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AN ALTERNATING BOUNDARY INTEGRAL BASED METHOD FOR A CAUCHY PROBLEM FOR KLEIN-GORDON EQUATION

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Резюме. Розглядається чисельне розв'язування задачі Коші для рівняння Клейна-Гордона у двозв'язній плоскій області. Зважаючи на некоректність цієї лінійної оберненої задачі, використано альтернуючий метод, який володіє регуляризуючими властивостями. Це приводить до розв'язування двох мішаних крайових задач на кожній ітерації. Ці мішані задачі наближено розв'язуються методом граничних інтегральних рівнянь. Приведено результати чисельних експериментів.

Abstract. We consider the numerical solution of a Cauchy problem for the Klein-Gordon equation in a planar double connected domain. Due to the ill-posedness of this linear inverse problem the alternating method with regularization properties is used. It leads to two mixed well-posed boundary value problems on every iteration. These problems are solved by boundary integral equation method. Numerical examples are presented.

1. Introduction

Let D be a double connected domain in $\mathbb{R}^2$ with inner and outer boundaries Γ_1 and Γ_2, respectively. We suppose that $\Gamma_1, \Gamma_2 \in C^3$ (see Fig. 1). Let ν denote the outward unit normal on boundary.

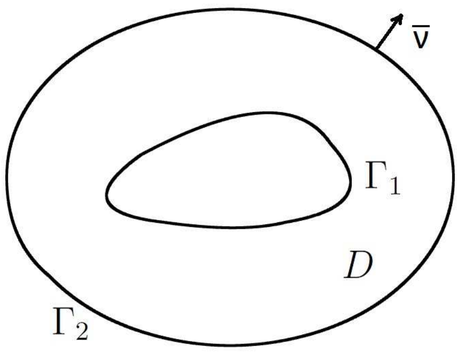


Fig. 1. An example of a double connected domain

Given the sufficiently smooth continuous functions f_1 and f_2, we consider the Cauchy problem of finding a function $u \in C^2(D) \cap C^1(\bar{D})$ which satisfies

Key words. Klein-Gordon equation; Cauchy problem; Double connected domain; Single- and double layer potentials; Integral equations; Alternating method.

the Klein-Gordon equation

$$\Delta u - \varkappa^2 u = 0 \quad \text{in} \quad D \tag{1}$$

and the boundary value conditions

$$u = f \quad \text{and} \quad \frac{\partial u}{\partial \nu} = g \quad \text{on} \quad \Gamma_2. \tag{2}$$

In (1) $\varkappa > 0$ is a given constant. In particular we are interested in finding the Cauchy data on the inner boundary Γ_1.

For the uniqueness of a solution to the Cauchy problem (1), (2) see, for example, [2]. The solution does not in general depend continuously on the data, i.e. the problem is ill-posed in the sense of Hadamard, thus making classical methods inappropriate.

We shall employ the so-called alternating iterative method proposed in [6] and successfully applied in several engineering problems, see for example [5] and [8]. The use of the alternating method with an integral equation approach for the Laplace equation is discussed in [3]. The details of alternating procedure for the case of the Klein-Gordon equation are listed in section 4. In each iteration, mixed direct problems are solved in the solution domain D. There are the Dirichlet-Neumann mixed boundary value problem

$$\Delta w - \varkappa^2 w = 0 \quad \text{in} \quad D, \tag{3}$$

$$w = h \quad \text{on} \quad \Gamma_1, \qquad \frac{\partial w}{\partial \nu} = g \quad \text{on} \quad \Gamma_2 \tag{4}$$

and Neumann-Dirichlet mixed boundary value problem

$$\Delta v - \varkappa^2 v = 0 \quad \text{in} \quad D, \tag{5}$$

$$\frac{\partial v}{\partial \nu} = p \quad \text{on} \quad \Gamma_1, \qquad v = f \quad \text{on} \quad \Gamma_2. \tag{6}$$

For the direct problems in this study, we propose and investigate a numerical method based on the potential theory. Instead, the problems are each reduced to boundary integral equations over Γ_1 and Γ_2. This approach makes the implementation of the alternating method very efficient.

2. Integral equation method for the mixed problems

2.1. Dirichlet-Neumann mixed problem

The problem (3), (4) will be solved by reducing to the system of integral equations of the first kind. We represent the solution $w \in C^2(D) \cap C^1(\bar{D})$ as a combination of a single- and a double-layer potential

$$w(x) = \int\limits_{\Gamma_1} \varphi_1(y)\Phi(x,y)\mathrm{d}s(y) + \int\limits_{\Gamma_2} \varphi_2(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)}\mathrm{d}s(y), \quad x \in D, \tag{7}$$

where φ_1 and φ_2 are unknown continuous densities, $\Phi(x,y) = \frac{1}{2\pi}K_0(\varkappa|x-y|)$ is a fundamental solution of the equation (3) in term of the modified Hankel function K_0 [1].

From the continuity of the single-layer potential and the normal derivative of the double-layer potential we obtain for the problem (3), (4) the following system of integral equations of the first kind

$$\begin{cases} \int\limits_{\Gamma_1} \varphi_1(y)\Phi(x,y)\mathrm{d}s(y) + \int\limits_{\Gamma_2} \varphi_2(y)\dfrac{\partial\Phi(x,y)}{\partial\nu(y)}\mathrm{d}s(y) = h(x), \quad x\in\Gamma_1, \\ \int\limits_{\Gamma_1} \varphi_1(y)\dfrac{\partial\Phi(x,y)}{\partial\nu(x)}\mathrm{d}s(y) + \\ \qquad + \dfrac{\partial}{\partial\nu(x)}\int\limits_{\Gamma_2} \varphi_2(y)\dfrac{\partial\Phi(x,y)}{\partial\nu(y)}\mathrm{d}s(y) = g(x), \quad x\in\Gamma_2. \end{cases} \tag{8}$$

It is known that modified Bessel functions have the following asymptotic properties [1] $K_0(z) \sim \ln\frac{1}{z}, \quad z\to 0$ and $K_1(z)\sim\frac{1}{z}, \quad z\to 0$. Thus, we obtained the system of integral equations of the first kind which contains kernels with logarithmic singularity as well as a hypersingularity.

Using the Maue type expression [7] the second equation from (8) could be rewritten in the following way

$$\int\limits_{\Gamma_1} \varphi_1(y)\Phi(x,y)\mathrm{d}s(y) + \int\limits_{\Gamma_2} \frac{\partial\varphi_2}{\partial\theta}(y)\frac{\partial\Phi(x,y)}{\partial\theta(x)}\mathrm{d}s(y) -$$
$$-\varkappa^2 \int\limits_{\Gamma_2} \varphi_2(y)\Phi(x,y)[\nu(x)\cdot\nu(y)]\mathrm{d}s(y) = g(x), \quad x\in\Gamma_2,$$

where θ denotes the unit tangential vector for Γ_2.

For the future numerical implementation we consider a parametrization of the system (8). We assume that the domain boundaries have the parametric representations

$$\Gamma_i = \{x_i(t) = (x_{i1}(t), x_{i2}(t)), \quad t\in[0,2\pi]\}, \qquad i=1,2,$$

where $x_i : \mathbb{R}\to\mathbb{R}^2$ are C^3 and 2π–periodic with $|x_i'(t)|>0$ for all $t\in[0,2\pi]$. As a result of the parametrization of the system (8) we obtain

$$\begin{cases} \dfrac{1}{2\pi}\int\limits_0^{2\pi} [\mu_1(\tau)H_{11}(t,\tau) + \mu_2(\tau)H_{12}(t,\tau)]\,\mathrm{d}\tau = h(t), \\ \dfrac{1}{2\pi}\int\limits_0^{2\pi} \left[\mu_1(\tau)H_{21}(t,\tau) + \mu_2'(\tau)\hat{H}_{22}(t,\tau) + \mu_2(\tau)H_{22}(t,\tau)\right]\mathrm{d}\tau = g(t), \end{cases} \tag{9}$$

where $\mu_i(t) = \varphi_i(x_i(t))$, $i=1,2$, $h(t) = h(x_1(t))$, $g(t) = 2g(x_2(t))|x_2'(t)|$. The representation of kernels of the obtained system is listed below

$$H_{11}(t,\tau) = K_0(\varkappa|r_{11}(t,\tau)|)|x_1'(\tau)|,$$
$$H_{12}(t,\tau) = \varkappa K_1(\varkappa|r_{12}(t,\tau)|)\frac{r_{12}(t,\tau)\cdot\nu_2(\tau)}{|r_{12}(t,\tau)|}|x_2'(\tau)|,$$
$$H_{21}(t,\tau) = -2\varkappa K_1(\varkappa|r_{21}(t,\tau)|)\frac{r_{21}(t,\tau)\cdot\nu_2(t)}{|r_{21}(t,\tau)|}|x_1'(\tau)||x_2'(t)|,$$

$$\hat{H}_{22}(t,\tau) = -2\varkappa K_1(\varkappa|r_{22}(t,\tau)|)\frac{[r_{22}(t,\tau)\cdot x_2'(t)]}{|r_{22}(t,\tau)|},$$
$$H_{22}(t,\tau) = 2\varkappa^2 K_0(\varkappa|r_{22}(t,\tau)|)[x_2'(t)\cdot x_2'(\tau)].$$

Here we introduced the notation $r_{ij}(t,\tau) = x_i(t) - x_j(\tau)$.

Next we express the system of integral equations (9) in the specific form to be able to apply the trigonometrical quadrature rules. The system of integral equations in the following form is ready for application of the numerical methods

$$\begin{cases} \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}[\mu_1(\tau)(H_{11}^1(t,\tau)\ln\frac{4}{e}\sin^2\frac{t-\tau}{2}+ \\ \qquad + H_{11}^2(t,\tau)) + \mu_2(\tau)H_{12}(t,\tau)]\mathrm{d}\tau = h(t), \\ \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}[\mu_1(\tau)H_{21}(t,\tau) + \mu_2'(\tau)\cot\frac{\tau-t}{2}+ \\ \qquad + \mu_2(\tau)(H_{22}^1(t,\tau)\ln\frac{4}{e}\sin^2\frac{t-\tau}{2} + H_{22}^2(t,\tau))]\mathrm{d}\tau = g(t). \end{cases} \tag{10}$$

Here kernels are represented as follows

$$H_{11}^1(t,\tau) = -\frac{1}{2}I_0(\varkappa|x_1(t)-x_1(\tau)|)|x_1'(\tau)|,$$
$$H_{22}^1(t,\tau) =$$
$$= \varkappa^2\left[\frac{I_0(\varkappa|r_{22}(t,\tau)|)+I_2(\varkappa|r_{22}(t,\tau)|)}{2|r_{22}(t,\tau)|^2}r_{22}(t,\tau)\cdot x_2'(t)r_{22}(t,\tau)\cdot x_2'(\tau)\right.$$
$$- \; I_0(\varkappa|r_{22}(t,\tau)|)r_{22}(t,\tau)\cdot\nu_2(t)|x_2'(t)|x_2'(t)\cdot x_2'(\tau)+$$
$$\left.+ \; \frac{I_1(\varkappa|r_{22}(t,\tau)|)}{\varkappa|r_{22}(t,\tau)|^3}r_{22}(t,\tau)\cdot\nu_2(t)|x_2'(t)|r_{22}(t,\tau)\cdot\nu_2(\tau)|x_2'(\tau)|\right],$$
$$H_{ii}^2(t,\tau) = H_{ii}(t,\tau) - H_{ii}^1(t,\tau)\ln\frac{4}{e}\sin^2\frac{t-\tau}{2}, \quad t\neq\tau, \quad i=1,2$$

with diagonal terms

$$H_{22}^1(t,t) = -\frac{1}{2}\varkappa^2|x_2'(t)|^2, \quad H_{11}^2(t,t) = -\frac{1}{2}\ln\frac{e\varkappa^2|x_1'(t)|^2}{4}|x_1'(t)| - \gamma|x_1'(t)|$$

and

$$H_{22}^2(t,t) = \varkappa^2\ln\frac{e\varkappa^2|x_2'(t)|^2}{4}|x_2'(t)|^2-$$
$$-\frac{1}{6}+\frac{1}{3}\frac{x_2'(t)\cdot x_2'''(t)}{|x_2'(t)|^2}+\frac{1}{2}\frac{|x_2''(t)|^2}{|x_2'(t)|^2}-\frac{(x_2'(t)\cdot x_2''(t))^2}{|x_2'(t)|^4}+\varkappa^2(\frac{1}{2}-\gamma)|x_2'(t)|^2,$$

where I_0 and I_1 are the modified Bessel functions and γ is the Euler constant [1].

For $m\in\mathbb{N}\cup\{0\}$ and $0<\alpha<1$, by $C^{m,\alpha}[0,2\pi]$ we denote the space of m-times uniformly Hölder continuously differentiable and 2π-periodic functions furnished with the usual Hölder norm. Using the Riesz theory [7] we can conclude that for given functions $h \in C^{m+1,\alpha}[0,2\pi]$, $g \in C^{m,\alpha}[0,2\pi]$ the system of integral equations (10) provides a unique solution $\mu_1 \in C^{m,\alpha}[0,2\pi]$ and $\mu_2 \in C^{m+1,\alpha}[0,2\pi]$.

Clearly, we have according to (7) the following representation for the normal derivative on the boundary Γ_1

$$\frac{\partial w}{\partial \nu}(x) = -\frac{1}{2}\varphi_1(x) +$$

$$+\int_{\Gamma_1} \varphi_1(y)\frac{\partial \Phi(x,y)}{\partial \nu(x)}\mathrm{d}s(y) + \int_{\Gamma_2} \varphi_2(y)\frac{\partial^2 \Phi(x,y)}{\partial \nu(x)\partial \nu(y)}\mathrm{d}s(y), \quad x \in \Gamma_1,$$

Taking into account the parametric representation of Γ_i, $i = 1,2$ and by some transformation in the kernels we obtain

$$\begin{aligned} \frac{\partial w}{\partial \nu}(x_1(t)) &= -\frac{1}{2}\mu_1(t) + \\ +\frac{1}{2\pi}\int_0^{2\pi} \Bigg[\mu_1(\tau)\left(L_{11}(t,\tau)\ln\frac{4}{e}\sin^2\frac{t-\tau}{2}\mathrm{d}\tau + L_{12}(t,\tau)\right) &+ \\ +\mu_2(\tau)L_2(t,\tau)\Bigg]\mathrm{d}\tau, &\quad t \in [0;2\pi] \end{aligned} \tag{11}$$

with kernels

$$L_{11}(t,\tau) = \frac{\varkappa}{2}I_1(\varkappa|r_{11}(t,\tau)|)\frac{r_{11}(t,\tau)\cdot\nu_1(t)}{|r_{11}(t,\tau)|}|x_1'(\tau)|,$$

$$L_{12}(t,\tau) = L_1(t,\tau) - L_{11}(t,\tau)\ln\frac{4}{e}\sin^2\frac{t-\tau}{2}, \quad t \neq \tau,$$

$$L_{12}(t,t) = \frac{x_1''(t)\cdot\nu_1(t)}{2|x_1'(t)|}.$$

2.2. Neumann-Dirichlet mixed problem

For solving the mixed boundary value problem (5), (6) we use the similar boundary integral equations approach as described in the previous section.

The solution to the problem (5), (6) inside the domain could be represented as the following sum of potentials

$$v(x) = \int_{\Gamma_1}\varphi_1(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)}\mathrm{d}s(y) + \int_{\Gamma_2}\varphi_2(y)\Phi(x,y)\mathrm{d}s(y), \quad x \in D.$$

As in the previous section, using the boundary conditions, we obtain the system of integral equations which after parametrization and all needed transformations is represented like

$$\begin{cases} \frac{1}{2\pi}\int\limits_0^{2\pi}[\mu_1'(\tau)\cot\frac{\tau-t}{2}+\mu_1(\tau)(\tilde{H}_{11}^1(t,\tau)\ln\frac{4}{e}\sin^2\frac{t-\tau}{2}+ \\ \qquad +\tilde{H}_{11}^2(t,\tau))+\mu_2(\tau)\tilde{H}_{12}(t,\tau)]\mathrm{d}\tau=p(t), \\ \frac{1}{2\pi}\int\limits_0^{2\pi}[\mu_1(\tau)\tilde{H}_{21}(t,\tau)+ \\ \qquad +\mu_2(\tau)(\tilde{H}_{22}^1(t,\tau)\ln\frac{4}{e}\sin^2\frac{t-\tau}{2}+\tilde{H}_{22}^2(t,\tau))]\mathrm{d}\tau=f(t). \end{cases} \tag{12}$$

Here the kernels are smooth functions and their differential properties are dependent from smoothness of the boundaries Γ_i. Using approach described earlier in this section, one can check the existence and uniqueness of the solution to the system (12).

Again we have the following way to calculate the function values on the inner boundary Γ_1

$$v(x)=\frac{1}{2}\varphi_1(x)+\int\limits_{\Gamma_1}\varphi_1(y)\frac{\partial\Phi(x,y)}{\partial\nu(y)}\mathrm{d}s(y)+\int\limits_{\Gamma_2}\varphi_2(y)\Phi(x,y)\mathrm{d}s(y),\quad x\in\Gamma_1.$$

The corresponding formula for the function values in terms of parametric representation of the boundary curve Γ_1 can be obtained

$$v(x_1(t))=\frac{1}{2}\mu_1(t)+$$
$$+\frac{1}{2\pi}\int\limits_0^{2\pi}\left[\mu_1(\tau)\left(\tilde{L}_{11}(t,\tau)\ln\frac{4}{e}\sin^2\frac{t-\tau}{2}+\tilde{L}_{12}(t,\tau)\right)+\right.$$
$$\left.+\mu_2(\tau)\tilde{L}_2(t,\tau)\right]\mathrm{d}\tau.$$

3. Numerical solution of integral equations

3.1. Quadrature method

To discretize our integral equations of the first kind we suggest quadrature method. Let $M\in\mathbb{N}$ and $t_j=\frac{j\pi}{M},\quad j=0,\dots,2M-1$. For approximation of corresponding integrals we use the following trigonometrical quadratures [4, 7]

$$
\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} f(\tau)\mathrm{d}\tau &\approx \frac{1}{2M}\sum_{j=0}^{2M-1} f(t_j),\\
\frac{1}{2\pi}\int_0^{2\pi} f(\tau)\ln\frac{4}{e}\sin^2\frac{t-\tau}{2}\mathrm{d}\tau &\approx \sum_{j=0}^{2M-1} R_j(t)f(t_j),\\
\frac{1}{2\pi}\int_0^{2\pi} f'(\tau)\cot\frac{\tau-t}{2}\mathrm{d}\tau &\approx \sum_{j=0}^{2M-1} T_j(t)f(t_j).
\end{aligned} \tag{13}
$$

Here the weight functions R_j and T_j are defined as

$$
R_j(t) = -\frac{1}{M}\left[\frac{1}{2} + \sum_{i=1}^{M-1}\frac{1}{i}\cos i(t-t_j) + \frac{\cos M(t-t_j)}{2M}\right]
$$

and

$$
T_j(t) = -\frac{1}{M}\sum_{i=1}^{M-1} i\cos i(t-t_j) - \frac{1}{2}\cos M(t-t_j).
$$

After application quadrature formulas (13) and performing collocation using the nodes of interpolation we obtain the system of linear equations with respect to unknown $\tilde{\mu}_\ell(t_j) \approx \mu_\ell(t_j)$, $\ell = 1, 2$, $j = 0, \ldots, 2M-1$

$$
\begin{cases}
\displaystyle\sum_{j=0}^{2M-1}\tilde{\mu}_1(t_j)H_{11}^1(t_k,t_j)R_j(t_k) + \frac{1}{2M}\sum_{j=0}^{2M-1}\tilde{\mu}_1(t_j)H_{11}^2(t_k,t_j) + \\
\qquad \displaystyle + \frac{1}{2M}\sum_{j=0}^{2M-1}\tilde{\mu}_2(t_j)H_{12}(t_k,t_j) = h(t_k), \quad k = 0,\ldots,2M-1,\\
\displaystyle\frac{1}{2M}\sum_{j=0}^{2M-1}\tilde{\mu}_1(t_j)H_{21}(t_k,t_j) + \sum_{j=0}^{2M-1}\tilde{\mu}_2(t_j)T_j(t_k) - \\
\qquad \displaystyle - \sum_{j=0}^{2M-1}\tilde{\mu}_2(t_j)H_{22}^1(t_k,t_j)R_j(t_k) - \\
\qquad \displaystyle - \frac{1}{2M}\sum_{j=0}^{2M-1}\tilde{\mu}_2(t_j)H_{22}^2(t_k,t_j) = g(t_k), \quad k = 0,\ldots,2M-1.
\end{cases} \tag{14}
$$

Finally, we have the following representation for approximate solution to Dirichlet-Neumann mixed problem (3), (4) in the domain D

$$
w(x) \approx \frac{1}{2M}\sum_{j=0}^{2M-1}\tilde{\mu}_1(t_j)K_0(\varkappa|x - x_1(t_j)|)|x_1'(t_j)| +
$$

$$
+\frac{1}{2M}\sum_{j=0}^{2M-1}\tilde{\mu}_2(t_j)\varkappa K_1(\varkappa|x - x_2(t_j)|)\frac{[(x - x_2(t_j))\cdot\nu_2(t_j)]}{|x - x_2(t_j)|}|x_2'(t_j)|, \quad x \in D.
$$

Taking into account (11) the numerical approximation for the normal derivative on Γ_1 can be calculated as

$$\frac{\partial w}{\partial \nu}(x_1(t_k)) \approx -\frac{1}{2}\tilde{\mu}_1(t_k) + \sum_{j=0}^{2M-1} \tilde{\mu}_1(t_j) L_{11}(t_k, t_j) R_j(t_k) +$$

$$+\frac{1}{2M}\sum_{j=0}^{2M-1} \tilde{\mu}_1(t_j) L_{12}(t_k, t_j) + \frac{1}{2M}\sum_{j=0}^{2M-1} \tilde{\mu}_2(t_j) L_2(t_k, t_j), \quad k = 0, \ldots, 2M-1.$$

Numerical solution of the system (12) is realized in the similar way.

3.2. Numerical experiments for mixed problems

Let's choose the domain with following boundaries (see Fig. 2)

$$\Gamma_1 = \{x(t) = (0.5\cos(t) + 0.5\cos(2t) - 0.25,\ \sin(t)), t \in [0, 2\pi]\}$$

and

$$\Gamma_2 = \{x(t) = (0.3\cos(t) + 0.25,\ 0.2\sin(t)), t \in [0, 2\pi]\}.$$

The boundary conditions for the Dirichlet-Neumann problem are given as

$$h(x) = 0.5x_1, \quad x \in \Gamma_1, \qquad g(x) = 0.05x_2^2, \quad x \in \Gamma_2$$

and for the Neumann-Dirichlet problem we choose

$$p(x) = e^{-x_2}, \quad x \in \Gamma_1, \qquad f(x) = 0.25\sin(x_1 + x_2), \quad x \in \Gamma_2.$$

For both problems we state $\varkappa = 1$.

The maximum norm errors of the obtained numerical solution values on Γ_1 for the Dirichlet-Neumann problem (3), (4) and calculated values of the normal derivative on Γ_1 for the Neumann-Dirichlet problem (5), (6) are listed for various values of the mesh size M in the Table 1. Note, that as the "exact" solutions we use the approximation solutions obtained by our numerical method with $M = 128$.

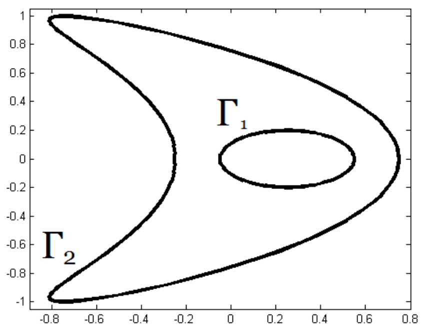


Fig. 2. Solution domain 1

Tabl. 1. Errors of the numerical solutions for the mixed problems

M	$\|\|\frac{\partial w}{\partial \nu} - \frac{\partial w_{ex}}{\partial \nu}\|\|_{C(\Gamma_1)}$	$\|\|v - v_{ex}\|\|_{C(\Gamma_1)}$
4	$1.631718 \cdot 10^{-3}$	$5.145063 \cdot 10^{-3}$
8	$2.131915 \cdot 10^{-5}$	$3.133429 \cdot 10^{-4}$
16	$8.192651 \cdot 10^{-10}$	$4.243675 \cdot 10^{-9}$
32	$3.295214 \cdot 10^{-14}$	$5.041247 \cdot 10^{-13}$

4. An alternating method for the Cauchy problem

4.1. An alternating procedure

To obtain the solution to Cauchy problem (1), (2) we use the alternating iterative procedure.

Each iteration of alternating procedure requires solving one of the mixed boundary value problems and finding Cauchy data on the inner domain boundary. These problems are numerically solved by application of integral equations method described in the above sections.

In problem definitions (3), (4) and (5), (6) functions f and g are the same as in the Cauchy problem (1), (2).

The functions p and h will be substituted with solution approximations during the alternating procedure run.

The alternating procedure of solving Cauchy problem (1), (2) runs as follows

- The first approximation $u^{(0)}$ to the solution is obtained by solving the problem (5), (6), with $p = p_0$, where p_0 is an arbitrary initial guess.
- Having constructed $u^{(2k)}$, we find $u^{(2k+1)}$ by solving (3), (4), with $h = u^{(2k)}\big|_{\Gamma_1}$.
- To obtain $u^{(2k+2)}$ the problem (5), (6) is solved with $p = \left.\dfrac{\partial u^{(2k+1)}}{\partial \nu}\right|_{\Gamma_1}$.

The following result about the convergence of alternating procedure can be obtained using the similar approach as in [3].

Theorem 1. *Suppose that Cauchy problem (1), (2) with appropriate input data f and g has a bounded solution. Let u_k be the k-th approximate solution in the alternating procedure. Then the following is true:*

$$\lim_{k\to\infty} \|u - u_k\|_{L^2(D)} = 0$$

for any sufficiently smooth initial data element p_0 which starts the procedure.

Also we have to note that alternating procedure which is applied to solve Cauchy problem is a regularizing method [3].

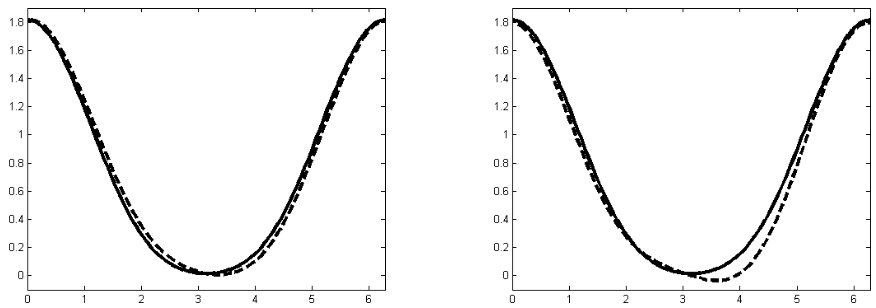
4.2. Numerical experiments for the Cauchy problem

In the numerical experiments we will use the solutions to the mixed problems for generating the input functions for problem (1), (2); i.e. we solve the mixed problem with predefined input functions, calculate the Cauchy data on both boundaries and as a result we got the input data for (1), (2) as well as the solution and it's normal derivative values on the inner boundary (the approximate solution will be compared with this values for checking the results). Please also note that the constant $\varkappa$ is set to one in the following numerical experiments.

Example 1. In the first example we will use the same domain as on Fig. 2. We generate input data for Cauchy problem by solving mixed problem (3), (4) with

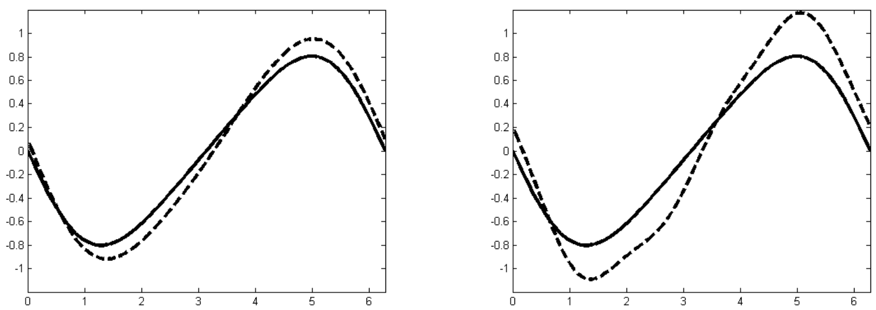
$$h(x) = 6(x_1^2 + x_2^2), \quad x \in \Gamma_1, \qquad g(x) = 3\sin(x_1 + x_2), \quad x \in \Gamma_2.$$

With $M = 128$ and zero initial guess which starts the alternating procedure, we obtain the results reflected in Fig. 3 and Fig. 4 for function and normal derivative reconstructions in case of exact input and input data with noise. The solid line (—) denotes the graph of exact solution and the dashed line (- - -) denotes the numerical solution obtained by alternating procedure.



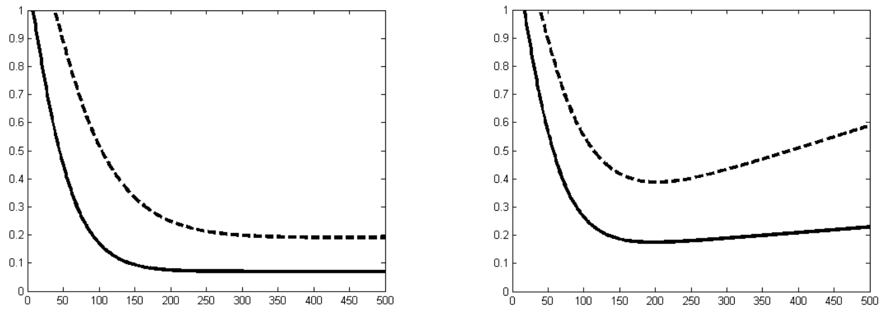
Exact data, $n = 500$ Data with 3% noise, $n = 185$

FIG. 3. Function values on the inner boundary Γ_1 for Ex. 1



Exact data, $n = 500$ Data with 3% noise, $n = 181$

FIG. 4. Normal derivative values on the inner boundary Γ_1 for Ex. 1



Exact data Data with 3% noise

FIG. 5. C-error of function (—) and normal derivative (- - -) on Γ_1 for Ex. 1

Example 2. Assume that boundaries have the following parametric representations (see Fig. 6)

$$\Gamma_1 = \{x(t) = (0.5\cos(t),\ 0.5\sin(t)), \quad t \in [0, 2\pi]\}$$

and

$$\Gamma_2 = \{x(t) = (2\cos(t),\ \sin(t)), \quad t \in [0, 2\pi]\}.$$

To obtain input functions for this numerical example we solve the mixed boundary value problem (5), (6) with

$$p(x) = x_1 + x_2, \quad x \in \Gamma_1,$$
$$f(x) = 0.5x_1, \quad x \in \Gamma_2.$$

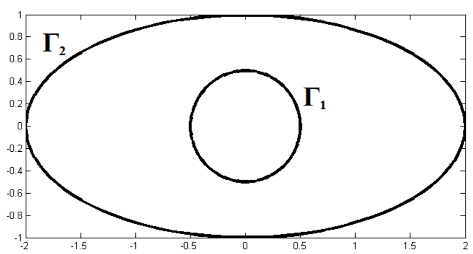
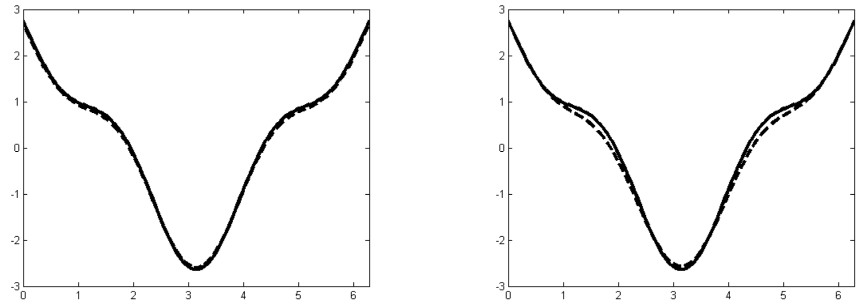


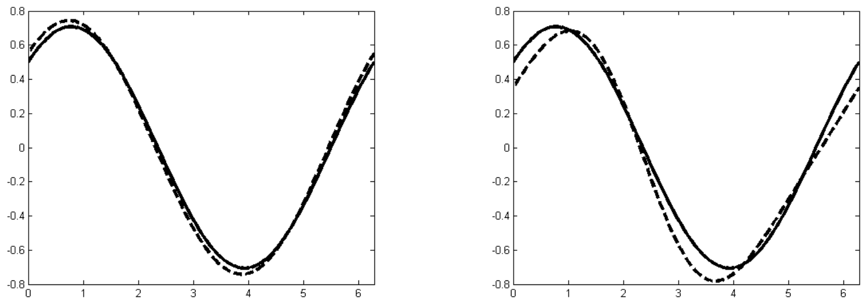
FIG. 6. Solution domain 2

The results of Cauchy data reconstruction on Γ_1 are presented in Fig. 7 and Fig. 8. The corresponding C-errors on every iteration step are reflected in Fig. 9



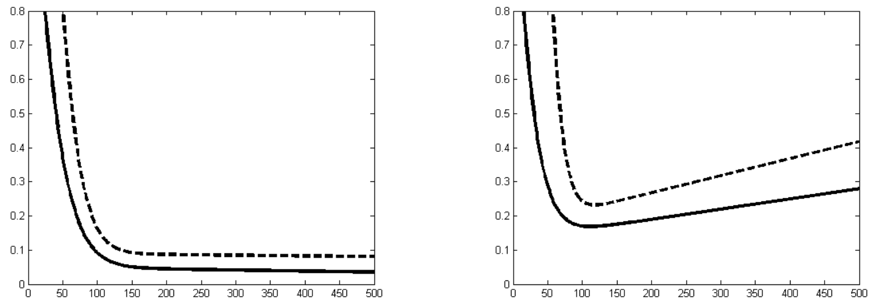
Exact data, $n = 500$ Data with 3% noise, $n = 110$

FIG. 7. Function values on the inner boundary Γ_1 for Ex. 2



Exact data, $n = 500$ Data with 3% noise, $n = 121$

FIG. 8. Normal derivative values on the inner boundary Γ_1 for Ex. 2



Exact data Data with 3% noise

FIG. 9. C-error of solution function (—) and normal derivative (- - -) on Γ_1 for Ex. 2

As one can observe from the above numerical examples, a satisfactory quality for the reconstruction of the boundary function and the normal derivative on the inner boundary is obtained with a reasonable stability against noisy data.

Bibliography

1. Abramowitz M. Handbook of Mathematical Functions With Formulas, Graphs and Mathematical Tables / M. Abramowitz, I. Stegun. – New York: Dover Publications, 1972.
2. Calderón A.-P. Uniqueness in the Cauchy problem for partial differential equations / A.-P. Calderón // Amer. J. Math. – 1958. – Vol. 80. – P. 16-36.
3. Chapko R. An alternating boundary integral based method for a Cauchy problem for the Laplace equation in semi-infinite regions / Roman Chapko, B. Thomas Johanson // Inverse problems and imaging. – 2008. – Vol. 2, No 3. – P. 317-333.
4. Chapko R. On a quadrature method for a logarithmic integral equation of the first kind / R. Chapko, R. Kress // In Agarwal, ed.: World Scientific Series in Applicable Analysis. – 1993. – Vol. 2. – Contributions in Numerical Mathematics. – World Scientific, Singapore – P. 127-140.
5. Jourhmane M. An alternating method for an inverse Cauchy problem / M. Jourhmane, A. Nachaoui // Numer. Algorithms. – 1999. – Vol. 21. – P. 247-260.
6. Kozlov V. A. On iterative procedures for solving ill-posed boundary value problems that preserve differential equations / V. A. Kozlov, V. G. Maz'ya // Algebra i Analiz. – 1989. – Vol. 1. – P. 144-170.
7. Kress R. Linear integral equations / R. Kress. – New-York: Springer-Verlag, 1989. – 367 p.
8. Lesnic D. An iterative boundary element method for solving numerically the Cauchy problem for the Laplace equation / D. Lesnic, L. Elliot, D. B. Ingham // Eng. Anal. Bound. Elem. – 1997. – Vol. 20. – P. 123-133.

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