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# MODIFIED NEWTON METHOD FOR ANTENNA POWER SYNTHESIS PROBLEM WITH FIXED NORM OF THE PATTERN

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Резюме. Постановка задачі синтезу антен за потужністю доповнена суттєвою фізичною умовою рівності норм заданої і одержаної діаграм. За допомогою методу Лагранжа задача зведена до безумовної мінімізації з невідомим параметром (множником Лагранжа), який відповідає за виконання згаданої умови. Рівнянням Ейлера для цієї задачі є нелінійне інтенальне рівняння типу Гаммерштейна з кубічною залежністю підінтегральної функції від модуля шуканої функції і лінійної залежністю від її аргумента (фазового множника). Рівняння розв'язується модифікованим методом Ньютона. Числові результати продемонстровані на прикладі лінійної антени, яка описується інтегральним перетворенням Фур'є фінітної функції. Виявлено і проаналізовано чисельно процес галуження розв'язків задачі.

Abstract. The problem formulation of the antenna synthesis according to the prescribed power radiation pattern is generalized by taking into account the important physical restriction on the norm of the synthesized pattern. By the Lagrange method, the problem is reduced to an unconditional variational problem with unknown parameter (Lagrange multiplier), which provides the condition of the norm equality. The Lagrange–Euler equation for this problem is a nonlinear integral equation of the Hammerstein type with cubic dependence of the integrand on modulus of the unknown function. The argument (phase) factor of this function is involved in the integrand linearly. The modified Newton method is used to solve this nonlinear equation. Numerical results are demonstrated on the example of the linear antenna. The solution branching is observed numerically and analyzed.

## 1. Introduction

The antenna power synthesis problem [1] belongs to the phase optimization problems in which the argument (phase) of a desired complex function is not the given function, but it is an additional parameter to be optimized. In contrast to the more usual synthesis problem according to the prescribed amplitude pattern [2], the main term in the functional of the power problem to be minimized, is the mean square difference between not the modulus (amplitudes) of obtained and desired radiation patterns, but between their squared values (powers). Increasing the algebraic degree of the unknown function leads to the higher nonlinearity of the problem and causes new theoretical and computational complications.

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*Key words.* Antenna power synthesis problem, Nonlinear integral equation, Modified Newton method, Branching of solutions

The possibility to provide the local irregularities in the desired pattern more precisely than in the synthesis problem by the given amplitude radiation pattern is one of the advantages of the power synthesis.

In its earlier formulation the power synthesis had some disadvantage connected with absence of any restriction onto the norm of radiation pattern. In this case the radiation pattern is involved in all terms of nonlinear integral equation as an linear multiplier, what admits, in particular, the existence of zero solution. Moreover, it is turned out that this solution is unique at certain combinations of parameters.

To avoid the above disadvantage, and to take into account some physical requirements on the synthesized radiation pattern, we supplement the functional to be optimized by additional condition describing the norm equality of the desired and obtained radiation patterns. Some modifications of this condition were used in [3]–[5] in other formulations of the antenna synthesis problems. Such conditional minimization problem can be reduced to the unconditional one by the Lagrange multipliers method.

The Lagrange-Euler equation for resulting functional is an nonlinear integral equation of the Hammerstein type. It contains the unknown function in the integrand as a cubic algebraic term.

The equation is numerically solved by the modified Newton method. Since the problem has nonunique solutions, they can be separated only by the appropriate choice of the initial approximations. Different types of solutions were found and analyzed. Their branching points were observed numerically as well. Of course, such an approach does not investigate the branching process completely. This question is a subject of special studies. For this purpose the approach based on the complex polynomial presentation of the solutions [6] can be appied.

Some results of this paper were announced in [7].

## 2. Problem formulation

The current $u$ on the antenna and radiation pattern $f$ generated by it, are connected by the relation

$$f = Au, \tag{1}$$

where $A$ is a linear bounded operator. The antenna synthesis problem according to the prescribed power radiation pattern $F^2$ consists in minimization of the functional [1]

$$\sigma_\alpha(u) = ||F^2 - |f|^2||_2^2 + \alpha||u||_1^2, \tag{2}$$

where $\|\cdot\|_1$, $\|\cdot\|_2$ are the mean square norms in the spaces of the currents and radiation patterns, respectively, $\alpha > 0$ is a given positive coefficient (weight factor); further we assume $\|F\|_2^2 = 1$. We supplement this functional by the condition

$$||f||_2^2 = 1. \tag{3}$$

Using the Lagrange multipliers method, we reduce the problem (2)-(3) to minimization of the functional

$$\sigma_{\alpha,\mu}(u) = ||F^2 - |f|^2||_2^2 + \alpha||u||_1^2 - \mu||f||_2^2 \tag{4}$$

with undefined coefficient (Lagrange multiplier) $\mu$. Fixing $\mu$, we denote by $u_\mu$ and $f_\mu$ the current $u$ minimizing $\sigma_{\alpha,\mu}(u)$ and radiation pattern $f$ generated by it, respectively. Then the condition (3) may be considered as the transcendental equation for determining $\mu$. Another way to solve the problem is to find $u$, $f$ and $\mu$ simultaneously.

The Lagrange-Euler equation for the functional (4) can be written in the form

$$\alpha f - 2AA^*[(F^2 - |f|^2)f] - \mu AA^* f = 0. \tag{5}$$

Here $A^*$ is the operator adjoint to $A$. After $f$ and $\mu$ is found from (5), (3), the desired field distribution $u$ is calculated as

$$u = \frac{1}{\alpha}\left(2A^*[(F^2 - |f|^2)f] + \mu A^* f\right). \tag{6}$$

Equation (3) may be supplemented to (5), and they together may be considered as the equation system for determining $f$ and $\mu$. The modified Newton method described in [9] in the context of similar systems of equation, can be applied to system (5), (3). In order to use it, we convert equations (5), (3) to the convenient form

$$\Phi(f,\mu) \equiv \alpha f - 2AA^*[(F^2 - |f|^2)f] - \mu AA^* f = 0, \tag{7}$$

$$\Psi(f) \equiv ||f||^2 - 1 = 0. \tag{8}$$

The next approximation to the unknown $f$ and $\mu$ is calculated in the used method as

$$f_{p+1} = f_p + \delta f'_p + i\delta f''_p, \tag{9}$$

$$\mu_{p+1} = \mu_p + \delta\mu_p, \tag{10}$$

where $\delta f'_p$, $\delta f''_p$, $\delta\mu_p$ are found from the linear equation system

$$\begin{bmatrix} \begin{matrix}[\alpha - \mu_p AA^* - 2AA^*(F^2 \\ -|f_p|^2) + 4AA^*(f'_p)]\end{matrix} & 4AA^*(f'_0 f''_0) & -AA^*(f'_0) \\ 4AA^*(f'_p f''_p) & \begin{matrix}[\alpha - \mu_p AA^* - 2AA^*(F^2 - \\ |f_p|^2) + 4AA^*(f''_p)]\end{matrix} & -AA^*(f''_0) \\ 2f'_p & -2f''_p & 0 \end{bmatrix} \times$$

$$\times \begin{bmatrix} \delta f'_p \\ \delta f''_p \\ \delta\mu_p \end{bmatrix} = \begin{bmatrix} -\Phi'_p \\ -\Phi''_p \\ -\Psi_p \end{bmatrix}. \tag{11}$$

In the case when the parameter $\mu$ is fixed, equation (8) does not participate in the system, then the last row in system (11), as well as the last column in its matrix are omitted.

## 3. Numerical results

The proposed approach has been tested on the example of the synthesis problem for the linear antenna of limited length, which is described by the Fourier transform operator mapping on the compactly bounded functions. The desired power pattern $F^2$ is assumed to be given also as a compactly bounded function. In this case the operators $A$, $A^*$, and the kernel $K(\xi_1, \xi_2)$ of the operator $AA^*$ have the forms

$$f(\xi) = Au \equiv \int_{-1}^{1} u(x) e^{icx\xi} dx, \tag{12}$$

$$A^* g = \frac{c}{2\pi} \int_{-1}^{1} g(\xi) e^{-icx\xi} d\xi, \tag{13}$$

$$K(\xi, \xi') = \frac{\sin c(\xi - \xi\prime)}{\pi(\xi - \xi\prime)}, \tag{14}$$

where $x$ is the normalized coordinate on the antenna, $\xi = \sin\vartheta / \sin\vartheta_0$ is the generalized angular coordinate in the far zone, $2\vartheta_0$ is the angle where the prescribed power pattern $F^2$ differs from zero, $c = ka \sin\vartheta_0$ is the characteristic physical parameter, $2a$ is the antenna length.

The Lagrange-Euler equation for the functional (4) for this example has the form

$$\alpha f(\xi') - \frac{2}{\pi} \int_{-1}^{1} \frac{\sin c(\xi - \xi\prime)}{\pi(\xi - \xi\prime)} [(F^2(\xi) - |f(\xi)|^2) f(\xi)] d\xi - \\ -\mu \int_{-1}^{1} \frac{\sin c(\xi - \xi\prime)}{\pi(\xi - \xi\prime)} f(\xi) d\xi = 0. \tag{15}$$

The numerical results are presented for the prescribed power patterns $F^2(\xi) \equiv 1/2$ and $F^2(\xi) \equiv \cos(\pi x/2)$, $|x| \le 1$; for $|x| > 1$ these functions equal zero.

The main result of the optimization is the mean-square deviation $\sigma_0 = ||F^2 - |f|^2||_2^2$ of the power patterns (the first term in functional (4)), two other terms have the auxiliary sense. Dependencies of $\sigma_0$ on the parameter $c$ are shown for these two prescribed patterns in Fig.1, respectively, for different solutions of equation (15). The results depend essentially on the parameter $\alpha$ (weight factor in functional (4)); its values are given in the figures. The solutions branch at some values of $c$; these values are denoted by $c_{n1}$ (the index $n$ values relate to different $\alpha$).

The real solutions to (15) exist for two given $F^2(\xi)$ and all values of $c$ (dashed lines in the figures). Different behaviour of $\sigma_0$ for different $F^2$ at small $c$ is

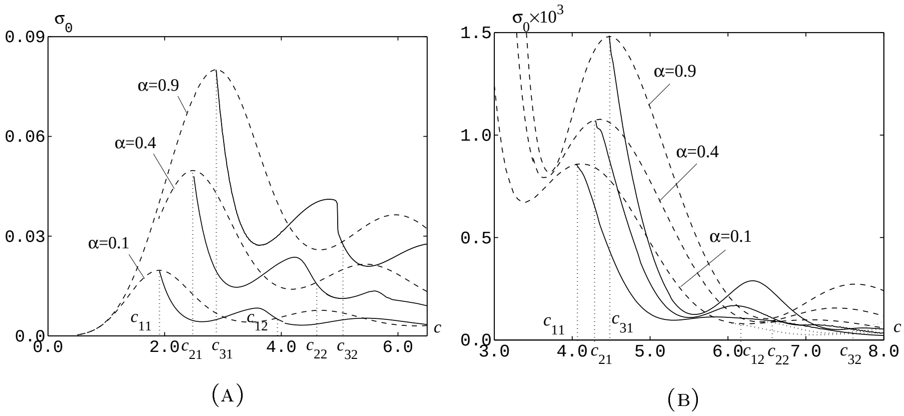


FIG. 1. The mean-square deviation of power patterns for different solutions to (15); (A) $F^2(\xi) \equiv 0.5$; (B) $F^2(\xi) = \cos^2(\pi\xi/2)$

explained by the fact, that the real solution to (15) is asymptotically constant at $c \to 0$. Therefore, the function $F^2(\xi) \equiv \text{const}$ can be easily approximated at small $c$. This property is inherent only to this given $F^2$.

Note, that at $\mu = 0$ and fixed $\alpha$ equation (15) has only zero solution at small $c$. This is explained by the fact that the second term in functional (4) is dominant at small $c$.

At the points $c = c_{n1}$ the complex solutions with odd phase functions $\arg f(-\xi) = -\arg f(\xi)$ (solid lines) branch off from the real solutions. The branching points approximately coincide with the first maximums of $\sigma_0$ as a function of $c$ for the real solution. It is easy to check from (6) that the current distribution on the antenna, which generates the pattern with odd $\arg f(\xi)$, is real, but it has zero points in the interval $x \in [-1, 1]$. This fact is important from the practical point of view, because in this case no phase transformer device is needed for its creation.

The next characteristic points in Fig.1 are the points $c = c_{n2}$ where two new complex solutions simultaneously arise with odd and even phases, respectively. They have the same $|f(\xi|$ and hence the same $\sigma_0(u)$. However, the current $u(x)$ is different for these solutions. One of them, corresponding the the odd $\arg f(\xi)$, is real and has zero points in $x \in [-1, 1]$, whereas the second one, with the even phase ($\arg f(-\xi) = \arg f(\xi)$) is even complex function (in some cases it also can have zeros on the antenna). The solutions with odd phase branch off from one of the same type (that is, with the odd phase), whereas the solution with even phase branch off from the real one; both them arise at the same point $c_{n2}$. Consequently, at least four solutions exist at $c > c_{n2}$: one real (that is, with zero phase), one with even phase, and two with odd phases. In Fig. 1 the results only for one solution with odd phase are presented.

Note that the evenness of the phase distributions $\arg f(\xi)$ and $\arg u(x)$ is caused by the symmetry of the given function $F^2(\xi)$ and both intervals $x \in [-1, 1]$ and $\xi \in [-1, 1]$.

As it is investigated so far, the solution behavior for the considered problem is qualitatively similar to that for the synthesis problem according to the amplitude radiation pattern (the rigorous solutions to this problem see in [8], [9]). However, this analogy can be not complete: the problem considered here has nonlinearity of the higher degree and can have additional solutions different in the behavior from those of the mentioned problem.

At fixed $c$ the current norm almost does not depend on $\alpha$ for the solutions of all types. This is caused by the fact that this norm is hardly affected by the radiation pattern norm, which is fixed in our formulation.

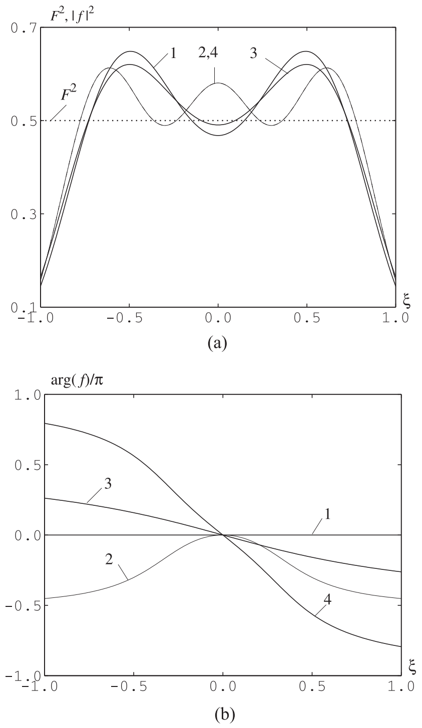


FIG. 2. Power (a) and phase (b) obtained patterns; $F^2(\xi) \equiv 0.5$; $\alpha = 0.4$; $c = 5$

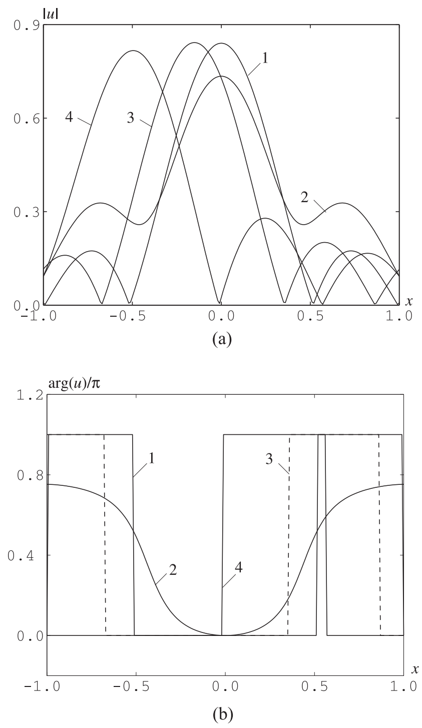


FIG. 3. Amplitude (a) and phase (b) distributions of the currents; $F^2(\xi) \equiv 0.5$; $\alpha = 0.4$; $c = 5$

The optimal directivity patterns $f(\xi)$ and the currents $u(x)$ which create them, corresponding to the solutions of different type for the desired function $F^2(\xi) \equiv \text{const}$, are presented in Figs. 2, 3; the parameters are shown in the captions. The curves are labeled as follows: (1) – real solution; (2) – first solution with odd phase; (3) – second solution with odd phase; (4) – solution with even phase. Analogous results for the case $F^2(\xi) \equiv \cos(\pi x/2)$ are shown in Figs. 4, 5. In this case the amplitude of power pattern in all solutions almost coincides with the desired one.

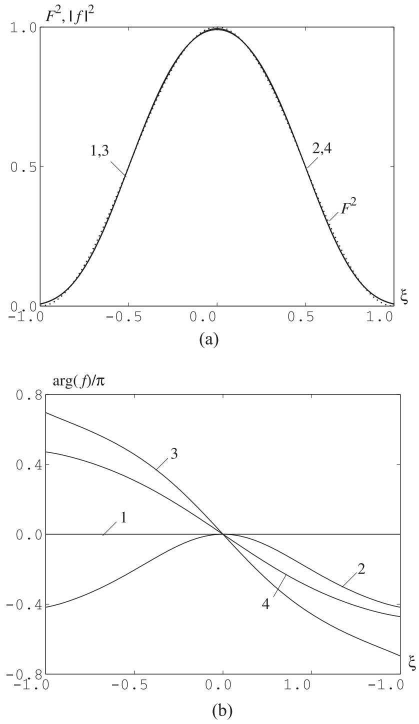


FIG. 4. Power (a) and phase (b) obtained patterns; $F^2(\xi) = \cos^2(\pi\xi/2)$; $\alpha = 0.9$; $c = 7$

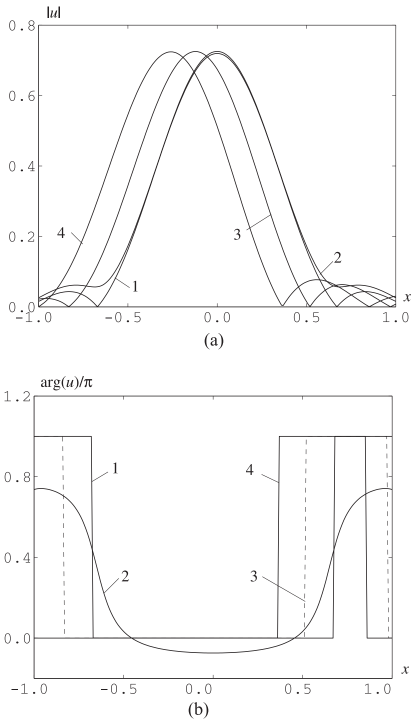


FIG. 5. Amplitude (a) and phase (b) distributions of the currents; $F^2(\xi) = \cos^2(\pi\xi/2)$; $\alpha = 0.9$; $c = 7$

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