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THREE-POINT DIFFERENCE SCHEMES OF HIGH-ORDER ACCURACY FOR SECOND-ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH BOUNDARY CONDITIONS OF THIRD KIND

Marta Król, Myroslav Kutniv

Резюме. Для нелінійних звичайних диференціальних рівнянь другого порядку з похідною в правій частині та крайовими умовами третього роду побудовано та обґрунтовано триточкові різницеві схеми високого порядку точності на нерівномірній сітці. Побудовано також апроксимацію потоку крайової задачі у вузлах сітки. Для обчислення розв'язку різницевих схем використовуються ітераційні методи. Доведено існування та єдиність розв'язку цих схем, встановлено оцінку точності. Ефективність триточкових різницевих схем шостого порядку точності проілюстрована на прикладах.

Abstract. Three-point difference schemes of hight-order accuracy on a non-equidistant grid for the second-order nonlinear ordinary differential equations with derivative in the right-hand side and boundary conditions of the third kind is constructed and justified. We also construct an approximation of flow for boundary value problem at grid nodes. Iterative methods were used to compute the solution of difference schemes. We prove the existence and uniqueness of the solution of this schemes and determine the order of accuracy. The efficiency of a three-point difference schemes of sixth-order accuracy is illustrated by an examples.

1. Introduction

An approach for construction of exact three-point difference scheme (ETDS) and three-point difference schemes (TDS) of high-order accuracy on a equidistant grid for the nonlinear problems of the form

$$\frac{d}{dx}\left[k(x)\frac{du}{dx}\right] = -f\left(x,u\right), \quad x \in (0,1), \quad u(0)=\mu_1, \quad u(1)=\mu_2$$

was suggested in [8, 7]. These results on a non-equidistant grid were generalized and developed in [6] and for monotone boundary value problems in [5, 1].

In the present paper the effective algorithmic implementation ETDS, proposed in [9], was developed via the truncated TDS for a nonlinear ODEs

$$\frac{d}{dx}\left[k(x)\frac{du}{dx}\right] = -f\left(x,u,\frac{du}{dx}\right), \quad x \in (0,1), \tag{1}$$

Key words. Nonlinear ordinary differential equations, boundary conditions of third kind, exact three-point difference schemes, three-point difference schemes of hight-order accuracy, iterative methods.

with a boundary conditions

$$k(0)\frac{du(0)}{dx} - \beta_1 u(0) = -\mu_1, \quad -k(1)\frac{du(1)}{dx} - \beta_2 u(1) = -\mu_2, \tag{2}$$

where $k(x), f(x,u,\xi)$ are given functions, a $\beta_1, \beta_2, \mu_1, \mu_2$ are given numbers. To find the coefficients and right-hand side of TDS at each node's $x_j, j = 1,2,...,N-1$ of the non-equidistance grid we need to solve two auxiliary initial value problems for nonlinear ODEs and two initial value problems for linear ODEs on the intervals $[x_{j-1}, x_j]$ (forward) and $[x_j, x_{j+1}]$ (backward). Moreover, to find right-hand sides difference boundary conditions we need to solve initial value problems for nonlinear and linear ODEs on the intervals $[x_0, x_1]$ (forward) and $[x_{N-1}, x_N]$ (backward). These initial value problems can be solved by executing only one step with an arbitrary one-step method order of accuracy $\bar{m} = 2[(m+1)/2]$ (m is a given the positive integer, $[\cdot]$ denotes the entire part of the number in this brackets). As a result the implementations ETDS which received from truncated TDS of rank $\bar{m}$, for which it is proved that it has an order of accuracy $\bar{m}$. Constructed approaching flow $k(x)du/dx$ at the grid nodes, the order of accuracy of which is the same as the solution, that is of $\bar{m}$.

2. Existence and uniqueness of the solution

Sufficient conditions for the existence and uniqueness of weak solutions of the problem (1), (2) are given by following statement.

Theorem 1. *Let the following assumptions be satisfied*

$$0 < c_1 \le k(x) \le c_2 \quad \forall x \in [0,1], \quad k(x) \in Q^1[0,1], \tag{3}$$

$$\begin{aligned} f_{u\xi}(x) &\equiv f(x,u,\xi) \in Q^0[0,1] \quad \forall u,\xi \in R^1, \\ f_x(u,\xi) &\equiv f(x,u,\xi) \in C(R^2) \quad \forall x \in [0,1], \end{aligned} \tag{4}$$

$$|f(x,u,\xi) - f_0(x)| \le c(|u|)[g(x) + |\xi|] \quad \forall x \in [0,1],\ u,\xi \in R^1, \tag{5}$$

$$[f(x,u,\xi) - f(x,v,\eta)](u-v) \le 0 \quad \forall x \in [0,1],\ u,v,\xi,\eta \in R^1, \tag{6}$$

$$\beta_1 > 0, \quad \beta_2 > 0. \tag{7}$$

Then the problem (1),(2) has a unique solution $u(x) \in W_2^1(0,1)$, *with* $u(x),\ k(x)\dfrac{du}{dx} \in C[0,1]$.

Here $c(t)$ is a continuous function, $f_0(x) \in L_2(0,1)$, $g(x) \in L_1(0,1)$, c_1, c_2, c_3 are some real constants, $Q^p[0,1]$ is the class of functions having p piece-wise continuous derivatives and a finite number of discontinuity points of first kind.

The proof can be found in [9].

3. Algorithmic implementation of the exact three-point difference schemes

On the interval $(0,1)$ we introduce the non-equidistant grid

$$\hat{\omega}_h = \{x_j \in (0,1),\ j = 1,2,...,N-1,\ h_j = x_j - x_{j-1} > 0,\ h_1 + h_2 + ... + h_N\}$$

such the discontinuity points of functions $k(x), f(x,u,\xi)$ coincide with the nodes of the grid $\hat{\omega}_h$. Denote by ρ the set of all discontinuity points and assume that N in such that $\rho \subseteq \hat{\omega}_h$. At discontinuity points the solution of problem (1),(2) should satisfy the continuity conditions

$$u(x_i - 0) = u(x_i + 0), \quad k(x)\frac{du}{dx}\bigg|_{x=x_i-0} = k(x)\frac{du}{dx}\bigg|_{x=x_i+0} \quad \forall x_i \in \rho.$$

For problem (1),(2) in paper [9] is constructed ETDS of the form

$$(au_{\bar{x}})_{\hat{x},j} = -\varphi(x_j, u), \quad j = 1, 2, ..., N-1, \tag{8}$$

$$\begin{gathered} \frac{1}{\hbar_0}\left(a_1 u_{x,0} - \beta_1 u_0\right) = -\varphi(x_0, u), \\ -\frac{1}{\hbar_N}\left(a_N u_{\bar{x},N} + \beta_2 u_N\right) = -\varphi(x_N, u), \end{gathered} \tag{9}$$

where

$$u_{\bar{x},j} = \frac{u_j - u_{j-1}}{h_j}, \quad u_{\hat{x},j} = \frac{u_{j+1} - u_j}{\hbar_j}, \quad u_{x,j} = \frac{u_{j+1} - u_j}{h_{j+1}},$$

$$a(x_j) = \left[\frac{1}{h_j} V_1^j(x_j)\right]^{-1}, \quad \hbar_j = \frac{h_j + h_{j+1}}{2}, \quad \hbar_0 = 0,5h_1, \quad \hbar_N = 0,5h_N,$$

$$\varphi(x_j, u) = [\hbar_j V_1^j(x_j)]^{-1} \int\limits_{x_{j-1}}^{x_j} V_1^j(\xi) f\left(\xi, u, \frac{du}{d\xi}\right) d\xi +$$

$$+ [\hbar_j V_2^j(x_j)]^{-1} \int\limits_{x_j}^{x_{j+1}} V_2^j(\xi) f\left(\xi, u, \frac{du}{d\xi}\right) d\xi,$$

$$\varphi(x_0, u) = [\hbar_0 V_1^1(x_1)]^{-1} \int\limits_{x_0}^{x_1} V_2^0(\xi) f\left(\xi, u, \frac{du}{d\xi}\right) d\xi + \hbar_0^{-1}\mu_1,$$

$$\varphi(x_N, u) = [\hbar_N V_1^N(x_N)]^{-1} \int\limits_{x_{N-1}}^{x_N} V_1^N(\xi) f\left(\xi, u, \frac{du}{d\xi}\right) d\xi + \hbar_N^{-1}\mu_2,$$

$$V_1^j(x) = \int\limits_{x_{j-1}}^{x} \frac{dt}{k(t)}, \quad V_2^j(x) = \int\limits_{x}^{x_{j+1}} \frac{dt}{k(t)}.$$

First of all, take into account, that since

$$(-1)^{\alpha+1} \int\limits_{x_{j+(-1)^\alpha}}^{x_j} V_\alpha^j(\xi) f\left(\xi, u(\xi), \frac{du}{d\xi}\right) d\xi =$$

$$= (-1)^\alpha V_\alpha^j(x_j) Z_\alpha^j(x_j, u) + Y_\alpha^j(x_j, u) - u_{j+(-1)^\alpha},$$

where $Y_\alpha^j(x,u), Z_\alpha^j(x,u), j = 2-\alpha,\ 3-\alpha, ..., N+1-\alpha,\ \alpha = 1,2$ are the solutions of the initial value problems

$$\frac{dY_\alpha^j(x,u)}{dx} = \frac{Z_\alpha^j(x,u)}{k(x)}, \quad \frac{dZ_\alpha^j(x,u)}{dx} = -f\left(x, Y_\alpha^j(x,u), \frac{Z_\alpha^j(x,u)}{k(x)}\right),$$
$$x_{j-2+\alpha} < x < x_{j-1+\alpha}, \tag{10}$$
$$Y_\alpha^j(x_{j+(-1)^\alpha}, u) = u_{j+(-1)^\alpha}, \quad Z_\alpha^j(x_{j+(-1)^\alpha}, u) = k(x)\frac{du}{dx}\bigg|_{x=x_{j+(-1)^\alpha}},$$

and $\bar{V}_\alpha^j(x) = (-1)^{\alpha+1}V_\alpha^j(x)$ are the solutions of the initial value problems

$$\frac{d\bar{V}_\alpha^j(x)}{dx} = \frac{1}{k(x)}, \quad x_{j-2+\alpha} < x < x_{j-1+\alpha},$$
$$\bar{V}_\alpha^j(x_{j+(-1)^\alpha}) = 0, \quad j = 2-\alpha, 3-\alpha, ..., N+1-\alpha, \quad \alpha = 1,2. \tag{11}$$

Obviously, the right-hand side of the ETDS can be written as

$$\varphi(x_j,u) = \frac{1}{\hbar_j}\sum_{\alpha=1}^{2}(-1)^\alpha\left[Z_\alpha^j(x_j,u) + (-1)^\alpha\frac{Y_\alpha^j(x_j,u) - u_{j+(-1)^\alpha}}{V_\alpha^j(x_j)}\right], \tag{12}$$

$$\varphi(x_0,u) = \frac{1}{\hbar_0}\left[Z_2^0(x_0,u) + \frac{Y_2^0(x_0,u) - u_1}{V_2^0(x_0)} + \mu_1\right], \tag{13}$$

$$\varphi(x_N,u) = \frac{1}{\hbar_N}\left[-Z_1^N(x_N,u) + \frac{Y_1^N(x_N,u) - u_{N-1}}{V_1^N(x_N)} + \mu_2\right]. \tag{14}$$

Therefore, to construct the ETDS (8), (9), (12)-(14) for $j = 2-\alpha, 3-\alpha, ..., N+1-\alpha$ it is necessary to solve initial value problems (10), (11) in the forward direction ($\alpha = 1$), and in the backward direction ($\alpha = 2$). We will solve then numerically by using one-step methods:

$$Y_\alpha^{(\bar{m})j}(x_j,u) = u_{j+(-1)^\alpha} + (-1)^{\alpha+1}h_{j-1+\alpha}\times$$
$$\times\,\Phi_1\left(x_{j+(-1)^\alpha}, u_{j+(-1)^\alpha}, \left(k\frac{du}{dx}\right)_{j+(-1)^\alpha}, (-1)^{\alpha+1}h_{j-1+\alpha}\right), \tag{15}$$

$$Z_\alpha^{(m)j}(x_j,u) = \left(k\frac{du}{dx}\right)_{j+(-1)^\alpha} + (-1)^{\alpha+1}h_{j-1+\alpha}\times$$
$$\times\,\Phi_2\left(x_{j+(-1)^\alpha}, u_{j+(-1)^\alpha}, \left(k\frac{du}{dx}\right)_{j+(-1)^\alpha}, (-1)^{\alpha+1}h_{j-1+\alpha}\right), \tag{16}$$

$$\bar{V}_\alpha^{(\bar{m})j}(x_j) = (-1)^{\alpha+1}h_{j-1+\alpha}\Phi_3(x_{j+(-1)^\alpha}, 0, (-1)^{\alpha+1}h_{j-1+\alpha}), \tag{17}$$

where $\Phi_1(x,u,v,h), \Phi_2(x,u,v,h), \Phi_3(x,u,h)$ are increment functions,

$$\left(k\frac{du}{dx}\right)_{j+(-1)^\alpha} = k(x)\frac{du}{dx}\bigg|_{x=x_{j+(-1)^\alpha}},$$

$Z_\alpha^{(m)j}(x_j,u)$ approximates the values $Z_\alpha^j(x_j,u)$ with an order of accuracy m, $Y_\alpha^{(\bar{m})j}(x_j,u)$ and $\bar{V}_\alpha^{(\bar{m})j}(x_j)$ approximate $Y_\alpha^j(x_j,u)$ and $\bar{V}_\alpha^j(x_j)$, respectively, with accuracy order $\bar{m}$.

If $k(x)$ and the right-hand side of the differential equation $f(x,u,\xi)$ are differentiated a sufficient number of times, then there exist expansions

$$\begin{aligned} Y_\alpha^j(x_j,u) =& Y_\alpha^{(\bar m)j}(x_j,u)+ \\ &+ [(-1)^{\alpha+1}h_{j-1+\alpha}]^{\bar m+1}\psi_\alpha^j(x_{j+(-1)^\alpha},u) + O(h_{j-1+\alpha}^{\bar m+2}), \end{aligned} \tag{18}$$

$$\begin{aligned} Z_\alpha^j(x_j,u) =& Z_\alpha^{(m)j}(x_j,u)+ \\ &+ [(-1)^{\alpha+1}h_{j-1+\alpha}]^{m+1}\tilde\psi_\alpha^j(x_{j+(-1)^\alpha},u) + O(h_{j-1+\alpha}^{m+2}), \end{aligned} \tag{19}$$

$$\begin{aligned} \bar V_\alpha^j(x_j) =& \bar V_\alpha^{(\bar m)j}(x_j)+ \\ &+ [(-1)^{\alpha+1}h_{j-1+\alpha}]^{\bar m+1}\bar\psi_\alpha^j(x_{j+(-1)^\alpha}) + O(h_{j-1+\alpha}^{\bar m+2}). \end{aligned} \tag{20}$$

If in the ETDS (8), (9), (12)-(14) the exact solutions of the corresponding initial value problems (10), (11) are approximated by numerical solutions, the following truncated TDS of rank $\bar m$ is obtained:

$$(a^{(\bar m)}y_{\bar x}^{(\bar m)})_{\hat x,j} = -\varphi^{(\bar m)}(x_j,y^{(\bar m)}), \quad j=1,2,\dots,N-1, \tag{21}$$

$$\begin{gathered} \frac{1}{\hbar_0}\left(a_1^{(\bar m)}y_{x,0}^{(\bar m)} - \beta_1 y_0^{(\bar m)}\right) = -\varphi^{(\bar m)}(x_0,y^{(\bar m)}), \\ -\frac{1}{\hbar_N}\left(a_N^{(\bar m)}y_{\bar x,N}^{(\bar m)} + \beta_2 y_N^{(\bar m)}\right) = -\varphi^{(\bar m)}(x_N,y^{(\bar m)}), \end{gathered} \tag{22}$$

where

$$a^{(\bar m)}(x_j) = \left[\frac{1}{h_j}V_1^{(\bar m)}(x_j)\right]^{-1}, \quad j=1,2,\dots,N,$$

$$\varphi^{(\bar m)}(x_j,u) = \hbar_j^{-1}\sum_{\alpha=1}^{2}(-1)^\alpha\left[Z_\alpha^{(m)j}(x_j,u) + (-1)^\alpha\frac{Y_\alpha^{(\bar m)j}(x_j,u)-u_{j+(-1)^\alpha}}{V_\alpha^{(\bar m)j}(x_j)}\right],$$

$$\varphi^{(\bar m)}(x_0,u) = \frac{1}{\hbar_0}\left[Z_2^{(m)0}(x_0,u) + \frac{Y_2^{(\bar m)0}(x_0,u)-u_1}{V_2^{(\bar m)0}(x_0)} + \mu_1\right],$$

$$\varphi^{(\bar m)}(x_N,u) = \frac{1}{\hbar_N}\left[-Z_1^{(m)N}(x_N,u) + \frac{Y_1^{(\bar m)N}(x_N,u)-u_{N-1}}{V_1^{(\bar m)N}(x_N)} + \mu_2\right].$$

We need the following assertion to prove the existence and uniqueness of a solution to TDS (21), (22) and to establish its accuracy.

Lemma 1. *Let*

$$0 < c_1 \le k(x) \le c_2 \quad \forall x\in[0,1], \quad k(x)\in Q^{m+1}[0,1],$$

$$f(x,u,\xi) \in \bigcup_{j=1}^{N} C^m\left([x_{j-1},x_j]\times R^2\right).$$

Then one has the following estimates

$$\left|a^{(\bar m)}(x_j) - a(x_j)\right| \le M\,|h|^{\bar m}, \quad j=1,2,\dots,N, \tag{23}$$

$$\varphi^{(\bar{m})}(x_j,u)-\varphi(x_j,u)= \\ =\left\{h_j^{m+1}\left[k(x)\left(\psi_1^j(x,u)-\bar{\psi}_1^j(x)k(x)\frac{du}{dx}\right)-\tilde{\psi}_1^j(x,u)\right]_{x=x_j+0}\right\}_{\hat{x}}+ \\ +O\left(\frac{h_j^{m+2}+h_{j+1}^{m+2}}{\hbar_j}\right),\quad j=1,2,\ldots,N-1, \tag{24}$$

$$\varphi^{(\bar{m})}(x_0,u)-\varphi(x_0,u)= \\ =\frac{h_1^{m+1}}{\hbar_0}\left[k(x)\left(\psi_1^1(x,u)-\bar{\psi}_1^1(x)k(x)\frac{du}{dx}\right)-\tilde{\psi}_1^1(x,u)\right]_{x=x_1}+ \\ +O\left(\frac{h_1^{m+2}}{\hbar_0}\right), \tag{25}$$

$$\varphi^{(\bar{m})}(x_N,u)-\varphi(x_N,u)= \\ =-\frac{h_N^{m+1}}{\hbar_N}\left[k(x)\left(\psi_1^N(x,u)-\bar{\psi}_1^N(x)k(x)\frac{du}{dx}\right)-\tilde{\psi}_1^N(x,u)\right]_{x=x_N}+ \\ +O\left(\frac{h_N^{m+2}}{\hbar_N}\right), \tag{26}$$

if m is odd and

$$\varphi^{(\bar{m})}(x_j,u)-\varphi(x_j,u)= \\ =\left\{h_j^{m}\left[k(x)\left(\psi_1^j(x,u)-\bar{\psi}_1^j(x)k(x)\frac{du}{dx}\right)\right]_{x=x_j+0}\right\}_{\hat{x}}+ \\ +O\left(\frac{h_j^{m+1}+h_{j+1}^{m+1}}{\hbar_j}\right),\quad j=1,2,\ldots,N-1, \tag{27}$$

$$\varphi^{(\bar{m})}(x_0,u)-\varphi(x_0,u)= \\ =\frac{h_1^{m}}{\hbar_0}\left[k(x)\left(\psi_1^1(x,u)-\bar{\psi}_1^1(x)k(x)\frac{du}{dx}\right)\right]_{x=x_1}+O\left(\frac{h_1^{m+1}}{\hbar_0}\right), \tag{28}$$

$$\varphi^{(\bar{m})}(x_N,u)-\varphi(x_N,u)= \\ =-\frac{h_N^{m}}{\hbar_N}\left[k(x)\left(\psi_1^N(x,u)-\bar{\psi}_1^N(x)k(x)\frac{du}{dx}\right)\right]_{x=x_N}+O\left(\frac{h_N^{m+1}}{\hbar_N}\right), \tag{29}$$

if m is even.

Proof. The estimate (23) follows from relation (20). Actually,

$$a^{(\bar{m})}(x_j)-a(x_j)=\frac{h_j[V_1^j(x_j)-V_1^{(\bar{m})j}(x_j)]}{V_1^j(x_j)V_1^{(\bar{m})j}(x_j)}=O(h_j^{\bar{m}}).$$

Let us prove (24)-(29). Note that

$$\begin{aligned}\varphi^{(\bar m)}(x_j,u)-\varphi(x_j,u)=\hbar_j^{-1}\sum_{\alpha=1}^{2}(-1)^{\alpha}\Big\{Z_\alpha^{(m)j}(x_j,u)-Z_\alpha^{j}(x_j,u)+\\ +(-1)^{\alpha}\left[\frac{Y_\alpha^{(\bar m)j}(x_j,u)-u_{j+(-1)^\alpha}}{V_\alpha^{(\bar m)j}(x_j)}-\frac{Y_\alpha^{j}(x_j,u)-u_{j+(-1)^\alpha}}{V_\alpha^{j}(x_j)}\right]\Big\}.\end{aligned}\tag{30}$$

$$\begin{aligned}\varphi^{(\bar m)}(x_0,u)-\varphi(x_0,u)=\frac{1}{\hbar_0}\Big\{Z_2^{(m)0}(x_0,u)-Z_2^{0}(x_0,u)+\\ +\frac{Y_2^{(\bar m)0}(x_0,u)-u_1}{V_2^{(\bar m)0}(x_0)}-\frac{Y_2^{0}(x_0,u)-u_1}{V_2^{0}(x_0)}\Big\},\end{aligned}\tag{31}$$

$$\begin{aligned}\varphi^{(\bar m)}(x_N,u)-\varphi(x_N,u)=\frac{1}{\hbar_N}\Big\{-Z_1^{(m)N}(x_N,u)+Z_1^{N}(x_N,u)+\\ +\frac{Y_1^{(\bar m)N}(x_N,u)-u_{N-1}}{V_1^{(\bar m)N}(x_N)}-\frac{Y_1^{N}(x_N,u)-u_{N-1}}{V_1^{N}(x_N)}\Big\}.\end{aligned}\tag{32}$$

From Lemma 3.4 (see [4, p.102]) and the equalities

$$V_\alpha^j(x_j)=\frac{h_{j-1+\alpha}}{k_{j+(-1)^\alpha}}+O(h_{j-1+\alpha}^2),$$

$$Y_\alpha^j(x_j,u)-u_{j+(-1)^\alpha}=(-1)^{\alpha+1}h_{j-1+\alpha}\left.\frac{du}{dx}\right|_{x=x_{j+(-1)^\alpha}}+O(h_{j-1+\alpha}^2),$$

$$j=2-\alpha,\ 3-\alpha,...,N+1-\alpha,\quad \alpha=1,2$$

we obtain

$$\begin{aligned}&Z_\alpha^{(m)j}(x_j,u)-Z_\alpha^{j}(x_j,u)=\\ &\quad=-[(-1)^{\alpha+1}h_{j-1+\alpha}]^{m+1}\tilde\psi_1^{j-1+\alpha}(x_{j+(-1)^\alpha},u)+O(h_{j-1+\alpha}^{m+2}),\end{aligned}\tag{33}$$

$$\begin{aligned}&\frac{Y_\alpha^{(\bar m)j}(x_j,u)-u_{j+(-1)^\alpha}}{V_\alpha^{(\bar m)j}(x_j)}-\frac{Y_\alpha^{j}(x_j,u)-u_{j+(-1)^\alpha}}{V_\alpha^{j}(x_j)}=-(-1)^{\alpha+1}h_{j-1+\alpha}^{\bar m}\times\\ &\times\left[k(x)\left(\psi_1^{j-1+\alpha}(x,u)-\bar\psi_1^{j-1+\alpha}(x)k(x)\frac{du}{dx}\right)\right]_{x=x_{j+(-1)^\alpha}}+\\ &+O(h_{j-1+\alpha}^{\bar m+1}),\quad j=2-\alpha,\ 3-\alpha,...,N+1-\alpha,\quad \alpha=1,2.\end{aligned}\tag{34}$$

Then the equalities (30)-(32) are reduced to estimates (25), (26), and

$$\begin{aligned}
&\varphi^{(\bar{m})}(x_j,u)-\varphi(x_j,u)=\frac{1}{\hbar_j}\Bigg\{h_{j+1}^{m+1}\times\\
&\times\left[k(x)\left(\psi_1^{j+1}(x,u)-\bar{\psi}_1^{j+1}(x)k(x)\frac{du}{dx}\right)-\tilde{\psi}_1^{j+1}(x,u)\right]_{x=x_{j+1}}-\\
&-h_j^{m+1}\left[k(x)\left(\psi_1^{j}(x,u)-\bar{\psi}_1^{j}(x)k(x)\frac{du}{dx}\right)-\tilde{\psi}_1^{j}(x,u)\right]_{x=x_{j-1}}\Bigg\}+\\
&+O\left(\frac{h_j^{m+2}+h_{j+1}^{m+2}}{\hbar_j}\right),
\end{aligned}\tag{35}$$

for odd m, and to (28), (29), and

$$\begin{aligned}
&\varphi^{(\bar{m})}(x_j,u)-\varphi(x_j,u)=\\
&=\frac{1}{\hbar_j}\Bigg\{h_{j+1}^{m}\left[k(x)\left(\psi_1^{j+1}(x,u)-\bar{\psi}_1^{j+1}(x)k(x)\frac{du}{dx}\right)\right]_{x=x_{j+1}}-\\
&-h_j^{m}\left[k(x)\left(\psi_1^{j}(x,u)-\bar{\psi}_1^{j}(x)k(x)\frac{du}{dx}\right)\right]_{x=x_{j-1}}\Bigg\}+\\
&+O\left(\frac{h_j^{m+1}+h_{j+1}^{m+1}}{\hbar_j}\right),
\end{aligned}\tag{36}$$

for even m.

Since

$$\begin{aligned}
&\left[k(x)\left(\psi_1^{j}(x,u)-\bar{\psi}_1^{j}(x)k(x)\frac{du}{dx}\right)\right]_{x=x_{j-1}}=\\
&\qquad=\left[k(x)\left(\psi_1^{j}(x,u)-\bar{\psi}_1^{j}(x)k(x)\frac{du}{dx}\right)\right]_{x=x_j}+O(h_j),\\
&\tilde{\psi}_1^{j}(x_{j-1},u)=\tilde{\psi}_1^{j}(x_j,u)+O(h_j),
\end{aligned}$$

it follows from (35) and (36) that the estimates (24) and (27) hold.

On the basis of the above-obtained results, one can prove the following assertion.

Theorem 2. *Let the assumptions of Theorem 1 and Lemma 1 hold. Then there exists an $h_0>0$ such that for all $\{h_j\}_{j=1}^N$ with $|h|=\max\limits_{1\le j\le N}h_j\le h_0$ and TDS (21), (22) has a unique solution, whose accuracy is characterized by the estimate*

$$\left\|y^{(\bar{m})}-u\right\|_{1,2,\hat{\bar{\omega}}_h}^{*}=\left[\left\|y^{(\bar{m})}-u\right\|_{0,2,\hat{\bar{\omega}}_h}^2+\left\|k\frac{dy^{(\bar{m})}}{dx}-k\frac{du}{dx}\right\|_{0,2,\hat{\bar{\omega}}_h}^2\right]^{1/2}\le M\,|h|^{\bar{m}},$$

where

$$\|u\|_{0,2,\hat{\omega}_h} = \left\{ \sum_{j=0}^{N} \hbar_j u_j^2 \right\}^{1/2},$$

$$k(x)\frac{dy^{(\bar{m})}}{dx}\bigg|_{x=x_j} = a_{j-1+\alpha} y^{(\bar{m})}_{\bar{x},j-1+\alpha} + Z_\alpha^{(m)j}\left(x_j, y^{(\bar{m})}\right) + $$

$$+ (-1)^\alpha \frac{Y_\alpha^{(\bar{m})j}\left(x_j, y^{(\bar{m})}\right) - y^{(\bar{m})}_{j+(-1)^\alpha}}{V_\alpha^{(\bar{m})j}(x_j)}, \quad \alpha = 1, 2,$$

$$j = 1, 2, ..., N-1,$$

$$k(x)\frac{dy^{(\bar{m})}}{dx}\bigg|_{x=x_0} = \beta_1 y_0^{(\bar{m})} - \mu_1, \quad k(x)\frac{dy^{(\bar{m})}}{dx}\bigg|_{x=x_N} = -\beta_2 y_N^{(\bar{m})} + \mu_2,$$

and M is a constant independent of $|h|$.

Proof. Let us consider the operators

$$B_h^{(\bar{m})} u_j = \begin{cases} -\dfrac{1}{\hbar_0}\left(a_1^{(\bar{m})} u_{x,0} - \beta_1 u_0\right), & j = 0, \\ -(a^{(\bar{m})} u_{\bar{x}})_{\hat{x},j}, & j = 1, 2, ..., N-1, \\ \dfrac{1}{\hbar_N}\left(a_N^{(\bar{m})} u_{\bar{x},N} + \beta_2 u_N\right), & j = N, \end{cases} \tag{37}$$

$$A_h^{(\bar{m})}(x_j, u) = \begin{cases} -\dfrac{1}{\hbar_0}\left(a_1^{(\bar{m})} u_{x,0} - \beta_1 u_0\right) - \varphi^{(\bar{m})}(x_0, u), & j = 0, \\ -(a^{(\bar{m})} u_{\bar{x}})_{\hat{x},j} - \varphi^{(\bar{m})}(x_j, u), & j = 1, 2, ..., N-1, \\ \dfrac{1}{\hbar_N}\left(a_N^{(\bar{m})} u_{\bar{x},N} + \beta_2 u_N\right) - \varphi^{(\bar{m})}(x_N, u), & j = N, \end{cases} \tag{38}$$

which is defined in the finite-dimensional space of grid functions $H(\hat{\bar{\omega}}_h)$ with the scalar products

$$(u, v)_{\hat{\bar{\omega}}_h} = \sum_{\xi \in \hat{\omega}_h} \hbar(\xi) u(\xi) v(\xi) + \hbar_0 u_0 v_0 + \hbar_N u_N v_N,$$

$$(u, v)_{\hat{\omega}_h^+} = \sum_{\xi \in \hat{\omega}_h^+} h(\xi) u(\xi) v(\xi), \quad \hat{\omega}_h^+ = \hat{\omega}_h \cup x_N,$$

and the norms

$$\|u\|_{0,2,\hat{\bar{\omega}}_h} = (u,u)^{1/2}_{\hat{\bar{\omega}}_h}, \quad \|u\|_{0,2,\hat{\omega}_h^+} = (u,u)^{1/2}_{\hat{\omega}_h^+},$$

$$\|u\|_{1,2,\hat{\bar{\omega}}_h} = \left(\|u\|^2_{0,2,\hat{\bar{\omega}}_h} + \|u_{\bar{x}}\|^2_{0,2,\hat{\omega}_h^+}\right)^{1/2}, \quad \|u\|_{B_h^{(\bar{m})}} = \left(B_h^{(\bar{m})} u, u\right)^{1/2}_{\hat{\bar{\omega}}_h}.$$

Because (see proof of Theorem 2 in [9])

$$(\varphi(x,u) - \varphi(x,v), u - v)_{\hat{\bar{\omega}}_h} \le -c_1 \int_0^1 \left\{ \frac{d}{d\eta}\left[\hat{u}(\eta) - \hat{v}(\eta) - u(\eta) + v(\eta)\right] \right\}^2 d\eta,$$

$$\hat{u}(\eta) = u(x_j)\frac{V_1^j(\eta)}{V_1^j(x_j)} + u(x_{j-1})\frac{V_2^{j-1}(\eta)}{V_1^j(x_j)}, \quad x_{j-1} \le \eta \le x_j,$$

then due to (23)-(29) $\exists h_0 > 0$ such that $\forall \{h_j\}_{j=1}^N$ with $|h| = \max\limits_{1\le j\le N} h_j \le h_0$ the following estimation holds:

$$0 < \tilde{c}_1 \le a^{(\bar{m})}(x) \quad \forall x \in \hat{\omega}_h^+,$$

$$\left(\varphi^{(\bar{m})}(x,u) - \varphi^{(\bar{m})}(x,v), u - v\right)_{\hat{\bar{\omega}}_h} \le 0.$$

Then, from the Green's first difference formula [3, p.26]) and inequality (see [3, p.39])

$$\gamma_1 \|u\|^2_{0,2,\hat{\bar{\omega}}_h} \le \left(u_{\bar{x}}^2, 1\right)_{\hat{\omega}_h^+} + \beta_1 u_0^2 + \beta_2 u_N^2, \ \gamma_1 > 0, \tag{39}$$

it follows that

$$\begin{aligned}
&\left(A_h^{(\bar{m})}(x,u) - A_h^{(\bar{m})}(x,v), u - v\right)_{\hat{\bar{\omega}}_h} = \left(a^{(\bar{m})}\left(u_{\bar{x}} - v_{\bar{x}}\right)^2, 1\right)_{\hat{\omega}_h^+} + \\
&+ \beta_1\left(u_0 - v_0\right)^2 + \beta_2\left(u_N - v_N\right)^2 - \\
&- \left(\varphi^{(\bar{m})}(x,u) - \varphi^{(\bar{m})}(x,v), u - v\right)_{\hat{\bar{\omega}}_h} \ge \|u - v\|^2_{B_h^{(\bar{m})}} = \\
&= \left(a^{(\bar{m})}\left(u_{\bar{x}} - v_{\bar{x}}\right)^2, 1\right)_{\hat{\omega}_h^+} + \beta_1\left(u_0 - v_0\right)^2 + \beta_2\left(u_N - v_N\right)^2 \ge \\
&\ge \min\{\tilde{c}_1, 1\}\left[\left((u_{\bar{x}} - v_{\bar{x}})^2, 1\right)_{\hat{\omega}_h^+} + \beta_1(u_0 - v_0)^2 + \beta_2(u_N - v_N)^2\right] \ge \\
&\ge \min\{\tilde{c}_1, 1\}\gamma_1\|u - v\|^2_{0,2,\hat{\bar{\omega}}_h}.
\end{aligned} \tag{40}$$

Therefore, if $|h| \le h_0$, then $A_h^{(\bar{m})}(x,u)$ is strongly monotone operator, and the TDS (21), (22) has a unique solution $y^{(\bar{m})}(x), x \in \hat{\bar{\omega}}_h$ (see [2, p.461]).

For error $z(x) = y^{(\bar{m})}(x) - u(x), x \in \hat{\bar{\omega}}_h$ of difference scheme (21), (22) will have a problem

$$\begin{aligned}
&-\left[a^{(\bar{m})}(x) z_{\bar{x}}(x)\right]_{\hat{x}} - \left(\varphi^{(\bar{m})}(x, y^{(\bar{m})}) - \varphi^{(\bar{m})}(x,u)\right) = \\
&\quad = \varphi^{(\bar{m})}(x,u) - \varphi(x,u) + \left[\left(a^{(\bar{m})}(x) - a(x)\right) u_{\bar{x}}(x)\right]_{\hat{x}}, \quad x \in \hat{\omega}_h,
\end{aligned} \tag{41}$$

$$\begin{aligned}
&-\frac{1}{\hbar_0}\left(a_1^{(\bar{m})} z_{x,0} - \beta_1 z_0\right) - \left(\varphi^{(\bar{m})}(x_0, y^{(\bar{m})}) - \varphi^{(\bar{m})}(x_0,u)\right) = \\
&\quad = \varphi^{(\bar{m})}(x_0,u) - \varphi(x_0,u) + \frac{1}{\hbar_0}\left(a_1^{(\bar{m})} - a_1\right) u_{x,0},
\end{aligned} \tag{42}$$

$$\begin{aligned}
&\frac{1}{\hbar_N}\left(a_N^{(\bar{m})} z_{x,N} - \beta_2 z_N\right) - \left(\varphi^{(\bar{m})}(x_N, y^{(\bar{m})}) - \varphi^{(\bar{m})}(x_N,u)\right) = \\
&\quad = \varphi^{(\bar{m})}(x_N,u) - \varphi(x_N,u) - \frac{1}{\hbar_N}\left(a_N^{(\bar{m})} - a_N\right) u_{\bar{x},N}.
\end{aligned} \tag{43}$$

From (41)-(43) we obtain

$$\left(A_h^{(\bar{m})}(x, y^{(\bar{m})}) - A_h^{(\bar{m})}(x, u), z\right)_{\hat{\bar{\omega}}_h} = \\ = \left(\left(\left(a^{(\bar{m})} - a\right) u_{\bar{x}}\right)_{\hat{x}}, z\right)_{\hat{\omega}_h} - \left(a_N^{(\bar{m})} - a_N\right) u_{\bar{x},N} z_N + \\ + \left(a_1^{(\bar{m})} - a_1\right) u_{x,0} z_0 + \left(\varphi^{(\bar{m})}(x, u) - \varphi(x, u), z\right)_{\hat{\bar{\omega}}_h}. \tag{44}$$

Using the relations (23)-(29), Cauchy-Bunyakovsky-Schwartz inequality, formula of summation by parts (see [3, p.25]), evaluate the expression on the right-hand side equality (44)

$$\left(\left(\left(a^{(\bar{m})} - a\right) u_{\bar{x}}\right)_{\hat{x}}, z\right)_{\hat{\omega}_h} - \left(a_N^{(\bar{m})} - a_N\right) u_{\bar{x},N} z_N + \left(a_1^{(\bar{m})} - a_1\right) u_{x,0} z_0 = \\ = \left(\left(a^{(\bar{m})} - a\right) \; u_{\bar{x}}, z_{\bar{x}}\right)_{\hat{\omega}_h^+} \leq \left\| a^{(\bar{m})} - a \right\|_{0,2,\hat{\omega}_h^+} \|u_{\bar{x}}\|_{0,2,\hat{\omega}_h^+} \|z_{\bar{x}}\|_{0,2,\hat{\omega}_h^+} \leq \\ \leq M\,|h|^{\bar{m}} \|z_{\bar{x}}\|_{0,2,\hat{\omega}_h^+} \leq \frac{M\,|h|^{\bar{m}}}{\tilde{c}_1} \|z\|_{B_h^{(\bar{m})}}, \tag{45}$$

$$\left(\varphi^{(\bar{m})}(x, u) - \varphi(x, u), z\right)_{\hat{\bar{\omega}}_h} \leq M\,|h|^{m+1} \|z_{\bar{x}}\|_{0,2,\hat{\omega}_h^+} \leq \\ \leq \frac{M\,|h|^{m+1}}{\tilde{c}_1} \|z\|_{B_h^{(\bar{m})}}, \tag{46}$$

if m is odd;

$$\left(\varphi^{(\bar{m})}(x, u) - \varphi(x, u), z\right)_{\hat{\bar{\omega}}_h} \leq M\,|h|^{m} \|z_{\bar{x}}\|_{0,2,\hat{\omega}_h^+} \leq \frac{M\,|h|^{m}}{\tilde{c}_1} \|z\|_{B_h^{(\bar{m})}}, \tag{47}$$

if m is even.

Taking into account the relations (40), (44)-(47) is the true estimation

$$\|z\|^2_{B_h^{(\bar{m})}} \leq \left(A_h^{(\bar{m})}(x, y^{(\bar{m})}) - A_h^{(\bar{m})}(x, u), z\right)_{\hat{\bar{\omega}}_h} \leq M\,|h|^{\bar{m}} \|z\|_{B_h^{(\bar{m})}}.$$

Hence it follows that $\|z\|_{B_h^{(\bar{m})}} \leq M\,|h|^{\bar{m}}$. So on the basis of equivalence of norms $\|\cdot\|_{1,2,\hat{\bar{\omega}}_h}$, $\|\cdot\|_{B_h^{(\bar{m})}}$, we obtain

$$\|z\|_{1,2,\hat{\bar{\omega}}_h} \leq M\,|h|^{\bar{m}}. \tag{48}$$

Due to (23), (48), (33), (34), (15)-(17) we have

$$\left| \left(k \frac{dz}{dx}\right)_0 \right| \leq \beta_1 |z_0| \leq M\,|h|^{\bar{m}}, \quad \left| \left(k \frac{dz}{dx}\right)_N \right| \leq \beta_2 |z_N| \leq M\,|h|^{\bar{m}}, \tag{49}$$

$$\left|\left(k\frac{dz}{dx}\right)_j\right| \le \left|a^{(\bar m)}_{j-1+\alpha} - a_{j-1+\alpha}\right|\left|y^{(\bar m)}_{\bar x, j-1+\alpha}\right| + |a_{j-1+\alpha}|\,|z_{\bar x, j-1+\alpha}| +$$
$$+\left|Z^{(m)j}_\alpha(x_j, y^{(\bar m)}) + (-1)^\alpha \frac{Y^{(\bar m)j}_\alpha(x_j, y^{(\bar m)}) - y^{(\bar m)}_{j+(-1)^\alpha}}{V^{(\bar m)j}_\alpha(x_j)} -\right.$$
$$\left. -Z^{(m)j}_\alpha(x_j, u) - (-1)^\alpha \frac{Y^{(\bar m)j}_\alpha(x_j, u) - u_{j+(-1)^\alpha}}{V^{(\bar m)j}_\alpha(x_j)}\right| +$$
$$+\left|Z^{(m)j}_\alpha(x_j, u) - Z^j_\alpha(x_j, u)\right| +$$
$$+\left|\frac{Y^{(\bar m)j}_\alpha(x_j, u) - u_{j+(-1)^\alpha}}{V^{(\bar m)j}_\alpha(x_j)} - \frac{Y^j_\alpha(x_j, u) - u_{j+(-1)^\alpha}}{V^j_\alpha(x_j)}\right| \le$$
$$\le M\,|h|^{\bar m} +$$
$$+\left|\Phi\left(x_{j+(-1)^\alpha}, y^{(\bar m)}_{j+(-1)^\alpha}, \left(k\frac{dy^{(\bar m)}}{dx}\right)_{j+(-1)^\alpha}, (-1)^{\alpha+1}h_{j-1+\alpha}\right) -\right.$$
$$\left. -\Phi\left(x_{j+(-1)^\alpha}, u_{j+(-1)^\alpha}, \left(k\frac{du}{dx}\right)_{j+(-1)^\alpha}, (-1)^{\alpha+1}h_{j-1+\alpha}\right)\right|,$$
$$\alpha = 1, 2, \quad j = 1, 2, ..., N-1,$$

where

$$\Phi(x, u, v, h) = v + h\Phi_2(x, u, v, h) - \frac{\Phi_1(x, u, v, h)}{\Phi_3(x, u, h)}.$$

Because

$$\Phi_1(x, u, v, 0) = \frac{v}{k(x)}, \quad \Phi_3(x, 0, 0) = \frac{1}{k(x)},$$

so using the Theorem on finite increments, we obtain

$$\Phi(x, u, v, h) = \Phi(x, u, v, 0) + h\frac{\partial \Phi(x, u, v, \bar h)}{\partial h} = h\frac{\partial \Phi(x, u, v, \bar h)}{\partial h}, \quad \bar h \in (0, h).$$

Then

$$\left|\left(k\frac{dz}{dx}\right)_j\right| \le M\,|h|^{\bar m} +$$
$$+ h_{j-1+\alpha}\left|\frac{\partial \Phi\left(x_{j+(-1)^\alpha}, y^{(\bar m)}_{j+(-1)^\alpha}, \left(k\frac{dy^{(\bar m)}}{dx}\right)_{j+(-1)^\alpha}, (-1)^{\alpha+1}\bar h_{j-1+\alpha}\right)}{\partial h} -\right.$$
$$\left. -\frac{\partial \Phi\left(x_{j+(-1)^\alpha}, u_{j+(-1)^\alpha}, \left(k\frac{du}{dx}\right)_{j+(-1)^\alpha}, (-1)^{\alpha+1}\bar h_{j-1+\alpha}\right)}{\partial h}\right| \le \qquad (50)$$

$$\leq M\,|h|^{\bar m} + h_{j-1+\alpha}\left|\frac{\partial^2\Phi(x_{j+(-1)^\alpha},\bar u_{j+(-1)^\alpha},\bar v_{j+(-1)^\alpha},\bar h_{j-1+\alpha})}{\partial h\partial u}\right|\left|z_{j+(-1)^\alpha}\right| +$$

$$+ h_{j-1+\alpha}\left|\frac{\partial^2\Phi(x_{j+(-1)^\alpha},\bar u_{j+(-1)^\alpha},\bar v_{j+(-1)^\alpha},\bar h_{j-1+\alpha})}{\partial h\partial v}\right|\left|\left(k\frac{dz}{dx}\right)_{j+(-1)^\alpha}\right| \leq$$

$$\leq M\,|h|^{\bar m} + |h|\,M_1\left|\left(k\frac{dz}{dx}\right)_{j+(-1)^\alpha}\right|, \quad \alpha = 1,2, \quad j = 1,2,...,N-1,$$

where $\bar u_j = u_j + \theta_j z_j,\ 0<\theta_j<1, \bar v_j = \left(k\frac{du}{dx}\right)_j + \eta_j\left(k\frac{dz}{dx}\right)_j,\ 0<\eta_j<1,\ j = 0,1,2,...,N$.

Consistently applying inequalities (49), (50), we obtain

$$\left|\left(k\frac{dz}{dx}\right)_j\right| \leq M\,|h|^{\bar m}, \quad j = 0,1,2,...,N.$$

Hence

$$\left\|k\frac{dz}{dx}\right\|_{0,2,\hat{\bar\omega}_h} \leq M\,|h|^{\bar m}.$$

Therefore, taking into account (48), we will have $\|z\|^*_{1,2,\hat{\bar\omega}_h} \leq M\,|h|^{\bar m}$.

For solving the nonlinear TDS order of accuracy $\bar m$ (21), (22) apply the iteration method.

Theorem 3. *Let the conditions of Theorem* 2 *are satisfied. Then*

$$\left|\varphi^{(\bar m)}(x,u) - \varphi^{(\bar m)}(x,v)\right| \leq \tilde L\,|u-v|,$$

there exist an $h_0>0$ *such that for all* $\{h_j\}_{j=1}^N$ *with* $|h|\leq h_0$,

$$0 < \tilde c_1 \leq a^{(\bar m)}(x),$$

$$\left(A_h^{(\bar m)}(x,u) - A_h^{(\bar m)}(x,v), u-v\right)_{\hat{\bar\omega}_h} \geq \|u-v\|^2_{B_h^{(\bar m)}},$$

the iteration method

$$B_h^{(\bar m)}\frac{y^{(\bar m,n)} - y^{(\bar m,n-1)}}{\tau} + A_h^{(\bar m)}(x, y^{(\bar m,n-1)}) = 0, \quad x\in\hat{\bar\omega}_h, \quad n = 1,2,..., \tag{51}$$

$$y^{(\bar m,0)}(x) = \frac{\mu_1+\mu_2+\mu_1\beta_2V_2^{(\bar m)}(x) + \mu_2\beta_1V_1^{(\bar m)}(x)}{\beta_1+\beta_2+\beta_1\beta_2V_1^{(\bar m)}(1)}, \quad x\in\hat{\bar\omega}_h,$$

$$V_1^{(\bar m)}(x_j) = \sum_{k=1}^{j} V_1^{(\bar m)k}(x_k), \quad V_2^{(\bar m)}(x_j) = \sum_{k=j+1}^{N} V_1^{(\bar m)k}(x_k)$$

with

$$\tau = \tau_0 = \left(1 + \frac{\tilde L}{\gamma_1\min\{\tilde c_1, 1\}}\right)^{-2}$$

converges and for the error we have

$$\left\|y^{(m,n)} - u\right\|^*_{1,2,\hat{\bar\omega}_h} \leq M(|h|^{\bar m} + q^n), \quad q = \sqrt{1-\tau_0}, \tag{52}$$

where the operators $B_h^{(\bar{m})}, A_h^{(\bar{m})}(x,u)$ are determined by the formulas (37), (38),

$$k(x)\frac{dy^{(\bar{m},n)}}{dx}\bigg|_{x=x_0} = \beta_1 y_0^{(\bar{m},n)} - \mu_1, \quad k(x)\frac{dy^{(\bar{m},n)}}{dx}\bigg|_{x=x_N} = -\beta_2 y_N^{(\bar{m},n)} + \mu_2,$$

$$\begin{aligned} k(x)\frac{dy^{(\bar{m},n)}}{dx}\bigg|_{x=x_j} =& a_{j-1+\alpha} y_{\bar{x},j-1+\alpha}^{(\bar{m},n)} + Z_\alpha^{(m)j}\left(x_j, y^{(\bar{m},n)}\right) \\ &+ (-1)^\alpha \frac{Y_\alpha^{(\bar{m})j}\left(x_j, y^{(\bar{m},n)}\right) - y_{j+(-1)^\alpha}^{(\bar{m},n)}}{V_\alpha^{(\bar{m})j}(x_j)}, \quad \alpha = 1,2, \\ & j = 1,2,...,N-1, \end{aligned}$$

and M is a constant independent of $|h|, m, n$.

Proof. According to Theorem 2 we have

$$\begin{aligned} \left\| y^{(\bar{m},n)} - u \right\|_{1,2,\hat{\bar{\omega}}_h}^* \leq \left\| y^{(\bar{m})} - u \right\|_{1,2,\hat{\bar{\omega}}_h}^* + \left\| y^{(\bar{m},n)} - y^{(\bar{m})} \right\|_{1,2,\hat{\bar{\omega}}_h}^* \leq \\ \leq M|h|^{\bar{m}} + \left\| y^{(\bar{m},n)} - y^{(\bar{m})} \right\|_{1,2,\hat{\bar{\omega}}_h}^* . \end{aligned} \tag{53}$$

Considering that the $f(x,u,\xi) \in \bigcup_{j=1}^{N} C^{\bar{m}}([x_{j-1}, x_i] \times R^2)$, we obtain

$$\left| \varphi^{(\bar{m})}(x,u) - \varphi^{(\bar{m})}(x,v) \right| \leq \tilde{L}|u-v|, \quad x \in \hat{\bar{\omega}}_h.$$

Using the Cauchy-Bunyakovsky-Schwartz inequality and (39) we get an estimate

$$\begin{aligned} \left(A_h^{(\bar{m})}(x,u) - A_h^{(\bar{m})}(x,v), w\right)_{\hat{\bar{\omega}}_h} \leq \|u-v\|_{B_h^{(\bar{m})}} \|w\|_{B_h^{(\bar{m})}} + \\ + \left\| \varphi^{(\bar{m})}(x,u) - \varphi^{(\bar{m})}(x,v) \right\|_{0,2,\hat{\bar{\omega}}_h} \|w\|_{0,2,\hat{\bar{\omega}}_h} \leq \\ \leq \|u-v\|_{B_h^{(\bar{m})}} \|w\|_{B_h^{(\bar{m})}} + \tilde{L}\|u-v\|_{0,2,\hat{\bar{\omega}}_h} \|w\|_{0,2,\hat{\bar{\omega}}_h} \leq \\ \leq \left(1 + \frac{\tilde{L}}{\gamma_1 \min\{\tilde{c}_1, 1\}}\right) \|u-v\|_{B_h^{(\bar{m})}} \|w\|_{B_h^{(\bar{m})}} . \end{aligned}$$

We put $w = \left(B_h^{(\bar{m})}\right)^{-1}\left(A_h^{(\bar{m})}(x,u) - A_h^{(\bar{m})}(x,v)\right)$, then

$$\begin{aligned} \left\| \left(B_h^{(\bar{m})}\right)^{-1}\left(A_h^{(\bar{m})}(x,u) - A_h^{(\bar{m})}(x,v)\right) \right\|_{B_h^{(\bar{m})}} \leq \\ \leq \left(1 + \frac{\tilde{L}}{\gamma_1 \min\{\tilde{c}_1, 1\}}\right) \|u-v\|_{B_h^{(\bar{m})}} . \end{aligned} \tag{54}$$

From (41), (54) it follows

$$\left(A_h^{(\bar{m})}(x,u) - A_h^{(\bar{m})}(x,v), \left(B_h^{(\bar{m})}\right)^{-1}\left(A_h^{(\bar{m})}(x,u) - A_h^{(\bar{m})}(x,v)\right)\right)_{\hat{\bar{\omega}}_h} \le$$

$$\le \left(1 + \frac{\tilde{L}}{\gamma_1 \min\{\tilde{c}_1, 1\}}\right)^2 \|u - v\|^2_{B_h^{(\bar{m})}} \le$$

$$\le \left(1 + \frac{\tilde{L}}{\gamma_1 \min\{\tilde{c}_1, 1\}}\right)^2 \left(A_h^{(\bar{m})}(x,u) - A_h^{(\bar{m})}(x,v), u - v\right)_{\hat{\bar{\omega}}_h}.$$

Therefore [3, p.353], the iteration method (51) converges in the space $H_{B_h^{(\bar{m})}}$. As the norms $\|\cdot\|_{1,2,\hat{\bar{\omega}}_h}$, $\|\cdot\|_{B_h^{(\bar{m})}}$ are equivalent, then the error can be estimated as

$$\left\|y^{(\bar{m},n)} - y^{(\bar{m})}\right\|_{1,2,\hat{\bar{\omega}}_h} \le M_1 q^n.$$

In addition

$$\left|\left(k\frac{dy^{(\bar{m},n)}}{dx}\right)_0 - \left(k\frac{dy^{(\bar{m})}}{dx}\right)_0\right| \le \beta_1 \left|y_0^{(\bar{m},n)} - y_0^{(\bar{m})}\right| \le$$

$$\le M_1 \left\|y^{(\bar{m},n)} - y^{(\bar{m})}\right\|_{1,2,\hat{\bar{\omega}}_h},$$

$$\left|\left(k\frac{dy^{(\bar{m},n)}}{dx}\right)_N - \left(k\frac{dy^{(\bar{m})}}{dx}\right)_N\right| \le \beta_2 \left|y_N^{(\bar{m},n)} - y_N^{(\bar{m})}\right| \le$$

$$\le M_2 \left\|y^{(\bar{m},n)} - y^{(\bar{m})}\right\|_{1,2,\hat{\bar{\omega}}_h},$$

$$\left|\left(k\frac{dy^{(\bar{m},n)}}{dx}\right)_j - \left(k\frac{dy^{(\bar{m})}}{dx}\right)_j\right| \le \left|a_{j-1+\alpha}^{(\bar{m})}\right| \left|y_{\bar{x},j-1+\alpha}^{(\bar{m},n)} - y_{\bar{x},j-1+\alpha}^{(\bar{m})}\right| +$$

$$+ \left|Z_\alpha^{(m)j}(x_j, y^{(\bar{m},n)}) - Z_\alpha^{(m)j}(x_j, y^{(\bar{m})})\right| + \frac{1}{\left|V_\alpha^{(\bar{m})j}(x_j)\right|} \left|y_{j+(-1)^\alpha}^{(\bar{m},n)} - y_{j+(-1)^\alpha}^{(\bar{m})}\right| +$$

$$+ \frac{1}{\left|V_\alpha^{(\bar{m})j}(x_j)\right|} \left|Y_\alpha^{(\bar{m})j}(x_j, y^{(\bar{m},n)}) - Y_\alpha^{(\bar{m})j}(x_j, y^{(\bar{m})})\right| \le$$

$$\le M_3 \left\|y^{(\bar{m},n)} - y^{(\bar{m})}\right\|_{1,2,\hat{\bar{\omega}}_h} +$$

$$+ \left[\left|\frac{\partial}{\partial u} Z_\alpha^{(m)j}(x_j, u)\right|_{u=\tilde{y}} + \frac{1}{\left|V_\alpha^{(\bar{m})j}(x_j)\right|} \left|\frac{\partial}{\partial u} Y_\alpha^{(\bar{m})j}(x_j, u)\right|_{u=\bar{y}}\right] \times$$

$$\times \left\|y^{(\bar{m},n)} - y^{(\bar{m})}\right\|_{0,2,\hat{\bar{\omega}}_h} \le$$

$$\le M_3 \left\|y^{(\bar{m},n)} - y^{(\bar{m})}\right\|_{1,2,\hat{\bar{\omega}}_h}, \quad j = 1, 2, ..., N,$$

$$\left\| k\frac{dy^{(\bar m,n)}}{dx} - k\frac{dy^{(\bar m)}}{dx} \right\|_{0,2,\hat{\bar\omega}_h} \le M \left\| y^{(\bar m,n)} - y^{(\bar m)} \right\|_{1,2,\hat{\bar\omega}_h}.$$

Hence we get that

$$\left\| y^{(\bar m,n)} - y^{(\bar m)} \right\|^*_{1,2,\hat{\bar\omega}_h} \le Mq^n. \tag{55}$$

From the inequality (53), (55) implies the following estimate (52).

From a practical point of view to find a solution TDS (21), (22) will eventually need to use an iteration method of Newton. Linearizing (21), (22) taking into account the equality

$$\varphi^{(\bar m)}(x_j, y^{(\bar m)}) = \hbar_j^{-1} \sum_{\alpha=1}^{2} \left[\frac{h_{j-1+\alpha}}{2} f\left(x_{j+(-1)^\alpha}, y^{(\bar m)}_{j+(-1)^\alpha}, \left.\frac{dy^{(\bar m)}}{dx}\right|_{x=x_{j+(-1)^\alpha}} \right) \right] + $$
$$+ O\left(\frac{h_j^2 + h_{j+1}^2}{\hbar_j} \right), \quad j = 1, 2, ..., N-1,$$

$$\varphi^{(\bar m)}(x_0, y^{(\bar m)}) = f\left(x_1, y_1^{(\bar m)}, \left.\frac{dy^{(\bar m)}}{dx}\right|_{x=x_1} \right) + \frac{1}{\hbar_0}\mu_1 + O(h_1),$$

$$\varphi^{(\bar m)}(x_N, y^{(\bar m)}) = f\left(x_{N-1}, y_{N-1}^{(\bar m)}, \left.\frac{dy^{(\bar m)}}{dx}\right|_{x=x_{N-1}} \right) + \frac{1}{\hbar_N}\mu_2 + O(h_N),$$

$$\left.\frac{dy^{(\bar m)}}{dx}\right|_{x=x_{j+(-1)^\alpha}} = y^{(\bar m)}_{\bar x, j-1+\alpha} + O\left(\frac{h^2_{j-1+\alpha}}{\hbar_j} \right),$$
$$j = 2-\alpha,\ 3-\alpha, ..., N+1-\alpha, \quad \alpha = 1, 2,$$

then modified Newton iteration method will be a form

$$\left(a^{(\bar m)} \nabla y_{\bar x}^{(\bar m,n)}\right)_{\hat x, j} + \frac{h_j}{2\hbar_j} \frac{\partial f\left(x_{j-1}, y_{j-1}^{(\bar m,n-1)}, \left.\frac{dy^{(\bar m,n-1)}}{dx}\right|_{x=x_{j-1}} \right)}{\partial u} \nabla y_{j-1}^{(\bar m,n)} +$$
$$+ \frac{h_{j+1}}{2\hbar_j} \frac{\partial f\left(x_{j+1}, y_{j+1}^{(\bar m,n-1)}, \left.\frac{dy^{(\bar m,n-1)}}{dx}\right|_{x=x_{j+1}} \right)}{\partial u} \nabla y_{j+1}^{(\bar m,n)} +$$
$$+ \frac{h_j}{2\hbar_j} \frac{\partial f\left(x_{j-1}, y_{j-1}^{(\bar m,n-1)}, \left.\frac{dy^{(\bar m,n-1)}}{dx}\right|_{x=x_{j-1}} \right)}{\partial \xi} \nabla y_{\bar x, j}^{(\bar m,n)} + \tag{56}$$
$$+ \frac{h_{j+1}}{2\hbar_j} \frac{\partial f\left(x_{j+1}, y_{j+1}^{(\bar m,n-1)}, \left.\frac{dy^{(\bar m,n-1)}}{dx}\right|_{x=x_{j+1}} \right)}{\partial \xi} \nabla y_{x, j}^{(\bar m,n)} =$$
$$= -\varphi^{(\bar m)}\left(x_j, y^{(\bar m,n-1)} \right) - \left(a^{(\bar m)} y_{\bar x}^{(\bar m,n-1)} \right)_{\hat x, j}, \quad j = 1, 2, ..., N-1,$$

$$\begin{gathered}\frac{1}{\hbar_0}\left(a_1^{(\bar{m},n)}\nabla y_{x,0}^{(\bar{m},n)}-\beta_1\nabla y_0^{(\bar{m},n)}\right)+\\+\frac{\partial f\left(x_1,y_1^{(\bar{m},n-1)},\left.\frac{dy^{(\bar{m},n-1)}}{dx}\right|_{x=x_1}\right)}{\partial u}\nabla y_1^{(\bar{m},n)}+\\+\frac{\partial f\left(x_1,y_1^{(\bar{m},n-1)},\left.\frac{dy^{(\bar{m},n-1)}}{dx}\right|_{x=x_1}\right)}{\partial \xi}\nabla y_{x,0}^{(\bar{m},n)}=\\=-\varphi^{(\bar{m})}\left(x_0,y^{(\bar{m},n-1)}\right)-\frac{1}{\hbar_0}\left(a_1^{(\bar{m},n)}y_{x,0}^{(\bar{m},n-1)}-\beta_1 y_0^{(\bar{m},n-1)}\right),\end{gathered}\tag{57}$$

$$\begin{gathered}-\frac{1}{\hbar_N}\left(a_N^{(\bar{m},n)}\nabla y_{\bar{x},N}^{(\bar{m},n)}+\beta_2\nabla y_N^{(\bar{m},n)}\right)+\\+\frac{\partial f\left(x_{N-1},y_{N-1}^{(\bar{m},n-1)},\left.\frac{dy^{(\bar{m},n-1)}}{dx}\right|_{x=x_{N-1}}\right)}{\partial u}\nabla y_{N-1}^{(\bar{m},n)}+\\+\frac{\partial f\left(x_{N-1},y_{N-1}^{(\bar{m},n-1)},\left.\frac{dy^{(\bar{m},n-1)}}{dx}\right|_{x=x_{N-1}}\right)}{\partial \xi}\nabla y_{\bar{x},N}^{(\bar{m},n)}=\\=-\varphi^{(\bar{m})}\left(x_N,y^{(\bar{m},n-1)}\right)+\frac{1}{\hbar_N}\left(a_N^{(\bar{m},n)}y_{\bar{x},N}^{(\bar{m},n-1)}+\beta_2 y_N^{(\bar{m},n-1)}\right),\end{gathered}\tag{58}$$

$$y_j^{(\bar{m},n)}=y_j^{(\bar{m},n-1)}+\nabla y_j^{(\bar{m},n)},\quad j=1,2,...,N-1,\quad n=1,2,...\tag{59}$$

4. Numerical examples

Example 1. Let us consider boundary value problem

$$\begin{gathered}\frac{d^2u}{dx^2}=\pi^2\exp(u),\quad x\in(0,1),\\ \frac{du(0)}{dx}-u(0)=-\frac{\pi}{\sqrt{3}}+\ln 1,5,\quad -\frac{du(1)}{dx}-u(1)=-\sqrt{3}\pi-\ln 2,\end{gathered}\tag{60}$$

with the exact solution

$$u(x)=-\ln\left(2\cos^2\left(\frac{\pi}{2}\left(x-\frac{1}{3}\right)\right)\right).$$

Since $f(x,u,\xi)=-\pi^2\exp(u)$ it follows that condition (5) is satisfied if we take $f_0(x)\equiv 0, c(t)=\pi^2\exp(t),\ g(x)\equiv 1$. Besides we have

$$[f(x,u,\xi)-f(x,v,\eta)]\,(u-v)=-\pi^2\exp(\theta u+(1-\theta)v)(u-v)^2\le 0,\quad 0<\theta<1.$$

Thus, due to Theorem 1 the problem has a unique solution.

For numerical solution of problem (60) on the equidistance grid $\bar{\omega}_h = \{x_j = jh, j = 0, 1, ..., N, h = 1/N\}$ we use TDS of the sixth order of accuracy ($m = 6$)

$$\begin{aligned} &y^{(6)}_{\bar{x}x,j} = -\varphi^{(6)}(x_j, y^{(6)}), \quad j = 1, 2, \ldots, N-1, \\ &\frac{2}{h}\left(y^{(6)}_{x,0} - \beta_1 y^{(6)}_0\right) = -\varphi^{(6)}(x_0, y^{(6)}), \\ &-\frac{2}{h}\left(y^{(6)}_{\bar{x},N} + \beta_2 y^{(6)}_N\right) = -\varphi^{(6)}(x_N, y^{(6)}), \end{aligned} \tag{61}$$

with

$$\varphi^{(6)}(x_j, u) = h^{-1}\sum_{\alpha=1}^{2}(-1)^{\alpha}\left[Z^{(6)j}_{\alpha}(x_j, u) + (-1)^{\alpha}\frac{Y^{(6)j}_{\alpha}(x_j, u) - u_{j+(-1)^{\alpha}}}{h}\right],$$

$$\varphi^{(6)}(x_0, u) = \frac{2}{h}\left[Z^{(6)0}_2(x_0, u) + \frac{Y^{(6)0}_2(x_0, u) - u_1}{h} + \mu_1\right],$$

$$\varphi^{(6)}(x_N, u) = \frac{2}{h}\left[-Z^{(6)N}_1(x_N, u) + \frac{Y^{(6)N}_1(x_N, u) - u_{N-1}}{h} + \mu_2\right],$$

$$\beta_1 = 1, \quad \beta_2 = 1, \quad \mu_1 = \frac{\pi}{\sqrt{3}} - \ln 1,5, \quad \mu_2 = \sqrt{3}\pi + \ln 2,$$

and $Y^{(6)j}_{\alpha}(x, u), Z^{(6)j}_{\alpha}(x, u)$ are numerical solutions of initial value problems

$$\begin{aligned} &\frac{dY^j_{\alpha}(x, u)}{dx} = Z^j_{\alpha}(x, u), \quad \frac{dZ^j_{\alpha}(x, u)}{dx} = -f\left(x, Y^j_{\alpha}(x, u), Z^j_{\alpha}(x, u)\right), \\ &\qquad x_{j-2+\alpha} < x < x_{j-2+\alpha}, \\ &Y^j_{\alpha}\left(x_{j+(-1)^{\alpha}}, u\right) = u_{j+(-1)^{\alpha}}, \quad Z^j_{\alpha}\left(x_{j+(-1)^{\alpha}}, u\right) = \left.\frac{du}{dx}\right|_{x=x_{j+(-1)^{\alpha}}}, \end{aligned} \tag{62}$$

$$j = 2-\alpha, 3-\alpha, ..., N+1-\alpha, \quad \alpha = 1, 2$$

computed by a explicit Runge-Kutta method of the sixth-order of accuracy (see Table 6.1 [10, p.189]).

To determine the solution of the difference scheme (61) the modified Newton method (56)-(59) will be used. System linear algebraic equations (56)-(58) for the unknowns $\nabla y^{(6,n)}(x)$, $x \in \hat{\omega}_h$ we solved by Gaussian elimination for linear system with a tridiagonal matrix.

Numerical results are given in Table 1. To evaluate the convergence rate in practice, we introduced the following quantities

$$er = \left\|z^{(6)}\right\|^*_{1,2,\bar{\omega}_h} = \left\|y^{(6)} - u\right\|^*_{1,2,\bar{\omega}_h}, \quad p = \log_2\frac{\left\|z^{(6)}\right\|^*_{1,2,\bar{\omega}_h}}{\left\|z^{(6)}\right\|^*_{1,2,\bar{\omega}_{h/2}}}.$$

In the following example the implementation of the TDS uses the $h - h/2$ a posteriori estimation to achieve a given accuracy EPS. The comparison with the true error Er shows that this accuracy is actually achieved.

Tabl. 1. Numerical results for problem (60).

N	***Er***	***p***
16	$0,2241 \cdot 10^{-5}$	
32	$0,3522 \cdot 10^{-7}$	6
64	$0,5514 \cdot 10^{-9}$	6
128	$0,8642 \cdot 10^{-11}$	6

Example 2. Let us consider the boundary value problem

$$\begin{aligned} \frac{d^2u}{dx^2} &= 3u\frac{du}{dx}, \quad x \in (0,1), \\ \frac{du(0)}{dx} &= -1,5/\cosh^2(0,75), \\ -\frac{du(1)}{dx} - u(1) &= 1,5/\cosh^2(0,75) + \tanh(0,75). \end{aligned} \tag{63}$$

The exact solution is $u(x) = \tanh\left(\frac{3(1-2x)}{4}\right)$.

The numerical results which have been obtained for difference scheme of order of accuracy 6 are given in Table 2

Tabl. 2. Numerical results for problem (63).

EPS	***N***	***Er***
10^{-4}	2048	$0,1323 \cdot 10^{-5}$
10^{-6}	2048	$0,4816 \cdot 10^{-7}$
10^{-8}	4096	$0,4078 \cdot 10^{-9}$

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Marta Król, Myroslav Kutniv,
Rzeszow University of Technology,
8, Powstancow Warszawy str., 35-959, Rzeszow, Polska

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