \left\{ y(t,\tau) \in L_2(Q_2): \quad \|y\|_{\mu,2}^2 = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} (ij)^{2\mu} \cdot |\langle y, \varphi_i\varphi_j\rangle|^2 < \infty \right\}.$$

It is easy to see that the functions from L_2^{μ} are generalization of a class of functions with dominating mixed partial derivative of degree 2μ.

Lemma 1. *At $\mu > \beta + \frac{1}{2}$ we have the following estimates*

$$\left\| \sum_{i=n+1}^{\infty}\sum_{j=1}^{n} y_{i,j} \cdot \varphi_i(t) \cdot \varphi_j(\tau) \right\|_C \le C \cdot n^{-\mu+\beta+\frac{1}{2}} \cdot \|y\|_{\mu,2},$$

$$\left\| \sum_{i=1}^{n}\sum_{j=n+1}^{\infty} y_{i,j} \cdot \varphi_i(t) \cdot \varphi_j(\tau) \right\|_C \le C \cdot n^{-\mu+\beta+\frac{1}{2}} \cdot \|y\|_{\mu,2}, \tag{9}$$

$$\left\| \sum_{i=n+1}^{\infty}\sum_{j=n+1}^{\infty} y_{i,j} \cdot \varphi_i(t) \cdot \varphi_j(\tau) \right\|_C \le C \cdot n^{-\mu+\beta+\frac{1}{2}} \cdot \|y\|_{\mu,2}. \tag{10}$$

Proof. An application of the Cauchy-Schwarz inequality provides

$$\left\| \sum_{i=n+1}^{\infty}\sum_{j=1}^{n} y_{i,j} \cdot \varphi_i(t) \cdot \varphi_j(\tau) \right\|_C = \left\| \sum_{i=n+1}^{\infty}\sum_{j=1}^{n} (i\cdot j)^{\mu} \cdot y_{i,j} \cdot \frac{\varphi_i(t)\cdot\varphi_j(\tau)}{(i\cdot j)^{\mu}} \right\|_C \le$$

$$\le \left\| \left\{ \sum_{i=n+1}^{\infty}\sum_{j=1}^{n} (i\cdot j)^{2\mu} \cdot |y_{i,j}|^2 \right\}^{\frac{1}{2}} \cdot \left\{ \sum_{i=n+1}^{\infty}\sum_{j=1}^{n} \frac{|\varphi_i(t)\cdot\varphi_j(\tau)|^2}{(i\cdot j)^{2\mu}} \right\}^{\frac{1}{2}} \right\|_C \le$$

$$\le C \cdot \left\| \left\{ \sum_{i=n+1}^{\infty}\sum_{j=1}^{n} \frac{(i\cdot j)^{2\beta}}{(i\cdot j)^{2\mu}} \right\}^{\frac{1}{2}} \right\|_C \cdot \|y\|_{\mu,2} \le$$

$$\le C \cdot \left(\sum_{i=n+1}^{\infty} \frac{1}{i^{2\mu-2\beta}} \right)^{\frac{1}{2}} \cdot \left(\sum_{j=1}^{n} \frac{1}{j^{2\mu-2\beta}} \right)^{\frac{1}{2}} \|y\|_{\mu,2} \le C \cdot n^{-\mu+\beta+\frac{1}{2}} \cdot \|y\|_{\mu,2}\,.$$

The relations (9) and (10) can be proved in the same way. Respectively we have

$$\left\|\sum_{i=1}^{n}\sum_{j=n+1}^{\infty} y_{i,j}\cdot\varphi_i(t)\cdot\varphi_j(\tau)\right\|_C \le$$
$$\le C\cdot\left(\sum_{i=1}^{n}\frac{1}{i^{2\mu-2\beta}}\right)^{\frac{1}{2}}\cdot\left(\sum_{j=n+1}^{\infty}\frac{1}{j^{2\mu-2\beta}}\right)^{\frac{1}{2}}\|y\|_{\mu,2}\le C\cdot n^{-\mu+\beta+\frac{1}{2}}\cdot\|y\|_{\mu,2}$$

and

$$\left\|\sum_{i=n+1}^{\infty}\sum_{j=n+1}^{\infty} y_{i,j}\cdot\varphi_i(t)\cdot\varphi_j(\tau)\right\|_C \le$$
$$\le C\cdot\left(\sum_{i=1}^{n}\frac{1}{i^{2\mu-2\beta}}\right)^{\frac{1}{2}}\cdot\left(\sum_{j=n+1}^{\infty}\frac{1}{j^{2\mu-2\beta}}\right)^{\frac{1}{2}}\|y\|_{\mu,2}\le$$
$$\le C\cdot n^{-2\mu+2\beta+1}\cdot\|y\|_{\mu,2}.$$

The main result of this section is given in the following theorem. □

Theorem 1. *Let for an orthonormal system* $\{\varphi_k(t)\}_{k=1^\infty}$ *the condition (5) is fulfilled. Assume that we have a sequence of noisy values (6) and a priori it is known that* $y\in L_2^{\mu}(Q_2)$ *at* $\mu>\beta+1/2$ *. Then for the summation method* $T_n^{\lambda}(y_\delta)$ *of degree* $\theta>\mu$ *at* $n\asymp\delta^{-\frac{2}{2\mu+2\beta+1}}$ *we have the following estimate*

$$\sup_{\|y\|_{\mu,2}\le 1}\ \sup_{\|\xi\|_{l_2}\le 1}\left\|y(t,\tau)-T_n^{\lambda}(y_\delta)(t,\tau)\right\|_C\le C\cdot\delta^{\frac{\mu-\beta-\frac{1}{2}}{\mu+\beta+\frac{1}{2}}}.$$

Proof. Taking into consideration (6) for $T_n^{\lambda}(y_\delta)$ we have

$$\left\|y(t,\tau)-T_n^{\lambda}(y_\delta)(t,\tau)\right\|_C=$$
$$=\left\|\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}y_{i,j}\cdot\varphi_i(t)\cdot\varphi_j(\tau)-\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_{i,j}\cdot y_{\delta,i,j}\cdot\varphi_i(t)\cdot\varphi_j(\tau)\right\|_C\le$$
$$\le\left\|\sum_{i=n+1}^{\infty}\sum_{j=1}^{n}y_{i,j}\cdot\varphi_i(t)\cdot\varphi_j(\tau)\right\|_C+\left\|\sum_{i=1}^{n}\sum_{j=n+1}^{\infty}y_{i,j}\cdot\varphi_i(t)\cdot\varphi_j(\tau)\right\|_C+$$
$$+\left\|\sum_{i=n+1}^{\infty}\sum_{j=n+1}^{\infty}y_{i,j}\cdot\varphi_i(t)\cdot\varphi_j(\tau)\right\|_C+ \tag{11}$$
$$+\left\|\sum_{i=1}^{n}\sum_{j=1}^{n}(1-\lambda_{i,j})\cdot y_{i,j}\cdot\varphi_i(t)\cdot\varphi_j(\tau)\right\|_C+$$
$$+\delta\cdot\left\|\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_{i,j}\cdot\xi_{i,j}\cdot\varphi_i(t)\cdot\varphi_j(\tau)\right\|_C.$$

First we estimate the fourth summand of the relation (11)

$$\Big\| \sum_{i=1}^{n} \sum_{j=1}^{n} (1-\lambda_{i,j}) \cdot y_{i,j} \cdot \varphi_i(t) \cdot \varphi_j(\tau) \Big\|_C \leq$$

$$\leq \Big\| \Big\{ \sum_{i=1}^{n} \sum_{j=1}^{n} (1-\lambda_{i,j})^2 \cdot |y_{i,j}|^2 \Big\}^{\frac{1}{2}} \Big\{ \sum_{i=1}^{n} \sum_{j=1}^{n} |\varphi_i(t) \cdot \varphi(\tau)|^2 \Big\}^{\frac{1}{2}} \Big\|_C \leq$$

$$\Big\| \Big\{ \sum_{i=1}^{n} \sum_{j=1}^{n} (ij)^{2\mu} \cdot |y_{i,j}|^2 \cdot (1-\lambda_{i,j})^2 \cdot \frac{1}{(ij)^{2\mu}} \Big\}^{\frac{1}{2}} \Big\|_C \cdot \Big\{ \sum_{i=1}^{n} \sum_{j=1}^{n} (ij)^{2\beta} \Big\}^{\frac{1}{2}} \leq$$

$$\leq \Big\| \Big\{ \sum_{i=1}^{n} \sum_{j=1}^{n} (ij)^{2\mu} \cdot |y_{i,j}|^2 \cdot \max_{1\leq i,j \leq n} \Big[\Big(\frac{ij}{n^2} \Big)^{2\theta} \cdot \frac{1}{(ij)^{2\mu}} \Big] \Big\}^{\frac{1}{2}} \Big\|_C \times \qquad (12)$$

$$\times \Big(\sum_{i=1}^{n} i^{2\beta} \Big) \cdot \Big(\sum_{j=1}^{n} j^{2\beta} \Big) \leq$$

$$\leq C \cdot n^{-2\mu} \cdot \|y\|_{\mu,2} \cdot n^{2\beta+1} = C \cdot n^{-2\mu+2\beta+1} \cdot \|y\|_{\mu,2} \,.$$

For the last summand of the relation (11) we have

$$\delta \cdot \Big\| \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{i,j} \cdot \xi_{i,j} \cdot \varphi_i(t) \cdot \varphi_j(\tau) \Big\|_C \leq$$

$$\leq \delta \cdot \Big\| \Big\{ \sum_{i=1}^{n} \sum_{j=1}^{n} |\lambda_{i,j}|^2 \cdot |\xi_{i,j}|^2 \Big\}^{\frac{1}{2}} \cdot \Big\{ \sum_{i=1}^{n} \sum_{j=1}^{n} |\varphi_i(t) \cdot \varphi(\tau)|^2 \Big\}^{\frac{1}{2}} \Big\|_C \leq \qquad (13)$$

$$\leq C \cdot \delta \cdot \Big\| \Big\{ \sum_{i=1}^{n} \sum_{j=1}^{n} |\xi_{i,j}|^2 \Big\}^{\frac{1}{2}} \Big\|_C \cdot n^{2\beta+1} \leq C \cdot \delta \cdot n^{2\beta+1} \,.$$

Summarizing the estimates (12), (13) and the results of the Lemma 1 for $\|y\|_{\mu,2} \leq 1$ we have

$$\big\| y(t,\tau) - T_n^{\lambda}(y_\delta)(t,\tau) \big\|_C \leq C \cdot n^{2\beta+1} (n^{-\mu-\beta-\frac{1}{2}} + \delta) \,.$$

If we choose n such that $n \asymp \delta^{-\frac{2}{2\mu+2\beta+1}}$, it follows

$$\big\| y(t,\tau) - T_n^{\lambda}(y_\delta)(t,\tau) \big\|_C \leq C \cdot \delta^{\frac{\mu-\beta-\frac{1}{2}}{\mu+\beta+\frac{1}{2}}} \,.$$

The theorem is proved. □

3. Class of functions of Sobolev type of smoothness

In this section we study the approximating properties of the summation method (7) on the class of functions of Sobolev type, which has the following form

$$W_2^{\mu}(Q_2) := \Big\{ y \in L_2(Q_2) : \quad \|y\|^2_{W_2^{\mu}} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \big(i^{2\mu} + j^{2\mu} \big) \cdot |y_{i,j}|^2 < \infty \Big\}.$$

To prove the main result of this section we need the following lemma.

Lemma 2. *For* $y \in W_2^\mu$ *at* $\mu > 2\beta + 1$ *we have the following estimates*

$$\left\| \sum_{i=n+1}^{\infty} \sum_{j=1}^{n} y_{i,j} \cdot \varphi_i(t) \cdot \varphi_j(\tau) \right\|_C \leq C \cdot n^{-\frac{\mu}{2}+\beta+\frac{1}{2}} \cdot \|y\|_{W_2^\mu},$$

$$\left\| \sum_{i=1}^{n} \sum_{j=n+1}^{\infty} y_{i,j} \cdot \varphi_i(t) \cdot \varphi_j(\tau) \right\|_C \leq C \cdot n^{-\frac{\mu}{2}+\beta+\frac{1}{2}} \cdot \|y\|_{W_2^\mu}, \tag{14}$$

$$\left\| \sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} y_{i,j} \cdot \varphi_i(t) \cdot \varphi_j(\tau) \right\|_C \leq C \cdot n^{-\mu+2\beta+\frac{1}{2}} \cdot \|y\|_{W_2^\mu}. \tag{15}$$

Proof. Applying Cauchy-Schwarz inequality for $y \in W_2^\mu(Q_2)$ at $\mu > 2\beta + 1$ we have the following

$$\left\| \sum_{i=n+1}^{\infty} \sum_{j=1}^{n} y_{i,j} \cdot \varphi_i(t) \cdot \varphi_j(\tau) \right\|_C =$$
$$= \left\| \sum_{i=n+1}^{\infty} \sum_{j=1}^{n} \left(i^{2\mu} + j^{2\mu}\right)^{\frac{1}{2}} \cdot y_{i,j} \cdot \frac{\varphi_i(t) \cdot \varphi_j(\tau)}{(i^{2\mu} + j^{2\mu})^{\frac{1}{2}}} \right\|_C \leq$$
$$\leq \left\| \left\{ \sum_{i=n+1}^{\infty} \sum_{j=1}^{n} \left(i^{2\mu} + j^{2\mu}\right) \cdot |y_{i,j}|^2 \right\}^{\frac{1}{2}} \left\{ \sum_{i=n+1}^{\infty} \sum_{j=1}^{n} \frac{|\varphi_i(t) \cdot \varphi_j(\tau)|}{\left(i^{2\mu} + j^{2\mu}\right)} \right\}^{\frac{1}{2}} \right\|_C \leq$$
$$\leq C \cdot \|y\|_{W_2^\mu} \cdot \left\| \left\{ \sum_{i=n+1}^{\infty} \sum_{j=1}^{n} \frac{i^{2\beta} j^{2\beta}}{2 i^\mu j^\mu} \right\}^{\frac{1}{2}} \right\|_C \leq$$
$$\leq C \cdot \|y\|_{W_2^\mu} \cdot \left\| \left(\sum_{i=n+1}^{\infty} \frac{1}{i^{\mu-2\beta}} \right)^{\frac{1}{2}} \cdot \left(\sum_{j=1}^{n} \frac{1}{j^{\mu-2\beta}} \right)^{\frac{1}{2}} \right\|_C \leq C \cdot n^{-\frac{\mu}{2}+\beta+\frac{1}{2}} \cdot \|y\|_{W_2^\mu}.$$

Similarly, for proofs of (14) and (15) respectively we have

$$\left\| \sum_{i=1}^{n} \sum_{j=n+1}^{\infty} y_{i,j} \cdot \varphi_i(t) \cdot \varphi_j(\tau) \right\|_C \leq$$
$$\leq C \cdot \|y\|_{W_2^\mu} \cdot \left\| \left(\sum_{i=1}^{n} \frac{1}{i^{\mu-2\beta}} \right)^{\frac{1}{2}} \cdot \left(\sum_{j=n+1}^{\infty} \frac{1}{j^{\mu-2\beta}} \right)^{\frac{1}{2}} \right\|_C \leq$$
$$\leq C \cdot n^{-\frac{\mu}{2}+\beta+\frac{1}{2}} \cdot \|y\|_{W_2^\mu}$$

and

$$\Big\| \sum_{i=n+1}^{\infty} \sum_{j=n+1}^{\infty} y_{i,j} \cdot \varphi_i(t) \cdot \varphi_j(\tau) \Big\|_C \leq$$

$$\leq C \cdot \|y\|_{W_2^\mu} \cdot \Big\| \Big(\sum_{i=n+1}^{\infty} \frac{1}{i^{\mu-2\beta}} \Big)^{\frac{1}{2}} \cdot \Big(\sum_{j=n+1}^{\infty} \frac{1}{j^{\mu-2\beta}} \Big)^{\frac{1}{2}} \Big\|_C \leq$$

$$\leq C \cdot n^{-\mu+2\beta+1} \cdot \|y\|_{W_2^\mu}.$$

The lemma is proved. □

Now we state the main result of the section.

Theorem 2. *Let for an orthonormal system* $\{\varphi_k(t)\}_{k=1^\infty}$ *the condition (5) is fulfilled. Assume that we have a sequence of noisy values (6) and a priori it is known that* $y \in W_2^\mu(Q_2)$ *for* $\mu > 2\beta + 1$. *Then for the summation method* $T_n^\lambda(y_\delta)$ *of degree* $\theta > \frac{\mu}{2}$ *at* $n \asymp \delta^{-\frac{2}{\mu+2\beta+1}}$ *we have the estimate*

$$\sup_{\|y\|_{\mu,2}\leq 1} \sup_{\|\xi\|_{l_2}\leq 1} \Big\| y(t,\tau) - T_n^\lambda(y_\delta)(t,\tau) \Big\|_C \leq C \cdot \delta^{\frac{\mu-2\beta-1}{\mu+2\beta+1}}.$$

Proof. To prove the theorem we need to estimate the summands of the relations (11) for $y \in W_2^\mu(Q_2)$, $\mu > 2\beta+1$. Using the fact that for $T_n^\lambda(y_\delta)$ the condition (8) is fulfilled and $\theta > \mu/2$ we estimate the fourth summand of the relation (11)

$$\Big\| \sum_{i=1}^{n} \sum_{j=1}^{n} (1-\lambda_{i,j}) \cdot y_{i,j} \cdot \varphi_i(t) \cdot \varphi_j(\tau) \Big\|_C \leq$$

$$\leq \Big\| \Big\{ \sum_{i=1}^{n} \sum_{j=1}^{n} (1-\lambda_{i,j})^2 \cdot |y_{i,j}|^2 \Big\}^{\frac{1}{2}} \Big\{ \sum_{i=1}^{n} \sum_{j=1}^{n} |\varphi_i(t) \cdot \varphi(\tau)|^2 \Big\}^{\frac{1}{2}} \Big\|_C \leq$$

$$\leq \Big\| \Big\{ \sum_{i=1}^{n} \sum_{j=1}^{n} (i^{2\mu} + j^{2\mu}) |y_{i,j}|^2 \frac{|1-\lambda_{i,j}|^2}{(i^{2\mu} + j^{2\mu})} \Big\}^{\frac{1}{2}} \Big\|_C \times$$

$$\times \Big\| \Big\{ \sum_{i=1}^{n} \sum_{j=1}^{n} |\varphi_i(t) \cdot \varphi(\tau)|^2 \Big\}^{\frac{1}{2}} \Big\|_C \leq \tag{16}$$

$$\leq C \cdot \Big\| \Big\{ \sum_{i=1}^{n} \sum_{j=1}^{n} (i^{2\mu} + j^{2\mu}) |y_{i,j}|^2 \cdot \max_{1\leq i,j\leq n} \Big[\frac{1}{(ij)^\mu} \cdot \Big(\frac{ij}{n^2} \Big)^{2\theta} \Big] \Big\}^{\frac{1}{2}} \Big\|_C \times$$

$$\times \Big\| \Big(\sum_{i=1}^{n} i^{2\beta} \Big)^{\frac{1}{2}} \Big(\sum_{j=1}^{n} j^{2\beta} \Big)^{\frac{1}{2}} \Big\|_C \leq$$

$$\leq C \cdot n^{-\mu} \cdot \Big\| \Big\{ \sum_{i=1}^{n} \sum_{j=1}^{n} (i^{2\mu} + j^{2\mu}) |y_{i,j}|^2 \Big\}^{\frac{1}{2}} \Big\|_C \cdot n^{2\beta+1} \leq C \cdot n^{-\mu+2\beta+1} \cdot \|y\|_{W_2^\mu}.$$

Taking into consideration that from (8) it follows that $|\lambda_{i,j}| < \infty$, then for the last summand of the relation (11) we have

$$\delta \cdot \left\| \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{i,j} \cdot \xi_{i,j} \cdot y_{i,j} \cdot \varphi_i(t) \cdot \varphi_j(\tau) \right\|_C \le C \cdot \cdot \delta n^{2\beta+1}. \tag{17}$$

Summarizing the estimates (16), (17) and the estimates from the Lemma 3.1 for $y \in W_2^{\mu}(Q_2)$ we obtain

$$\left\| y(t,\tau) - T_n^{\lambda}(y_\delta)(t,\tau) \right\|_C \le C \cdot n^{2\beta+1} \left(n^{-\frac{\mu}{2} - \beta - \frac{1}{2}} + \delta \right).$$

When choosing $n \asymp \delta^{-\frac{2}{\mu+2\beta+1}}$ it follows that

$$\left\| y(t,\tau) - T_n^{\lambda}(y_\delta)(t,\tau) \right\|_C \le C \cdot \delta^{\frac{\mu-2\beta-1}{\mu+2\beta+1}}.$$

Thus, the theorem is proved. □

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