UDC 519.6

THE ALTERNATING METHOD APPLIED TO TWO-POINT BOUNDARY VALUE PROBLEMS

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Резюме. Альтернуючий ітераційний метод Козлова-Мазьї, що був запропонований для обернених крайових задач для рівнянь в частинних похідних, застосовано до дво-точкової крайової задачі для звичайного диференціального рівняння другого порядку. Досліджено випадок лінійного диференціального оператора другого порядку. Зокрема, подано критерій збіжності як зв'язок між коефіцієнтами диференціального оператора і кінцевим моментом часу інтервалу. Для нелінійного диференціального оператора виведено деякі формули, за допомогою яких можна довести збіжність. Однак, як показали чисельні експерименти, знаходження критерію збіжності в нелінійному випадку є нетривіальною задачею.

Abstract. The alternating iterative method of Kozlov and Maz'ya, originally proposed for inverse boundary value problems for partial differential operators, is applied to a two-point boundary value problem for a second-order ordinary differential operator. The case of a linear second-order operator is investigated in detail. In particular, a criteria for convergence expressing a relationship between the coefficients of this operator and the final time of the interval is given. For nonlinear operators some formulas are derived on which a proof of convergence can be obtained. However, as is highlighted by a numerical example, finding criteria on the problem to guarantee convergence of the alternating method in the nonlinear case is nontrivial.

1. Introduction

The alternating iterative method was proposed in 1989 by V. A. Kozlov and V. G. Maz'ya [33] to solve some inverse ill-posed problems such as the Cauchy problem for a self-adjoint and strongly elliptic operator and data reconstruction for hyperbolic operators. An advantage with the alternating method is that one solves well-posed problems for the same type of governing partial differential operator in the solution domain as in the ill-posed problem and there is no parameter involved in the procedure. These properties have made the alternating method a popular choice in engineering applications and we give a brief survey on some of these results and applications before introducing the problem to be studied.

For general applications and implementation of the alternating method for Cauchy problems for time-independent operators (typically the Laplace operator), see [35, 23, 8, 16, 42, 40, 29, 24]. Relaxation to speed up the convergence has been introduced and examined in [29, 30, 25, 27]. Generalization of the alternating method to the Stokes system was undertaken in [7] and to

Key words. Heat equation, mixed problem, Rothe's method, boundary integral equation method, trigonometrical quadrature method.

the Helmholtz operator in [26], see also [10]. The alternating method for the Laplace equation was extended to unbounded domains in [13]. Convergence for some nonlinear operators was shown in [41, 4]. The various numerical implementations have mainly been performed using the boundary element method or integral equations, which is natural when only boundary data is updated. Implementation using the finite element method and error estimates suitable for adaptive methods were given in [5]. In that work it was also shown that the alternating method for elliptic problems can be interpreted as the minimization of a certain functional.

The aim of the present study is to show that the alternating method can be applied also to some two-point boundary value problems for a second-order operator. Specifically, we study

$$\begin{cases} u''(t) + f(t,u) = 0, \quad \text{in} \quad I, \\ u(0) = \varphi, \\ u(T) = \psi. \end{cases} \tag{1}$$

Here, $I = [0,T]$, where $T > 0$ is a real number, and $f : I \times X \to X$. We do not strive after the most general situation nor to have a method that can be compared with the many advanced numerical methods already presented in the literature for ordinary differential operators of the form (1). Instead, as pointed out above, we are interested solely in the alternating method and to add some more knowledge around this procedure, in particular, to give some classes of functions f for which the iterative method converges and to give some f for which there is no convergence. Thus, for simplicity, we concentrate on (1) when f is a continuous function, and where the space X is $\mathbb{R}^n$ or a Hilbert space; potentially X can be a Banach space. In fact, the main part of this study is devoted to the linear case when $f = Q(t)u$ with $Q(t) = A + B(t)$ being a smooth positive self-adjoint operator on X, and to show convergence of the alternating method in this case thereby generalizing the similar situation in [33] to time-dependent operators. One can of course have a higher order differential operator as well as different type of boundary conditions but we do not investigate that further here.

There are many applications leading to a model of the form (1), for example, deflection of cantilever beams under certain load [11], plate deflection theory [2], confinement of a plasma column using radiation pressure [47], heat transfer in fins [32], the study of tumour growth [1, 52], cell oxygen uptake [36, 39] and in modelling the distribution of heat sources in the human head [19, 44] to only mention a few.

Partly due to its many applications, there is an overwhelming literature on two-point boundary value problems and it is not within the scope of this study to give a general overview; instead below we point towards some references for (1) and within these the reader can find further references.

Existence and uniqueness of a solution to (1) is nontrivial. In the case $X = \mathbb{R}^n$, existence of a solution was settled in [48, 49]. For existence of a solution when X is a Banach space, see [12, 51, 43]. General references for second-order differential equations in Banach spaces are [18, Chapter 2], [21, Chapter 2 Section 7] and [46, Chapter 5 Section 3].

For general ideas on the numerical solution of (1), see [3, Chapter 11] and [31]. An excellent overview of both theoretical and numerical findings for (1), starting with the seminal paper of E. Picard [45], is given in the introduction in [14].

Let us then describe the method that we shall use to obtain a numerical approximation to (1). Following the original paper on the alternating method [33], the algorithm is:

(i) Make an initial guess η_0 of $u'(0)$. Then the first approximation u_0 is obtained by solving

$$\begin{cases} u_0''(t) + f(t, u_0) = 0, & \text{in } I, \\ u_0(0) = \varphi, \\ u_0'(0) = \eta_0. \end{cases} \tag{2}$$

(ii) Having obtained u_{2k}, the approximation u_{2k+1} for $k \geq 0$, is obtained by solving

$$\begin{cases} u_{2k+1}''(t) + f(t, u_{2k+1}) = 0, & \text{in } I, \\ u_{2k+1}(T) = \psi, \\ u_{2k+1}'(T) = u_{2k}'(T). \end{cases} \tag{3}$$

(iii) Then u_{2k+2} is obtained by solving

$$\begin{cases} u_{2k+2}''(t) + f(t, u_{2k+2}) = 0, & \text{in } I, \\ u_{2k+2}(0) = \psi, \\ u_{2k+2}'(0) = u_{2k+1}'(0). \end{cases} \tag{4}$$

The procedure then continues by iterating in the last two steps. Clearly, the initial value problems solved in each step are well-posed.

As mentioned above, we shall mainly concentrate on the linear case and in Section 2, we investigate the situation when $f(t,u) = Q(t)u$, with $Q(t) = A + B(t)$ being a self-adjoint linear smooth operator generating a (vector) sine and cosine function. Convergence of the alternating procedure is shown under a restriction on the final time T, see Theorem 2.2. We remark that the conditions on $Q(t)$ can be relaxed such that Q can be a differential operator on the space X, thus the results obtained can be applied to time-dependent hyperbolic problems as well. The results in Section 2 builds on Chapter 5 in [6], where the setting was $\mathbb{R}^n$.

To gain more insight and to be able to state a condition that is more easy to check for convergence of the alternating method, in Section 3 a linear and scalar equation is examined when $X = \mathbb{R}$ and $f(t,u) = q(t)u$. It is shown that provided that the smallest eigenvalue for some two-point boundary value problems in the interval I is greater than one then the method converges for $0 < T_1 < T$, see Theorem 3.4. In Section 3.1, we describe a class of functions f for which the alternating method diverges. In Section 4, we briefly investigate the nonlinear case. We derive some formulas for the iterates on which a proof of convergence can be based. However, this needs some monotonicity results for the solution and the function f. As is highlighted by a numerical example in Section 4.1, the alternating method can converge in the nonlinear case without the iterates being monotonically increasing (decreasing) towards the analytical solution. Thus, a full proof of the convergence in the nonlinear case seems intricate and beyond the scope of this study. In Section 4.1, we also suggest and briefly investigate a modification in the sense of linearization in the alternating procedure. This modification appears to converge for classes of functions where the original alternating method diverges. This merit further investigations of this linearization but it is not pursued here but deferred to future work.

2. The alternating procedure for second order linear equations

We start by first introducing some notation. The space $C(I;X)$ is the set of all continuous functions $v : I \to X$ and endowed with the usual supremum norm

$$\|v\|_\infty = \max_{0\leq t\leq T} |v(t)|.$$

Similarly, $C^k(I;X)$ is the space of k-times differentiable functions with the k-th derivative being continuous (supremum norm) and $k \geq 1$ an integer. The spectral radius of an operator Q is defined as usual,

$$r(Q) = \sup\{|\lambda|; \lambda \in \sigma(Q)\}.$$

We are interested in solving (1) in the case when $f(t,u) = Q(t)u$. We assume that

$$Q(t) = A + B(t), \tag{5}$$

where A is a linear operator generating a cosine function, i.e. a function $c(t)$ mapping into the space of bounded operators on X and satisfying $c(t+s) + c(t-s) = c(t)c(s)$, where $t, s \geq 0$, and $c(0) = I$, see further [18, Section 2] for criteria on A to guarantee existence of such a function $c(t)$. Moreover, $B(t)$ maps into the space of bounded linear operators on X and is twice strongly continuously differentiable and the domain of $B(t)$ has to contain the domain of A. Furthermore, $Q(t)$ is assumed to be self-adjoint and positive. This latter condition will in particular guarantee that in the case of $\mathbb{R}^n$, the initial value problems used in the alternating method will not be stiff.

We study the linear second-order differential equation with two-point boundary value conditions:

$$\begin{cases} u'' + Q(t)u = 0, & \text{in } I, \\ u(0) = \varphi, \\ u(T) = \psi, \end{cases} \tag{6}$$

where $u \in C^2([0,T];X)$ and Q as above; for the boundary data φ, $\psi \in X$.

It is known, see [38], that for problem (6) there exists functions $S(t)$ and $C(t)$, commonly denoted the (vector) sine and cosine function respectively, that satisfy

$$S(0) = C'(0) = 0, \quad S'(0) = C(0) = I.$$

Provided that the spectral radius $r(C^*(T)S'(T)) < 1$, then $S(T)$ has an inverse and the solution to (6) can be given as

$$u(t) = S(t)S(T)^{-1}(\psi - C(T)\varphi) + C(t)\varphi. \tag{7}$$

This will be verified in the next section.

For simplicity, we shall assume that X is a Hilbert space mainly to simplify the use of adjoint operators; most of the derivations can be justified also in a Banach space.

1. **Properties of the sine and cosine functions.** The solutions $S(t)$ and $C(t)$ need not be self-adjoint although $Q(t)$ is. By C^* and S^* we mean the adjoint of C and S respectively, i.e. the adjoint of $C(t)$ and $S(t)$ for $t \in I$. For the sake of completeness we include a proof of the following.

Lemma 1. *The solutions $S(t)$ and $C(t)$ to problem (6) satisfy the identities:*

$$S'^*(t)C(t) - S^*(t)C'(t) = I, \tag{8}$$

and

$$S'(t)C^*(t) - C'(t)S^*(t) = I. \tag{9}$$

The elements $S^*(t)$ and $C^*(t)$ are the adjoint operators of $S(t)$ and $C(t)$, and I is the identity.

Proof. Due to the smoothness assumption on Q, we can differentiate the left-hand side of equality (8) to formally obtain

$$\begin{aligned}
&\frac{d}{dt}(S'^*(t)C(t) - S^*(t)C'(t)) = \\
&\quad= S''^*(t)C(t) + S'^*(t)C'(t) - S'^*(t)C'(t) - S^*(t)C''(t) = \\
&\quad= S''^*(t)C(t) - S^*(t)C''(t) = \\
&\quad= -(Q(t)S(t))^*C(t) + S^*(t)Q(t)C(t) = \\
&\quad= -S^*(t)Q(t)C(t) + S^*(t)Q(t)C(t) = 0.
\end{aligned}$$

The equality (8) then follows by formally integrating this using the initial conditions for the $S(t)$ and $C(t)$ and their derivatives.

To prove (9), we first show that S^*S' and C^*C' are self-adjoint We have

$$\frac{d}{dt}(S^*S' - S'^*S) = S^*S'' - S''^*S = S^*QS - S^*QS = 0.$$

Again, formally integrating using that $S(0) = 0$, it follows that $S^*S' = S'^*S$. Similarly, one can show that $C^*C' = C'^*C$.

Define the following operator matrix

$$B = \begin{pmatrix} -C'^*(t) & C^*(t) \\ S'^*(t) & -S^*(t) \end{pmatrix}. \tag{10}$$

This matrix is a left inverse of

$$A = \begin{pmatrix} S(t) & C(t) \\ S'(t) & C'(t) \end{pmatrix},$$

that is $BA = I$, and this is straightforward to check by formal matrix multiplication using (8) together with $S^*S' = S'^*S$ and $C^*C' = C'^*C$. Thus, B is the inverse of A and using that therefore $BA = I$, i.e.

$$\begin{pmatrix} S(t) & C(t) \\ S'(t) & C'(t) \end{pmatrix} \begin{pmatrix} -C'^*(t) & C^*(t) \\ S'^*(t) & -S^*(t) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{11}$$

gives (9). □

We note that from (11) follows immediately that also SC^* and $S'C'^*$ are self-adjoint, which we state as a separate result.

Lemma 2. *The operators SC^* and $S'C'^*$ are self-adjoint.*

We then verify that provided $r(C^*(T)S'(T)) < 1$ then (7) is a well-defined solution to (6).

Lemma 3. *Assume that $r(C^*(T)S'(T)) < 1$. Then the inverse of $S(T)$ exists.*

Proof. This is a standard application of the Neumann series in combination with the relation (8). Indeed,

$$(I - C^*(T)S'(T)) \sum_{j=0}^{k-1} (C^*(T)S'(T))^j \quad = \quad (I - (C^*(T)S'(T))^k). \tag{12}$$

Letting k tend to infinity one can conclude, since $r(C^*(T)S'(T)) < 1$, that $(I - C^*(T)S'(T))$ has an inverse. Applying (8) the result follows. □

2. **Convergence of the alternating procedure for (6).** The alternating procedure for problem (6) was given in the introduction. For clarity, we state the steps again. The element u_{2k} satisfies the initial value problem

$$\begin{cases} u''_{2k} + Q(t)u_{2k} = 0, \text{ in } I, \\ u_{2k}(0) = \varphi, \\ u'_{2k}(0) = u'_{2k-1}(0), \end{cases} \tag{13}$$

where $u'_0(0) = \eta$ is arbitrary. The solution to this problem is given by

$$u_{2k}(t) = S(t)u'_{2k-1}(0) + C(t)\varphi. \tag{14}$$

The element u_{2k+1} is constructed as the solution to

$$\begin{cases} u''_{2k+1} + Q(t)u_{2k+1} = 0, \text{ in } I, \\ u_{2k+1}(T) = \psi, \\ u'_{2k+1}(T) = u'_{2k}(T), \end{cases} \tag{15}$$

with solution

$$u_{2k+1}(t) = (S(t)C^*(T) - C(t)S^*(T))u'_{2k}(T) + (C(t)S'^*(T) - S(t)C'^*(T))\psi. \tag{16}$$

To verify that this indeed is a solution one can use that SC^* and $S'C'^*$ are self-adjoint according to Lemma 2 together with (8)-(9).

We shall then establish convergence of the alternating algorithm (convergence was shown in [33] for time-independent operators).

Theorem 1. *Let u be a solution to problem (6) and let $C(t)$ and $S(t)$ be the fundamental cosine and sine solutions to this problem. Let u_k be the k-th approximate solution generated by the alternating procedure. If $r(C^*(T)S'(T)) < 1$, where r is the spectral radius, then*

$$\|u_{2k} - u\|_\infty \leq C_1 \delta^k$$

and

$$\|u_{2k+1} - u\|_\infty \leq C_2 \delta^k,$$

where C_1 and C_2 are positive constants and $\delta \in (r(C^*(T)S'(T)), 1)$.

Proof. The solution u_{2k+1} to (15) is given by (16) and this gives

$$\begin{aligned} u_{2k+1}(t) &= (S(t)C^*(T) - C(t)S^*(T))u'_{2k}(T) + \\ &\quad +(C(t)S'^*(T) - S(t)C'^*(T))\psi = \\ &= Z_1(t)u'_{2k}(T) + Z_2(t)\psi. \end{aligned} \tag{17}$$

In particular, calculating $u'_{2k-1}(0)$ and using that the solution to (13) is given by (14) tedious but straightforward calculations show that

$$\begin{aligned} u_{2k}(t) &= S(t)\sum_{j=0}^{k-1}(C^*(T)S'(T))^j\, C'^*(T)(C(T)\varphi - \psi) + \\ &\quad +S(t)(C^*(T)S'(T))^k\eta + C(t)\varphi. \end{aligned} \tag{18}$$

Using this expression in (17) one derives

$$\begin{aligned} u_{2k+1}(t) &= Z_1(t)S'^*(T)(C^*(T)S'(T))^k\eta + \\ &\quad +Z_1(t)S'^*(T)\sum_{j=0}^{k-1}(C^*(T)S'(T))^j C'^*(T)(C(T)\varphi - \psi) + \\ &\quad +Z_1(t)C'^*(T)\varphi + Z_2(t)\psi. \end{aligned}$$

Similar to the proof of Lemma 3 it follows from identity (8) that

$$\begin{aligned} \sum_{j=0}^{k-1}(C^*(T)S'(T))^j &= (I - C^*(T)S'(T))^{-1}(I - (C^*(T)S'(T))^k) = \\ &= S^{-1}(T)C'^*(T)^{-1}(I - (C^*(T)S'(T))^k). \end{aligned} \tag{19}$$

Employing this in (18) one obtains

$$\begin{aligned} u_{2k}(t) &= S(t)(I - (C^*(T)S'(T))^k)S^{-1}(T)C'^*(T)^{-1}C'^*(T)(C(T)\varphi - \psi) + \\ &\quad +S(t)(C^*(T)S'(T))^k\eta + C(t)\varphi = \\ &= S(t)(C^*(T)S'(T))^k(\eta - S^{-1}(T)(C(T)\varphi - \psi)) + \\ &\quad +S(t)S^{-1}(T)(C(T)\varphi - \psi) + C(t)\varphi. \end{aligned}$$

Similarly, using (19) in (17)

$$\begin{aligned} u_{2k+1}(t) &= Z_1(t)S_1'^*(T)(C^*(T)S'(T))^k(\eta - S^{-1}(T)(C(T)\varphi - \psi)) + \\ &\quad +S(t)S^{-1}(T)(C(T)\varphi - \psi) + C(t)\varphi. \end{aligned}$$

Next, from Lemma 3 the element $S^{-1}(T)$ exists and thus the solution to problem (6) is given by (7). Using this, we finally have

$$u_{2k}(t) - u(t) = S(t)(C^*(T)S'(T))^k(\eta - u'(0))$$

and similarly

$$u_{2k+1}(t) - u(t) = Z_1(t)S_1'^*(T)(C^*(T)S'(T))^k(\eta - u'(0)).$$

Taking norms and making use of the identity

$$r(Q) = \limsup_{k\to\infty} \|Q^k\|^{1/k}, \tag{20}$$

we get

$$\|u_{2k} - u\|_\infty \le \|S\|_\infty \|(C^*(T)S'(T))^k\| \|\eta - u'(0)\| \le C_1\delta^k$$

and

$$\|u_{2k+1} - u\|_\infty \le \|Z_1\|_\infty \|S_1'^*(T)\| \|(C^*(T)S'(T))^k\| \|\eta - u'(0)\| \le C_2\delta^k,$$

where $\delta \in (r(C^*(T)S'(T)), 1)$. Thus, the result follows. □

Remark 1. One can relax the conditions on the operator $Q = A + B(t)$. In fact, one can impose conditions such that $B(t)$ can be a differential operator and thus the problem studied can model for example the Dirichlet problem for a hyperbolic equation, see [37]. This then generalizes the results in [33] for the Dirichlet problem for hyperbolic operators to include time-dependent coefficients. Note though that the Dirichlet problem for the hyperbolic problem has only a unique solution when T is irrational, see [20]. Note also that generalizing to include equations with a term $V(t)u'$ is considerable more difficult in the Banach space setting, see [18, Chapter 8].

Remark 2. Consider the partial differential operator

$$\Delta u + f(u) = 0 \quad \text{in } \Omega$$

supplied with Dirichlet boundary conditions, where Ω is an annular smooth domain in $\mathbb{R}^n$. Searching for a radial solution, $u(r)$, leads to the equation

$$u''(r) + \frac{n-1}{r}u'(r) + f(u(r)) = 0.$$

Substituting $s = r^{2-n}$ gives

$$u''(s) + \rho(s)f(u(s)) = 0,$$

with boundary conditions $u(s_1) = \varphi$ and $u(s_2) = \psi$, see further [34]. Thus, with f of the above form, the results also apply to problems for the Laplace equation. Nonlinear functions f will be discussed in Section 4, thus the alternating method could potentially be applied to semi-linear problems for the Laplace operator.

3. A scalar equation

The results in the previous section are in an abstract setting and as remarked at the end of the previous section the operator $Q(t)$ could even in fact be a partial differential operator. Since the present study has as one of its aims to study the alternating method for ordinary differential equations, we simplify in this section and replace $Q(t)$ by $q^2(t)$, where $q(t)$ is a scalar real-valued function and $X = \mathbb{R}$, and study a classical scalar second-order two point boundary value problem,

$$\begin{cases} u'' + q^2(t)u = 0, \text{ in } I, \\ u(0) = \varphi, \\ u(T) = \psi, \end{cases} \tag{21}$$

where $q \in C(0,T)$. The condition for convergence of the alternating method stated in Lemma 1 is in the case of (21) reduced to $|c(T)s'(T)| <1$, where c and s are the usual fundamental solutions. To give conditions on the function q and final time T for which this condition is satisfied, we shall need the following two lemmas below. These essentially follow from classical comparison theorems for Sturm-Liouville operators; for completeness we give the proofs. For an overview of history and results on Sturm-Liouville comparison theory, see [15, 17, 50].

Lemma 4. *Let $a, b \in C[0,T]$, and let y be a nontrivial solution of*

$$\begin{cases} y'' + a^2(t)y = 0, \\ y'(0) = 0. \end{cases}$$

Suppose that y has its first positive zero at $t = T$, and let z be a nontrivial solution of the equation

$$\begin{cases} z'' + b^2(t)z = 0, \\ z'(0) = 0, \end{cases}$$

with $b^2(t) > a^2(t)$ on $(0,T)$. Then there exists τ with $0 < \tau < T$, such that $z(\tau) = 0$.

Proof. Without loss of generality we can assume that $y(0) = 1$. Therefore, by the assumption that y has no zeros in $0 < t < T$, we find that y is positive on this interval. Using the governing equation, it follows that y' is decreasing on $(0,T)$. Assume then that z has no zeros in $(0,T)$, for instance, that z is positive on $(0,T)$. Let $w = y'z - yz'$; then $w(0) = 0$ and using $y(T) = 0$ gives $w(T) = y'(T)z(T) \leq 0$ since y' is decreasing and z is positive. However,

$$w' = y''z + y'z' - y'z' - yz'' = yz(b^2 - a^2) > 0,$$

which is a contradiction. □

Similarly, one can show a result about zeros of the derivative.

Lemma 5. *Let $a, b \in C[0,T]$, and let y be a nontrivial solution of*

$$\begin{cases} y'' + a^2(t)y = 0, \\ y(0) = 0. \end{cases}$$

Suppose that y' has its first positive zero at $t = T$, and let z be a solution of

$$\begin{cases} z'' + b^2(t)z = 0, \\ z(0) = 0, \end{cases}$$

with $b^2(t) > a^2(t)$ *on* $(0, T)$. *Then there exists* τ *with* $0 < \tau < T$, *such that* $z'(\tau) = 0$.

Proof. We can assume that $y'(0) = 1$, and since by assumption y' does not have any zero on $(0, T)$ one can conclude that y is positive on $(0, T)$. Assume that z' has no zeros in $(0, T)$, for instance, that z' is positive on $(0, T)$. This gives that z is positive on $(0, T)$ since $z(0) = 0$. Let $w = y'z - yz'$, then $w(0) = 0$ and using that $y'(T) = 0$ together with the positiveness of y and z' imply $w(T) = -y(T)z'(T) \leq 0$. However,

$$w' = y''z + y'z' - y'z' - yz'' = yz(b^2 - a^2) > 0,$$

which is a contradiction. □

To derive properties of the fundamental solutions c and s, we shall use the above lemmas together with the following two eigenvalue problems to compare zeros of the solutions. Let λ_{DN} be the first eigenvalue of the following problem*

$$\begin{cases} u'' + \lambda q^2(t)u = 0, \text{ in } I, \\ u(0) = 0, \\ u'(T) = 0, \end{cases} \tag{22}$$

and let λ_{ND} be the first eigenvalue of the problem†

$$\begin{cases} u'' + \lambda q^2(t)u = 0, \text{ in } I, \\ u'(0) = 0, \\ u(T) = 0. \end{cases} \tag{23}$$

Lemma 6. *Let* λ_{DN} *and* λ_{ND} *be defined as above. If* $1 < \min\{\lambda_{DN}, \lambda_{ND}\}$, *then the alternating procedure converges on every interval* $[0, T_1]$, $0 < T_1 < T$.

Proof. The fundamental solution $c(t)$ satisfies $c(0) = 1$ and $c'(0) = 0$. Clearly, from the governing equation for this function, $c'(t)$ is non-positive on the interval $(0, T)$ implying that $c(t)$ is decreasing on this interval. Suppose that $c(T_1) = 0$ for some $0 < T_1 < T$. Then, from Lemma 4 with $T = T_1$ and $a^2 = q^2$ and $b^2 = \lambda_{ND}q^2$, we conclude that the solution to (23) is zero for $t = \tau$ with $0 < \tau < T_1$. However, then the eigenfunction solution to (23) would be identically zero, which is a contradiction. Therefore, we find that $c(t)$ do no change sign on $[0, T_1]$ and we can conclude that $0 < c(t) < 1$ on $[0, T_1]$. A similar conclusion can be made using Lemma 5 for $s'(t)$, and therefore $0 < c(t)s'(t) < 1$ on $[0, T_1]$. Thus, the condition for convergence in Theorem 1 is satisfied. □

It is then possible to state a convergence result for the alternating method involving a condition on the coefficient q and the final time T.

Theorem 2. *If* $T \leq (2 \max_{0 \leq t \leq T} |q(t)|)^{-1}\pi$, *then the alternating procedure converges as a geometric progression on the interval* $(0, T)$.

*The subscript DN refers to a Dirichlet condition at $t = 0$ and a Neumann condition at $t = T$.

†The subscript ND refers to a Neumann condition at $t = 0$ and a Dirichlet condition at $t = T$.

Proof. Put $a^2(t) = q^2(t)$ and $b^2(t) = \max_{0\le t\le T} q^2(t) = M^2$. Once can check that $1 < \min\{\lambda_{DN}, \lambda_{ND}\}$ in the interval $[0, T_1]$, where $T_1 \le \pi/2M$. Thus, the conclusion follows from Lemma 6. □

3. **Non-convergence for the alternating method.** As mentioned in the introduction, we are interested in a class of equations for which the alternating method do not converge. Guided by the results in the previous section, we can then give such a class of equations.

Consider the following problem:

$$\begin{cases} u'' - q^2(t)u = 0, \text{ in } I, \\ u(0) = \varphi, \\ u(T) = \psi, \end{cases} \tag{24}$$

where $q \in C[0, T]$. Let c and s be the fundamental solutions corresponding to this equation. Examining the proof of Theorem 1 it is clear that the alternating method do not converge if $|s'(T)c(T)| > 1$. We adjust T, if necessary, such that c and s do not have any zeros for $0 < t < T$. We shall then show that $|s'(T)c(T)| > 1$ holds for the fundamental solutions to (24).

Proposition 1. *Let c and t be the fundamental solutions corresponding to the equation* (24)*. Then $|s'(T)c(T)| > 1$.*

Proof. Since T is chosen such that c and s do not have any zeros in $0 < t < T$ and since $c(0) = 1$ we conclude that c is positive on $(0, T)$. Hence, it follows from the equation (24) that c'' is positive, which implies that c' is increasing on $(0, T)$. Thus, $c(T) > c(0) = 1$. In similar way, one can show that $s'(T) > s'(0) = 1$. □

Therefore, since $|s'(T)c(T)| > 1$, we can conclude that the alternating method applied to (24) will not converge.

4. Nonlinear operators

In this section we shall investigate the nonlinear case

$$\begin{cases} u''(t) + f(u(t)) = 0, \quad \text{in} \quad I, \\ u(0) = \varphi, \\ u(T) = \psi. \end{cases} \tag{25}$$

For simplicity, we assume that u takes values in $\mathbb{R}$. We shall further assume that there exists a unique solution to problem (25). The existence and uniqueness of a solution is a nontrivial matter, and there are plenty of results and conditions in the literature. A good place to start is Chapter 1 in [9]. From that chapter it follows that under a Lipschitz condition on f there exists a time-interval where existence and uniqueness of a solution to (25) holds. Note that only assuming that f is continuous and positive will not guarantee uniqueness, for counterexamples, see [22].

We shall write down the solution to each of the first four steps in the alternating method to be able to derive some general expressions for the generated elements η_k and ζ_k.

To generate an initial guess for the alternating method, let

$$\begin{cases} v''(t) = 0, & \text{in } I, \\ v(0) = \varphi, \\ v(T) = \psi, \end{cases} \tag{26}$$

that is

$$v(t) = \frac{T-t}{T}\varphi + \frac{t}{T}\psi.$$

Then define $\eta_0 = v'(0) = \frac{1}{T}(\psi - \varphi)$. With this initial guess, the first approximation u_0 in the alternating procedure is given by

$$\begin{cases} u_0''(t) + f(u_0(t)) = 0, & \text{in } I, \\ u_0(0) = \varphi, \\ u_0'(0) = \eta_0, \end{cases} \tag{27}$$

with formal solution

$$u_0(t) = \varphi + t\eta_0 - \int_0^t (t-\tau) f(u_0(\tau))\,d\tau = \frac{T-t}{T}\varphi + \frac{t}{T}\psi - \int_0^t (t-\tau) f(u_0(\tau))\,d\tau,$$

where in the last equality the expression for the element η_0 was used. The derivative of u_0 at $t = T$ is calculated from this as

$$u_0'(t) = \frac{1}{T}(\psi - \varphi) - \int_0^t f(u_0(\tau))\,d\tau,$$

giving

$$\zeta_1 = u_0'(T) = \frac{1}{T}(\psi - \varphi) - \int_0^T f(u_0(\tau))\,d\tau.$$

The next approximation u_1 is found from

$$\begin{cases} u_1''(t) + f(u_1(t)) = 0, & \text{in } I, \\ u_1(T) = \psi, \\ u_1'(T) = \zeta_1, \end{cases} \tag{28}$$

with solution

$$u_1(t) = \psi + (t-T)\zeta_1 + \int_t^T (t-\tau) f(u_1(\tau))\,d\tau.$$

Inserting the expression for ζ_1,

$$u_1(t) = \psi + \frac{t-T}{T}(\psi - \varphi) - (t-T)\int_0^T f(u_0(\tau))\,d\tau + \int_t^T (t-\tau) f(u_1(\tau))\,d\tau.$$

From this, the derivative of u_1 at zero is

$$\eta_2 = u_1'(0) = \frac{1}{T}(\psi - \varphi) - \int_0^T \left(f(u_0(\tau)) - f(u_1(\tau))\right)d\tau.$$

Then u_2 is constructed as the solution to

$$\begin{cases} u_2''(t) + f(u_2(t)) = 0, & \text{in } I, \\ u_2(0) = \varphi, \\ u_2'(0) = \eta_2, \end{cases} \tag{29}$$

and formally

$$\begin{aligned} u_2(t) &= \varphi + t\eta_2 - \int_0^t (t-\tau) f(u_2(\tau))\, d\tau = \\ &= \varphi + \frac{t}{T}(\psi - \varphi) - t\int_0^T (f(u_0(\tau)) - f(u_1(\tau)))\, d\tau - \\ &- \int_0^t (t-\tau) f(u_2(\tau))\, d\tau. \end{aligned}$$

Calculating the derivative at $t = T$ we obtain

$$\zeta_3 = u_2'(T) = \frac{1}{T}(\psi - \varphi) - \int_0^T (f(u_0(\tau)) - f(u_1(\tau)) + f(u_2(\tau)))\, d\tau. \quad (30)$$

Then u_3 is constructed,

$$\begin{cases} u_3''(t) + f(u_3(t)) = 0, \quad \text{in} \quad I, \\ u_3(T) = \psi, \\ u_3'(T) = \zeta_3, \end{cases} \quad (31)$$

having the solution

$$u_3(t) = \psi + (t-T)\zeta_3 + \int_t^T (t-\tau) f(u_3(\tau))\, d\tau$$

or by using the expression for ζ_3,

$$\begin{aligned} u_3(t) = \psi + \\ + (t-T)\left(\frac{1}{T}(\psi - \varphi) - \int_0^T (f(u_0(\tau)) - f(u_1(\tau)) + f(u_2(\tau)))\, d\tau\right) + \\ + \int_t^T (t-\tau) f(u_3(\tau))\, d\tau. \end{aligned}$$

From this expression, we have the derivative

$$\begin{aligned} \eta_4 = u_3'(0) = \frac{1}{T}(\psi - \varphi) - \\ - \int_0^T (f(u_0(\tau)) - f(u_1(\tau)) + f(u_2(\tau)) - f(u_3(\tau)))\, d\tau \Big). \end{aligned} \quad (32)$$

Note that (30) and (32) justifies the term alternating method, since the sign appear to alternate with each iteration.

We further observe that

$$\zeta_3 - \zeta_1 = \int_0^T (f(u_1(\tau)) - f(u_2(\tau)))\, d\tau$$

and

$$\eta_4 - \eta_2 = \int_0^T (f(u_3(\tau)) - f(u_2(\tau)))\, d\tau.$$

Continuing by iterating in the last two steps, a simple induction step reveals,

Proposition 2. *Let $\{\eta_{2k}\}_{k=0}^{\infty}$ and $\{\zeta_{2k+1}\}_{k=0}^{\infty}$ be generated from the alternating procedure. Then*

$$\eta_{2k+2} - \eta_{2k} = \int_0^T \left(f(u_{2k+1}(\tau)) - f(u_{2k}(\tau))\right) d\tau$$

and

$$\zeta_{2k+3} - \zeta_{2k+1} = \int_0^T \left(f(u_{2k+1}(\tau)) - f(u_{2k+2}(\tau))\right) d\tau.$$

Now, note that if f was a positive increasing function and if the approximations u_k generated by the alternating method satsfied $u_{k+1} \geq u_k$, then one can conclude that $\{\eta_{2k}\}$ will be an increasing sequence and $\{\zeta_{2k}\}$ a decreasing sequence. Thus, provided these could be bounded from above and below, one could establish a convergence proof. Another possibility is that the odd approximations $\{u_{2k+1}\}$ are all above each of the even approximations $\{u_{2k}\}$.

However, it appears rather difficult to find conditions on the function f and the final time T to have such conditions satisfied. In fact, in the next section, we shall take a rather simple function f and show numerically that the sequences $\{\eta_{2k}\}$ and $\{\zeta_{2k}\}$ do not need to be monotone, and still there appears to be convergence.

4. **A numerical example for a nonlinear problem.** Let

$$\begin{cases} u''(t) + \frac{1}{2}\sin(2u(t)) = 0, & \text{in } I, \\ u(0) = 0, \\ u(T) = \psi. \end{cases} \tag{33}$$

Here, $f(u) = \frac{1}{2}\sin(2u(t))$ is Lipschitz with constant $L = 1$. Hence, from [9, p. 5] there is a unique solution to (33) for $T < 2\sqrt{2}$. In fact, we assume that *psi* is chosen such that we have the following explicit expression for the solution,

$$u(t) = \arcsin\frac{e^{2t} - 1}{e^{2t} + 1}. \tag{34}$$

The initial guess is constructed as in the previous section. The initial value problems needed to be solved in each iteration step of the alternating procedure are solved with the Matlab function ODE45 (Matlab version R2013b on a computer with Windows 8.2 and an Intel(R) Core(TM) i3-3217U Central Unit Processor (CPU) at 1.8GHz).

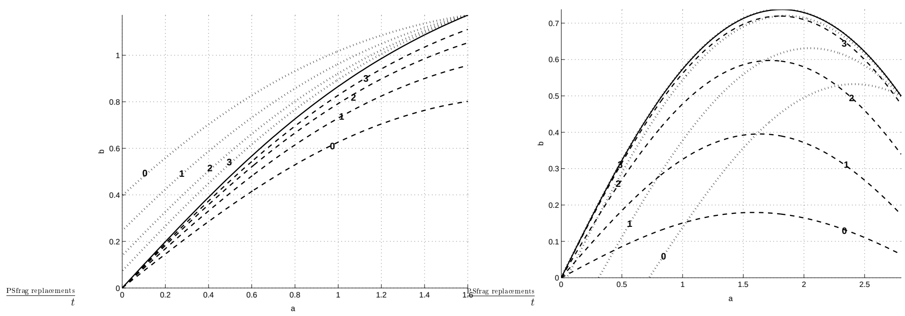
In Fig. 1(a) we present the results obtained after 8 iterations (that is u_7 is the final approximation; the corresponding value for k for the solution u_{2k} and u_{2k+1}, respectively, is marked out on each approximation) obtained with $T = 1.6$ and ψ generated from (34). As can be seen from this figure there is convergence towards the solution to (33). Moreover, a monotone behaviour of the approximations, expected due to Proposition 2, is present. In fact, with these solutions together with the function f and Proposition 2, the sequences η_k and ζ_k should both be positive and decreasing. This has been checked for and is the case in the numerical simulations.

Increasing T there is convergence of the similar kind up to about $T = 1.8$, where the method starts to become slower and eventually does not converge.

Choosing instead $T = 2.8$ and taking $\psi = 0.5$, one can see that monotonicity is no longer present in the sense that some even iterates u_{2k} intersects some odd iterates u_{2m+1}; this is shown in Fig. 1(b). In this case, we used the Matlab function bvpc4 to generate an approximation to the solution to (33) to test convergence against with $\psi = 0.5$ formula (34) does not give the sought solution.

One can also change sign of the function f and run the procedure with $-f$. This causes problems with the alternating method and only for small values of T there seems to be convergence. For example, the method diverges for $T = 1$ and $\psi = 1$ as is highlighted in Fig. 2(a). Note that changing sign was shown in the linear case in Section 3.1 to generate non-convergent sequences in the alternating procedure.

We remark that we have also tried a linearization in the alternating method in the sense that f is instead evaluated on the solution from the previous step. This new linearized procedure does not give any significant improvement for (33). However, changing to $-f$ this linearized procedure appears to converge for $T = 1$ and $\psi = 1$ as shown in Fig. 2(b).



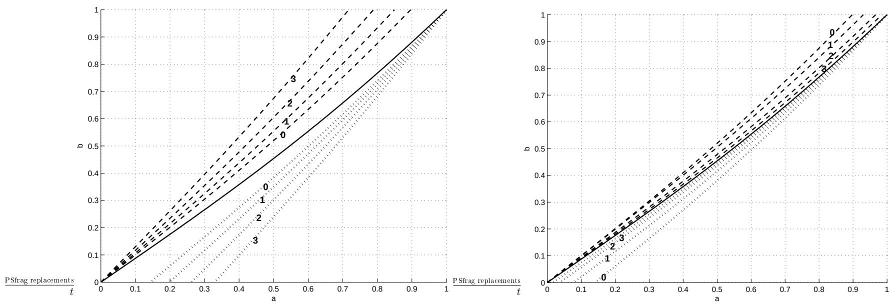
a) $T = 1.6$ and $\psi = u(1.6)$, u from (4.34)

b) $T = 2.8$ and $\psi = 0.5$

FIG. 1. The solutions u_{2k} (- - -) and u_{2k+1} ($\cdots$), and the analytical solution u (—) for various T and ψ.

5. Conclusion

The alternating method [33] was investigated for two-point boundary value problems for second order time-dependent differential operators. Convergence was established in the linear case extending [33] to the time-dependent case with the operators taking values in a Hilbert space (potentially the similar analysis can be carried over to the Banach space setting). In the scalar case, a criteria involving the coefficients of the operator and the final time were given to guarantee convergence. It was also shown that changing sign of a term in the differential operator generates equations for which the alternating method does not converge. Moreover, for nonlinear operators, expressions were derived



a) $T = 1$ and $\psi = 1$ with $-f(u)$ b) $T = 1$, $\psi = 1$ with $-f(u)$ and linearization

FIG. 2. The solutions u_{2k} (- - -) and u_{2k+1} ($\cdots$), and the analytical solution u (—) for various T and ψ.

on which a proof of convergence can potentially be obtained. However, as was highlighted by numerical examples, to pin-point precise criteria on the operator and final time to have a proof of convergence also in the nonlinear case seem difficult. A linearization was suggested such that linear differential equations were solved at each iteration step and this linearization turned out to converge in some cases where the orignal alternating method did not converge. This merits further investigations and is deferred to future work.

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Received 20.03.2014