Журнал обчислювальної та прикладної математики | 2014 № 2 (116) | Journal of Numerical & Applied Mathematics

UDC 519.6

ON THE BOUNDARY INTEGRAL EQUATIONS METHOD FOR EXTERIOR BOUNDARY VALUE PROBLEMS FOR INFINITE SYSTEMS OF ELLIPTIC EQUATIONS OF SPECIAL KIND

Yuriy Muzychuk

Резюме. В тривимірних обмежених областях з ліпшицевою межею розглянуто зовнішні граничні задачі для нескінченних систем еліптичних рівнянь спеціального трикутного вигляду зі змінними коефіцієнтами. Сформульовано варіаційні постановки задач Діріхле, Неймана та Робіна та встановлено їхню коректність у відповідних просторах Соболєва. За допомогою введеного поняття q-згортки отримано аналоги першої та другої формул Гріна та побудовано інтегральні зображення розв'язків розглянутих задач у випадку сталих коефіцієнтів. Досліджено властивості інтегральних операторів та коректність отриманих систем граничних інтегральних рівнянь.

Abstract. Boundary value problems for infinite triangular systems of elliptic equations with variable coefficients are considered in exterior 3d Lipschitz domains. Variational formulations of Dirichlet, Neumann and Robin problems are received and their well-posedness in corresponding Sobolev spaces is established. Via the introduced q-convolution the analogues of the first and the second Green's formulae are obtained and integral representations of the generalized solutions for formulated problems in the case of constant coefficients are built. We investigate the properties of integral operators and well-posedness of received systems of boundary integral equations.

1. Introduction

The method of boundary integral equations (BIEs) can be applied to a wide class of boundary value problems (BVPs) for elliptic partial differential equations (PDEs). Theoretical aspects of this method have been well investigated in the literature, see, e.g.,[1, 2], and the references therein. The main advantage of the BIEs method is the reduction by one of the dimension of the problem by switching to unknown functions that are defined only on the domain's boundary. It is particularly suited for exterior problems in unbounded domains. Numerous engineering applications confirm the efficiency of this method.

In the case of initial-boundary value problems for evolution equations, the BIEs method can be used both for the BVP investigations and for their effective numerical solution, see, e.g.,[3, 4, 5, 6]. But since the time and space variables are intertwined in the kernel of boundary integral operators it makes the application of this method more complicated. Therefore when solving the

Key words. Boundary value problems; boundary integral equations; elliptic equation; infinite system; variational formulation.

BIEs that depend on the time and space variables, besides the Galerkin or collocation methods, specialized approaches for consideration of the time variable are used. Such composite methods have been studied in the works cited above. They have certain characteristics that define the features of the algorithm implementation. For instance, usage of the so-called Convolution Quadrature method [5] or the Laguerre transform of the time variable [7, 8] leads to solving sequences of BIEs.

In [9] the BIEs method was used for finding solutions of interior BVPs for infinite triangular systems which one could obtain from evolution equations by the Laguerre transform in the time domain. The idea of this method lies in the generalization of the concept of the potential on infinite sequences of functions that depend only on the space variables. Herewith the convolutions of the Cauchy data of the unknown solution with fundamental solution of the infinite system and its normal derivative are used. Application of such convolution of the infinite sequences to the particular problem leads to a sequence of BIEs that has he same operator of the left-hand side and the expression in the right-hand side contains solutions of the previous BIEs. In this paper we extend this approach for exterior problems.

Traditionally the BIEs method is used for BVPs with constant coefficients, since in case of variable coefficients PDE's fundamental solutions are generally not explicitly available. Still on the stage of investigation of the well-posedness of BVPs we will consider a system with variable coefficients. Note that such problems can be treated as some generalization of BVPs that arise as a result of the application of the Laguerre transform to the non-stationary problems.

The paper is organized as follows. In Section 2 we formulate the Dirichlet, Neumann and Robin BVPs for some kind of infinite triangular system consisting of elliptic PDEs with variable coefficients. We consider these problems in appropriate Sobolev spaces and show their well-posedness. Then we introduce the notion of sequences and a new operation on them - the q-convolution of sequences. In this section we also consider variational formulations of the corresponding BVPs and arrive at the analogues of the first and the second Green's formulae. In Section 4 we obtain the integral representation of the solution of the BVPs with constant coefficients and establish a relationship between the Cauchy data of some generalized solution and corresponding BIEs which we study in the following Section 5.

2. Formulation of the BVPs and basic relations

Let $\Omega \subset \mathbb{R}^3$ be a bounded and simply connected domain with Lipschitz boundary Γ and $\Omega^+ := \mathbb{R}^3 \setminus \overline{\Omega}$ be an exterior domain. We consider an infinite system in Ω^+

$$\begin{cases} Pu_0 = f_0 \ , \\ c_{1,0}u_0 + Pu_1 = f_1 \ , \\ c_{2,0}u_0 + c_{2,1}u_1 + Pu_2 = f_2 \ , \\ \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \\ c_{k,0}u_0 + c_{k,1}u_1 + ... + c_{k,k-1}u_{k-1} + Pu_k = f_k \ , \\ \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \end{cases} \tag{1}$$

where $u_0, u_1, \ ..., \ u_k, \ ...$ are unknown functions, $c_{i,j}$ $(i, j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$ are some given measurable and bounded in Ω^+ functions with $c_{i,j} = 0$ when $j \geq i$; f_i $(i \in \mathbb{N}_0)$ are given in Ω^+ functions (functionals). In a formal second order differential operator

$$(Pu)(x) := -\sum_{i,j=1}^{3} \frac{\partial}{\partial x_j}\left[a_{i,j}(x)\frac{\partial u(x)}{\partial x_i}\right] + a_0(x)u(x), \quad x \in \Omega^+, \tag{2}$$

the functions $a_{i,j}$ $(i, j = 1, 2, 3)$ and a_0 are measurable and bounded and satisfy the conditions:

$$a_{i,j}(x) = a_{j,i}(x) \ \ (i, j = 1, 2, 3) \text{ for almost all } x \in \Omega^+,$$

$$\sum_{i,j=1}^{3} a_{i,j}(x)\xi_i\xi_j \geq \alpha \sum_{i=1}^{3} \xi_i^2 \ \text{ for arbitrary } \xi_1, \xi_2, \xi_3 \in \mathbb{R} \text{ and almost all } x \in \Omega^+, \tag{3}$$

with some constant $\alpha > 0$ and

$$a_0(x) > 0 \text{ for almost all } x \in \Omega^+. \tag{4}$$

Let the unit normal vector $\bar{\nu}(x) = (\nu_1(x), \nu_2(x), \nu_3(x))$ to Γ be directed outwards of Ω^+. We investigate BVPs for system (1) that consist in finding its solutions that satisfy one of the following conditions on the boundary Γ

(i) Dirichlet condition:

$$u_k|_\Gamma = \tilde{h}_k, \ k \in \mathbb{N}_0, \tag{5}$$

(ii) Neumann condition:

$$\partial_{\bar{\nu}} u_k|_\Gamma = \tilde{g}_k, \ k \in \mathbb{N}_0, \tag{6}$$

(iii) Robin condition:

$$\left(\partial_{\bar{\nu}} u_k - (b_{k,0}u_0 + b_{k,1}u_1 + ... + b_{k,k-1}u_{k-1} + b_{k,k}u_k)\right)|_\Gamma = \tilde{g}_k, \ k \in \mathbb{N}_0, \tag{7}$$

where $\tilde{h}_i, \ \tilde{g}_i$ $(i \in \mathbb{N}_0)$ are given functions (functionals) on Γ, $b_{i,j} \in L^\infty(\Gamma)$ $(i, j \in \mathbb{N}_0)$ are given functions on Γ with $b_{i,j} = 0$ when $j > i \geq 0$, $b_{i,i} \geq \tilde{b}_i > 0$, $\tilde{b}_i$ – constants. In other words, we will consider the Dirichlet problem (1), (5), the Neumann problem (1), (6) and the Robin problem (1), (7).

Note that the triangular form of system (1) allows us to consequently find the unknown functions $u_k, \ k \in \mathbb{N}_0$. This way when solving the k-th equation ($k \geq 1$) we assume that all solutions $u_i, \ 0 \leq i \leq k-1$, have been found on previous steps and move them to the right hand side of the equation. For instance, we will use this approach for the investigation of the well-posedness of the previously mentioned BVPs. But it isn't suitable for their numerical solution with usage of potentials since it requires additional calculation of volume potentials for combinations of functions $u_i, \ 0 \leq i \leq k-1$, found on previous steps. The method introduced in [9] regarding the interior problems for system (1) allows us to avoid this and build an efficient algorithm for their numerical solution.

We will use the Lebesgue space $L_2(\Omega^+)$ and Sobolev spaces $H^1(\Omega^+)$ and $H_0^1(\Omega^+)$ of real-valued scalar functions and dual to them $\widetilde{H}^{-1}(\Omega^+) :=$

$\left(H^1(\Omega^+)\right)'$ and $H^{-1}(\Omega^+) := \left(H_0^1(\Omega^+)\right)'$, correspondingly. Under $\mathcal{D}(\Omega^+)$ and $\mathcal{D}'(\Omega^+)$ we will understand the spaces of all test functions and distributions on them.

The following bilinear form

$$a_{\Omega^+}(u,v) := \int_{\Omega^+} \left[\sum_{i,j=1}^{3} a_{i,j}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_j} + a_0(x) u(x) v(x) \right] dx \tag{8}$$

is well defined for any functions $u, v \in H^1(\Omega^+)$. It is known, see, e.g.,[10] and [3, 6] for the case of constant coefficients, one can consider $a_{\Omega^+}(\cdot,\cdot)$ as an inner product and introduce in $H^1(\Omega^+)$ a new norm $|||u||| := (a_{\Omega^+}(u,u))^{1/2}$, which is equivalent to the usual one under the conditions (3) and (4). It is obvious that this form is $H^1(\Omega^+)$-elliptic.

In $H^1(\Omega^+)$ we will consider the following subspace

$$H^1(\Omega^+, P) := \left\{ \ u \in H^1(\Omega^+) \ \middle| \ Pu \in L_2(\Omega^+) \ \right\}, \tag{9}$$

equipped with the norm

$$||u||_{H^1(\Omega^+,P)} := \left(\ ||u||^2_{H^1(\Omega^+)} + ||Pu||^2_{L_2(\Omega^+)} \ \right)^{1/2}. \tag{10}$$

Let $\gamma_0^+ : H^1(\Omega^+) \to H^{1/2}(\Gamma)$ be the trace operator and $\gamma_1^+ : H^1(\Omega^+, P) \to H^{-1/2}(\Gamma)$ be the conormal derivative operator, which coincides with the conormal derivative

$$\partial_{\bar{\nu}} u(x) := \sum_{i,j=1}^{3} a_{i,j}(x) \frac{\partial u(x)}{\partial x_i} \nu_j(x), \ x \in \Gamma$$

in case of functions from $H^2(\Omega^+)$, a sufficiently smooth boundary Γ and continuous on $\overline{\Omega^+}$ coefficients $a_{i,j}$ $(i,j = 1,2,3)$. It is known ([1], Theorem 4.4), that for functions $u \in H^1(\Omega^+, P)$ and $v \in H^1(\Omega^+)$ the first Green's formula holds

$$(Pu, v)_{\Omega^+} = a_{\Omega^+}(u,v) + \langle \gamma_1^+ u, \gamma_0^+ v \rangle_\Gamma. \tag{11}$$

where $(\cdot,\cdot)_{\Omega^+}$ and $\langle \cdot,\cdot \rangle_\Gamma$ denote the $L_2(\Omega^+)$ the inner product and the duality between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, correspondingly. If $u \in H^1(\Omega^+)$ then the form $a_{\Omega^+}(\cdot,\cdot)$ can also be used for the definition of $Pu \in H_0^{-1}(\Omega^+)$

$$\langle Pu, v \rangle_{\Omega^+,1,0} := a_{\Omega^+}(u,v), \ \forall v \in H_0^1(\Omega^+). \tag{12}$$

Here $\langle \cdot,\cdot \rangle_{\Omega^+,1,0}$ denotes the duality between $H^{-1}(\Omega^+)$ and $H_0^1(\Omega^+)$.

Let X be an arbitrary linear space over the field of real numbers, $\mathbb{Z}$ – the set of integers. By X^∞ we denote a linear space of mappings $\mathbf{u} : \mathbb{Z} \to X$ satisfying $u(k) = 0$ when $k < 0$. For any element $\mathbf{u} \in X^\infty$ we have $u_k \equiv (\mathbf{u})_k := \mathbf{u}(k), k \in \mathbb{Z}$, and will write it as $\mathbf{u} := (u_0, u_1, ..., u_k, ...)^\top$. Henceforth we will call elements of X^∞ sequences.

We will use triangular matrix operators $\mathbf{C} : (L_2(\Omega^+))^\infty \to (L_2(\Omega^+))^\infty$ and $\mathbf{B} : (L_2(\Gamma))^\infty \to (L_2(\Gamma))^\infty$ that act as $(\mathbf{Cu})_k = \sum_{l=0}^{k} c_{k,l} \cdot (\mathbf{u})_l$, $k \in \mathbb{N}_0$, and $(\mathbf{Bu})_k = \sum_{l=0}^{k} b_{k,l} \cdot (\mathbf{u})_l$, $k \in \mathbb{N}_0$, where $c_{k,l}$ and $b_{k,l}$ are the coefficients of the system (1) and of the Robin boundary condition (7), correspondingly.

The following denotations of sequences are used

$$\mathbf{a}_{\Omega^+}(\mathbf{u},\mathbf{v}) := (a_{\Omega^+}(u_0,v_0),\ a_{\Omega^+}(u_1,v_1),\ ...)^\top,\ \ \mathbf{u},\mathbf{v} \in (H^1(\Omega^+))^\infty,$$

and

$$(\mathbf{u},\mathbf{v})_X := ((u_0,v_0)_X,\ (u_1,v_1)_X,\ ...)^\top,\ \ \mathbf{u},\mathbf{v} \in (X)^\infty,$$

where X is some Hilbert space. In the same manner we will denote sequences for duality pairing. For example, if $\mathbf{u} \in H^{-1/2}(\Gamma)$ and $\mathbf{v} \in H^{1/2}(\Gamma)$ we will use the notation $\langle \mathbf{u},\mathbf{v}\rangle_\Gamma := (\langle u_0,v_0\rangle_\Gamma, \langle u_1,v_1\rangle_\Gamma, ...)^\top$. Analogously, linear functionals on sequences will be treated as component-wise. For the sequence $\mathbf{u} \in (H^1(\Omega^+))^\infty$ we introduce the definition of an exterior trace as a sequence of traces of its components, i.e. $\gamma_0^+\mathbf{u} := (\gamma_0^+u_0, \gamma_0^+u_1, ...)^\top$ will be called an exterior trace of the sequence $\mathbf{u}$ on the surface Γ. If $\mathbf{u} \in (H^1(\Omega^+,P))^\infty$ the sequence $\gamma_1^+\mathbf{u} := (\gamma_1^+u_0, \gamma_1^+u_1, ...)^\top$ will denote an exterior conormal derivative of the sequence $\mathbf{u}$ on the domain's boundary.

Taking into account previous definitions, generalized solutions of the Dirichlet, Neumann and Robin BVPs for system (1) can be defined in the following way.

Definition 1. Let $\mathbf{f} \in (H^{-1}(\Omega))^\infty$ and $\tilde{\mathbf{h}} \in (H^{1/2}(\Gamma))^\infty$. Sequence $\mathbf{u} \in (H^1(\Omega^+))^\infty$ is called a *generalized solution of the Dirichlet problem* (1), (5) if it satisfies the variational equality

$$\mathbf{a}_{\Omega^+}(\mathbf{u},\mathbf{v}) + (\mathbf{C}\mathbf{u},\mathbf{v})_{\Omega^+} = \langle \mathbf{f},\mathbf{v}\rangle_{\Omega^+,1,0},\ \ \forall \mathbf{v} \in (H_0^1(\Omega^+))^\infty, \tag{13}$$

and the boundary condition

$$\gamma_0^+\mathbf{u} = \tilde{\mathbf{h}} \quad \text{on } \Gamma. \tag{14}$$

Definition 2. Let $\mathbf{f} \in (\widetilde{H}^{-1}(\Omega))^\infty$ and $\tilde{\mathbf{g}} \in (H^{-1/2}(\Gamma))^\infty$. Sequence $\mathbf{u} \in (H^1(\Omega^+))^\infty$ is called a *generalized solution of the Neumann problem* (1), (6) if it satisfies the variational equality

$$\mathbf{a}_{\Omega^+}(\mathbf{u},\mathbf{v}) + (\mathbf{C}\mathbf{u},\mathbf{v})_{\Omega^+} = \langle \mathbf{f},\mathbf{v}\rangle_{\Omega^+,1} - \langle \tilde{\mathbf{g}},\gamma_0^+\mathbf{v}\rangle_\Gamma,\ \ \forall \mathbf{v} \in (H^1(\Omega^+))^\infty. \tag{15}$$

Here $\langle\cdot,\cdot\rangle_{\Omega^+,1}$ denotes the duality between $\widetilde{H}^{-1}(\Omega^+)$ and $H^1(\Omega^+)$.

Definition 3. Let $\mathbf{f} \in (\widetilde{H}^{-1}(\Omega))^\infty$ and $\tilde{\mathbf{g}} \in (H^{-1/2}(\Gamma))^\infty$. Sequence $\mathbf{u} \in (H^1(\Omega^+))^\infty$ is called a *generalized solution of the Robin problem* (1), (7) if it satisfies the variational equality

$$\begin{aligned} &\mathbf{a}_{\Omega^+}(\mathbf{u},\mathbf{v}) + (\mathbf{C}\mathbf{u},\mathbf{v})_{\Omega^+} + \langle \mathbf{B}\gamma_0^+\mathbf{u},\gamma_0^+\mathbf{v}\rangle_\Gamma = \\ &\qquad = \langle \mathbf{f},\mathbf{v}\rangle_{\Omega^+,1} - \langle \tilde{\mathbf{g}},\gamma_0^+\mathbf{v}\rangle_\Gamma,\ \ \forall \mathbf{v} \in (H^1(\Omega^+))^\infty. \end{aligned} \tag{16}$$

Theorem 1. *The Dirichlet boundary value problem* (1), (5) *has a unique generalized solution.*

Proof. The triangular form of the system (13) gives us opportunity to consider its equations one after another and apply the same standard procedure for investigation of variational equations (see, e.g. [2]) on each step of the proof. Let's start with the first equation:

$$a_{\Omega^+}(u_0,v) = \langle f_0,v\rangle_{\Omega^+,1,0},\ \ \forall v \in H_0^1(\Omega^+).$$

According to the trace theorem for each function $\tilde{h}_k \in H^{1/2}(\Gamma)$ there exists a (non-unique) element $\tilde{u}_k \in H^1(\Omega^+)$ that $\gamma_0^+ \tilde{u}_k = \tilde{h}_k$. Therefore, we can obtain the following variational equation for the difference $u_0 - \tilde{u}_0 =: w \in H_0^1(\Omega^+)$

$$a_{\Omega^+}(w, v) = \langle \tilde{f}_0, v \rangle_{\Omega^+,1,0} := \langle f_0, v \rangle_{\Omega^+,1,0} - a_{\Omega^+}(\tilde{u}_0, v), \quad \forall v \in H_0^1(\Omega^+). \tag{17}$$

Due to the $H^1(\Omega^+)$-ellipticity of the bilinear form and the boundedness of the functional $\tilde{f}_0$ on $H_0^1(\Omega^+)$ according to the Lax-Milgram theorem this equation has a unique solution $w \in H_0^1(\Omega^+)$. This proves existence of the unique function $u_0 \in H^1(\Omega^+)$ that is a generalized solution of the first problem.

When considering the second variational equation we move the function u_0 into the right hand side of the corresponding equation and for the difference $u_1 - \tilde{u}_1 =: w \in H_0^1(\Omega^+)$ we arrive at the variational equation that differs from (17) only by the right hand side. Therefore, by using the previous considerations we prove the assertion of the theorem for the solution u_1. Obviously, acting this way on each succeeding step we will obtain the variational equation (17) with the following right hand side

$$\langle \tilde{f}_k, v \rangle_{\Omega^+,1,0} := \langle f_k, v \rangle_{\Omega^+,1,0} - \sum_{i=0}^{k-1} (c_{k,i} \tilde{u}_i, v)_{\Omega^+} - a_{\Omega^+}(\tilde{u}_k, v),$$
$$\forall v \in H_0^1(\Omega^+), \quad k \in \mathbb{N}.$$

Here $u_i \, (i = \overline{0, k-1})$ are generalized solutions of the problems considered on the previous steps. As can be seen $\tilde{f}_k \in H^{-1}(\Omega^+)$. Hence, there exists a unique generalized solution of the current BVP. Therefore, for each BVP with an arbitrary index $k \in \mathbb{N}$ the generalized solution $u_k \in H^1(\Omega^+)$ exists and is unique. □

Theorem 2. *The Robin boundary value problem* (1), (7) *has a unique generalized solution.*

Proof. Let's consider the first equation of system (16):

$$a_{\Omega^+}(u_0, v) + b_{\Gamma,0}(u_0, v) = \langle f_0, v \rangle_{\Omega^+,1} - \langle \tilde{g}_0, \gamma_0^+ v \rangle_\Gamma, \quad \forall v \in H^1(\Omega^+). \tag{18}$$

Here the bilinear form $b_{\Gamma,k}(\cdot, \cdot)$ $(k \in \mathbb{N}_0)$ is expressed through traces of elements of space $H^1(\Omega^+)$ on the boundary Γ:

$$b_{\Gamma,k}(u, v) = \int_\Gamma b_{k,k}(x) \gamma_0^+ u(x) \gamma_0^+ v(x) dS_x, \; u, v \in H^1(\Omega^+).$$

As long as $b_{k,k} \in L^\infty(\Gamma)$ and $\gamma_0 u, \gamma_0 v \in H^{1/2}(\Gamma) \subset L_2(\Gamma)$, such integral exists. Expression

$$\tilde{a}_{\Omega^+}(u, v) := a_{\Omega^+}(u, v) + b_{\Gamma,0}(u, v), \quad u, v \in H^1(\Omega^+), \tag{19}$$

can be treated as some bilinear form for $u, v \in H^1(\Omega^+)$. Obviously, it is $H^1(\Omega^+)$-elliptic.

On the other hand, taking into account the estimate

$$|\langle \tilde{g}_0, \gamma_0^+ v \rangle_\Gamma| \le ||\tilde{g}_0||_{H^{-1/2}(\Gamma)} ||\gamma_0^+ v||_{H^{1/2}(\Gamma)} \le C ||\tilde{g}_0||_{H^{-1/2}(\Gamma)} ||v||_{H^1(\Omega^+)}$$

the functional

$$\langle \tilde{f}_0, v\rangle_{\Omega^+,1} := \langle f_0, v\rangle_{\Omega^+,1} - \langle \tilde{g}_0, \gamma_0^+ v\rangle_\Gamma$$

is an element of $\widetilde{H}^{-1}(\Omega^+)$. Then, according to the Lax-Milgram theorem there exists a unique solution $u_0 \in H^1(\Omega^+)$ of the equation (18).

Next we follow the scheme, used in the proof of the previous theorem. Let's consider the equation with an arbitrary index $k \in \mathbb{N}$. After moving all items that contain functions u_i ($i = \overline{0, k-1}$) into the right hand side, this equation takes the form:

$$a_{\Omega^+}(u_k, v) + b_{\Gamma,k}(u_k, v) = \langle \tilde{f}_k, v\rangle_{\Omega^+,1}, \quad \forall v \in H^1(\Omega^+), \quad k \in \mathbb{N}, \tag{20}$$

where

$$\langle \tilde{f}_k, v\rangle_{\Omega^+,1} := \langle f_k, v\rangle_{\Omega^+,1} - \langle \tilde{g}_k, \gamma_0^+ v\rangle_\Gamma - \sum_{i=0}^{k-1} (c_{k,i} u_i, v)_{\Omega^+} - \sum_{i=0}^{k-1} \langle b_{k,i} \gamma_0^+ u_i, \gamma_0^+ v\rangle_\Gamma.$$

Clearly, $\tilde{f}_k \in \widetilde{H}^{-1}(\Omega^+)$. Since the obtained variational equation differs from (18) only in the right hand side, we arrive at the conclusion that there exists its unique solution $u_k \in H^1(\Omega^+)$. Thus we've shown the existence and the uniqueness of each component of the solution of variational system (16). □

As a conclusion of the previous theorem we obtain

Theorem 3. *The Neumann boundary value problem* (1), (6) *has a unique generalized solution.*

Note that condition (4) is a characteristic feature of PDEs obtained from the evolution equations by means of the Laguerre transform. Without such constraint the bilinear form will be just coercive. In this case the existence and the uniqueness of the solutions of BVPs for system (1) can be investigated according to the Fredholm theory, see, e.g., [1, 2], or by considering the variational formulations in corresponding weighted Sobolev spaces [11].

We shall now use the well known procedure (see, e.g. [12], chapter 7) to transform variational problems to the equivalent ones in the operator form. We first consider the variational equation (13) and suppose that the sequence $\mathbf{u} \in (H^1(\Omega^+))^\infty$ is its solution. Bearing in mind (12), we can rewrite it in the following way:

$$\langle \mathbf{Pu}, \mathbf{v}\rangle_{\Omega^+,1,0} + (\mathbf{Cu}, \mathbf{v})_{\Omega^+} = \langle \mathbf{f}, \mathbf{v}\rangle_{\Omega^+,1,0}, \quad \forall \mathbf{v} \in (H_0^1(\Omega^+))^\infty, \tag{21}$$

where the matrix operator $\mathbf{P}$ acts on $\forall \mathbf{u} \in (H^1(\Omega^+))^\infty$ by the rule:

$$(\mathbf{Pu})_k = Pu_k, \quad k \in \mathbb{N}_0.$$

Taking into account the embedding of spaces $H_0^1(\Omega^+) \subset L_2(\Omega^+) \subset H^{-1}(\Omega^+)$, the equality (21) may be presented as

$$\langle \mathbf{Pu}, \mathbf{v}\rangle_{\Omega^+,1,0} + \langle \mathbf{Cu}, \mathbf{v}\rangle_{\Omega^+,1,0} = \langle \mathbf{f}, \mathbf{v}\rangle_{\Omega^+,1,0}, \quad \forall \mathbf{v} \in (H_0^1(\Omega^+))^\infty.$$

After introducing the notation

$$\mathbf{G} := \mathbf{P} + \mathbf{C}, \tag{22}$$

the previous equality can be given in the form of the operator equation

$$\mathbf{Gu} = \mathbf{f} \quad \text{in} \quad (H^{-1}(\Omega^+))^\infty. \tag{23}$$

Thus, the generalized solution of the Dirichlet problem (1), (5) is the solution of the operator equation (23) and satisfies the same boundary condition (5) or its sequence analogue (14). And vice versa, it is easy to see, that the solution of (23), (14) is a generalized solution of the Dirichlet problem (1), (5).

In order to get the operator equation for the Neumann and the Robin problems we will use the Green's formula in the form of (11) instead of (12). We will consider the generalized solutions in space $(H^1(\Omega^+, P))^\infty$ and assume $\mathbf{f} \in (L_2(\Omega^+))^\infty$. Thus, let the sequence $\mathbf{u} \in (H^1(\Omega^+, P))^\infty$ be the generalized solution of the Robin problem (1), (7) i.e. it satisfies the variational equation (16). If we apply the formula (11) to this equation, we get

$$(\mathbf{Pu}, \mathbf{v})_{\Omega^+} - \langle \gamma_1^+ \mathbf{u}, \gamma_0^+ \mathbf{v} \rangle_\Gamma + (\mathbf{Cu}, \mathbf{v})_{\Omega^+} + \\ + \langle \mathbf{B}\gamma_0^+ \mathbf{u}, \gamma_0^+ \mathbf{v} \rangle_\Gamma = (\mathbf{f}, \mathbf{v})_{\Omega^+} - \langle \tilde{\mathbf{g}}, \gamma_0^+ \mathbf{v} \rangle_\Gamma,$$

or

$$(\mathbf{Gu} - \mathbf{f}, \mathbf{v})_{\Omega^+} + \langle \mathbf{B}\gamma_0^+ \mathbf{u} - \gamma_1^+ \mathbf{u} + \tilde{\mathbf{g}}, \gamma_0^+ \mathbf{v} \rangle_\Gamma = 0, \ \ \forall \mathbf{v} \in (H^1(\Omega^+))^\infty. \tag{24}$$

After substitution of an arbitrary element $\mathbf{v} \in (\mathcal{D}(\Omega^+))^\infty$ into (24) we come to the following equality

$$\langle \mathbf{Gu} - \mathbf{f}, \mathbf{v} \rangle_{\Omega^+} = 0,$$

where $\langle \cdot, \cdot \rangle_{\Omega^+}$ is based on the duality between $\mathcal{D}'(\Omega^+)$ and $\mathcal{D}(\Omega^+)$. Thus,

$$\mathbf{Gu} = \mathbf{f} \ \ \text{in} \ (\mathcal{D}'(\Omega^+))^\infty.$$

Since $\mathbf{f} \in (L_2(\Omega^+))^\infty$, the previous equation can be understood as

$$\mathbf{Gu} = \mathbf{f} \ \ \text{in} \ (L_2(\Omega^+))^\infty. \tag{25}$$

Therefore, after substitution of any sequence $\mathbf{v} \in (H^1(\Omega^+))^\infty$ into (24) we arrive at the relation

$$\langle \mathbf{B}\gamma_0^+ \mathbf{u} - \gamma_1^+ \mathbf{u} + \tilde{\mathbf{g}}, \gamma_0^+ \mathbf{v} \rangle_\Gamma = 0 \ \ \forall \mathbf{v} \in (H^1(\Omega^+))^\infty,$$

that, by taking into account that values of the trace operator $\gamma_0^+ : H^1(\Omega^+) \to H^{1/2}(\Gamma)$ fill in the whole space $H^{1/2}(\Gamma)$, is an equivalent form of the Robin boundary condition

$$\gamma_1^+ \mathbf{u} - \mathbf{B}\gamma_0^+ \mathbf{u} = \tilde{\mathbf{g}} \ \ \text{in} \ \ (H^{-1/2}(\Gamma))^\infty. \tag{26}$$

Thus, we have shown that the generalized solution of the Robin problem can be characterized by the operator equation (25) and the boundary condition (26). Analogously it can be shown that the generalized solution of the Neumann problem can be characterized by the same operator equation (25) and the Neumann boundary condition

$$\gamma_1^+ \mathbf{u} = \tilde{\mathbf{g}} \quad \text{in} \ \ (H^{-1/2}(\Gamma))^\infty. \tag{27}$$

Conversely, it is obvious that every solution of the problem (25), (26) (resp. (27)) satisfies the variational problem (16) (resp. (15)).

Note that boundary conditions (26) and (27), as in the theory of elliptic equations, will be referred to as the natural boundary conditions.

3. BVPs in convolution terms

As we have outlined in the introduction, all theoretical and practical aspects of the BIEs method are well known in case of its application to the BVPs for the first equation of the system (1) considered separately as well as for this system as a whole but with a finite number of equations. Henceforth our goal will be to obtain a formula for the solutions of BVPs and appropriate BIEs for the infinite system. Similarly to the previous section, we will use the fact that system (1) is triangular and will develop a recurrent process of the calculation of the components of the solution. To avoid additional volume potentials in the solution representation we will move the components that were found on the previous steps to the right-hand side of the current equation. For this purpose we introduce the following convolution operation on sequences.

Let X, Y and Z be arbitrary linear spaces and $q : X \times Y \to Z$ – some mapping.

Definition 4. By the q-convolution of sequences $\mathbf{u} \in X^\infty$ and $\mathbf{v} \in Y^\infty$ we understand a sequence $\mathbf{w} \in Z^\infty$ that is defined according to the following rule

$$\mathbf{w} = \mathbf{u} \underset{q}{\circ} \mathbf{v}, \tag{28}$$

where $w_n \equiv (\mathbf{u} \underset{q}{\circ} \mathbf{v})_n := \sum_{i=0}^{n} q\left(u_{n-i}, v_i\right)$, when $n \geq 0$, and $w_n = 0$ when $n < 0$.

We will simplify the notation of the q-convolution for some mappings. For instance, in case of $q(u, v) := \langle u, v\rangle_{\Omega^+,1,0}$ we will write $\mathbf{u} \underset{\Omega^+,1,0}{\circ} \mathbf{v} := \mathbf{u} \underset{q}{\circ} \mathbf{v}$.

Consider a sequence $\mathbf{u} \in \left(H^1(\Omega^+)\right)^\infty$ that satisfies the equation (23). Let's substitute it into this equation and, treating the result as equality of elements from $(H^{-1}(\Omega^+))^\infty$ and taking

$$q(w, v) = \langle w, v\rangle_{\Omega^+,1,0},\ v \in H_0^1(\Omega^+),\ w \in H^{-1}(\Omega^+),$$

we apply the q-convolution with an arbitrary sequence $\mathbf{v} \in \left(H_0^1(\Omega^+)\right)^\infty$ to both sides of this equality. After that we arrive at the following variational equation

$$(\mathbf{Gu}) \underset{\Omega^+,1,0}{\circ} \mathbf{v} = \mathbf{f} \underset{\Omega^+,1,0}{\circ} \mathbf{v}, \quad \forall \mathbf{v} \in \left(H_0^1(\Omega^+)\right)^\infty. \tag{29}$$

Thus, the generalized solution of the Dirichlet problem (1), (5) can be characterized by the variational equality (29) and the boundary condition (14).

Now we assume that sequence $\mathbf{u} \in \left(H^1(\Omega^+, P)\right)^\infty$ satisfies the operator equation (25). We apply the q-convolution with some arbitrary sequence $\mathbf{v} \in \left(H^1(\Omega^+)\right)^\infty$ to both of its sides as elements of $(L_2(\Omega^+))^\infty$, taking $q(w, v) = (w, v)_{\Omega^+}$, $v \in H^1(\Omega^+)$, $w \in L_2(\Omega^+)$. As a result we get

$$(\mathbf{Gu}) \underset{\Omega^+}{\circ} \mathbf{v} = \mathbf{f} \underset{\Omega^+}{\circ} \mathbf{v}, \quad \forall \mathbf{v} \in \left(H^1(\Omega^+)\right)^\infty. \tag{30}$$

Thus, the generalized solution of the Robin boundary value problem can be characterized by the variational equality (30) and the boundary condition (26).

Obviously, this property also holds for the generalized solution of the Neumann boundary value problem.

Let's obtain for operator **G** the analogue of the first Green's formula using the q-convolution of sequences. At first note that the component of the q-convolution in the left hand side of (30) with an arbitrary index $k \in \mathbb{N}_0$ after application of the first Green's formula (11) can be written as

$$\left((\mathbf{Gu}) \underset{\Omega^+}{\circ} \mathbf{v} \right)_k = \sum_{i=0}^{k} a_{\Omega^+}(u_i,\ v_{k-i}) + \sum_{i=0}^{k} \langle \gamma_1^+ u_i, \gamma_0^+ v_{k-i} \rangle_\Gamma + + \sum_{i=1}^{k} \left(\sum_{j=0}^{i-1} c_{i,j} u_j,\ v_{k-i} \right)_{\Omega^+}. \tag{31}$$

Henceforth we assume that the sum expressions are equal to zero if their last index is less than the first one i.e. in case of $k = 0$ the last item in the previous formula is absent.

Consider a sequence $\left(\Phi_0^+(\mathbf{u},\mathbf{v}),\ \ \Phi_1^+(\mathbf{u},\mathbf{v}),\ ...,\ \ \Phi_k^+(\mathbf{u},\mathbf{v}),\ ...\right)^\top$, components of which are such expressions:

$$\begin{aligned} &\Phi_0^+(\mathbf{u},\mathbf{v}) := a_{\Omega^+}(u_0,\ v_0), \\ &\Phi_k^+(\mathbf{u},\mathbf{v}) := \sum_{i=0}^{k} a_{\Omega^+}(u_i,\ v_{k-i}) + \sum_{i=1}^{k} \left(\sum_{j=0}^{i-1} c_{i,j} u_j,\ v_{k-i} \right)_{\Omega^+},\ k \in \mathbb{N}_0. \end{aligned} \tag{32}$$

Definition 5. Sequence

$$\mathbf{\Phi}^+(\mathbf{u},\mathbf{v}) = \left(\Phi_0^+(\mathbf{u},\mathbf{v}),\ \ \Phi_1^+(\mathbf{u},\mathbf{v}),\ ...,\ \ \Phi_k^+(\mathbf{u},\mathbf{v}),\ ...\right)^\top,\ \mathbf{u},\mathbf{v} \in \left(H^1(\Omega^+) \right)^\infty,$$

defined by the formula (32) is called a bilinear form associated with operator **G**.

Such notation of the bilinear form gives us ability to present the relation (31) in the following way

$$\begin{gathered} (\mathbf{Gu}) \underset{\Omega^+}{\circ} \mathbf{v} = \mathbf{\Phi}^+(\mathbf{u},\mathbf{v}) +\ \gamma_1^+ \mathbf{u} \underset{\Gamma}{\circ} \gamma_0^+ \mathbf{v}, \\ \forall\, \mathbf{u} \in \left(H^1(\Omega^+, P) \right)^\infty,\ \ \mathbf{v} \in \left(H^1(\Omega^+) \right)^\infty, \end{gathered} \tag{33}$$

and treat it as the first Green's formula for the operator **G**. Note that for the left part of the variational equality (29) we can analogously obtain the expression

$$(\mathbf{Gu}) \underset{\Omega^+,1,0}{\circ} \mathbf{v} = \mathbf{\Phi}^+(\mathbf{u},\mathbf{v}), \quad \forall\, \mathbf{u} \in \left(H^1(\Omega^+) \right)^\infty,\ \mathbf{v} \in \left(H_0^1(\Omega^+) \right)^\infty, \tag{34}$$

when using the equality (12).

In general, due to the triangular structure of operator **C**, definition of the second Green's formula may be complicated. In order to apply the classical approach, see, e.g. [2], we need an additional condition on the operator **C**

$$(\mathbf{Cu}) \underset{\Omega^+}{\circ} \mathbf{v} = (\mathbf{Cv}) \underset{\Omega^+}{\circ} \mathbf{u}, \quad \forall\, \mathbf{u},\mathbf{v} \in (L_2(\Omega^+))^\infty, \tag{35}$$

which provides the symmetry of the operator $\mathbf{G}$ with regard to the operation of q-convolution. Then applying (33) twice to the couple of sequences $\mathbf{u}, \mathbf{v} \in \left(H^1(\Omega^+, P)\right)^\infty$ we arrive at the following variational equality.

Theorem 4. *For sequences* $\boldsymbol{u}, \boldsymbol{v} \in \left(H^1(\Omega^+, P)\right)^\infty$ *the following equality holds:*

$$(\boldsymbol{Gu}) \underset{\Omega^+}{\circ} \boldsymbol{v} - (\boldsymbol{Gv}) \underset{\Omega^+}{\circ} \boldsymbol{u} = \gamma_1^+ \boldsymbol{u} \underset{\Gamma}{\circ} \gamma_0^+ \boldsymbol{v} - \gamma_1^+ \boldsymbol{v} \underset{\Gamma}{\circ} \gamma_0^+ \boldsymbol{u}. \tag{36}$$

We treat it as the second Green's formula for the operator $\mathbf{G}$. Further in this paper we suppose the operator $\mathbf{C}$ satisfies (35).

4. Integral representation of the solution

Green's formulae and fundamental solutions of the operator $\mathbf{G}$ are the key ingredients of the integral representation of the solutions of the BVPs. As usual we call the sequence $\widetilde{\mathbf{E}}(x,y) = \left(\widetilde{E}_0(x,y),\ \widetilde{E}_1(x,y),\ ...\right)^\top,\ x,y \in \mathbb{R}^3$, a fundamental solution of the operator $\mathbf{G}$, if it satisfies the equation

$$\mathbf{G}\widetilde{\mathbf{E}} = \boldsymbol{\delta}_y \text{ in } \left(\mathcal{D}'(\mathbb{R}^3)\right)^\infty,$$

where $\boldsymbol{\delta}_y(x) = (\delta_y(x),\ \delta_y(x),\ ...)^\top$ and $\delta_y(\cdot) := \delta(\cdot - y)$ is Dirac's delta-function. Henceforth we also assume this operator has constant coefficients and particularly

$$P := -\Delta + \kappa^2. \tag{37}$$

The condition (35) can be rewritten in the form

$$\sum_{k=1}^{n}\sum_{i=0}^{k-1} c_{k,i}\xi_i\eta_{n-k} = \sum_{k=1}^{n}\sum_{i=0}^{k-1} c_{k,i}\eta_i\xi_{n-k}, \quad \forall n \in \mathbb{N},\ \forall \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^\infty. \tag{38}$$

The last feature is natural for system (1) which is obtained as a result of the Laguerre transformation with parameter $\sigma > 0$ of the heat ($\kappa = \sqrt{\sigma}$) or the wave ($\kappa = \sigma$) equation [8]. Note that γ_1^+ now denotes a normal derivative operator. We also recall the well-known fundamental solution of the operator P:

$$\widetilde{E}_0(x,y) := \frac{e^{-\kappa|x-y|}}{4\pi|x-y|},\ x,y \in \mathbb{R}^3. \tag{39}$$

In [13] and references therein the construction of such solutions for the operator $\mathbf{G}$ with constant coefficients has been considered. For instance, if system (1) corresponds to the wave equation, then the fundamental solution's components for the operator $\mathbf{G}$ have the following presentation

$$\widetilde{E}_i(x,y) := \frac{e^{-\kappa|x-y|}}{4\pi|x-y|}\mathcal{L}_i(\kappa|x-y|),\ \ \forall i \in \mathbb{N}_0,\ \ x,y \in \mathbb{R}^3, \tag{40}$$

where $\mathcal{L}_i$ denotes the Laguerre polynomial [16].

By using the q-convolution we build sequences that in analogy to the theory of elliptic equations can be also called potentials. For that we use a sequence $\mathbf{E}(x,y) = (E_0(x,y),\ E_1(x,y),\ ...)^\top$, where

$$E_i(x,y) := \widetilde{E}_i(x,y) - \widetilde{E}_{i-1}(x,y),\ i \in \mathbb{N},\ \ E_0(x,y) = \widetilde{E}_0(x,y),\ x,y \in \mathbb{R}^3, \tag{41}$$

It was shown in [9] that $\mathbf{E}$ is the solution of the equation

$$\mathbf{GE} = \overline{\boldsymbol{\delta}_y} \text{ in } \left(\mathcal{D}'(\mathbb{R}^3)\right)^\infty, \tag{42}$$

where $\overline{\boldsymbol{\delta}_y}(x) = (\,\delta_y(x),\ 0,\ 0,\ \dots)^\top$.

Definition 6. Let $\boldsymbol{\lambda} \in \left(H^{1/2}(\Gamma)\right)^\infty$ and $\boldsymbol{\mu} \in \left(H^{-1/2}(\Gamma)\right)^\infty$. Sequences

$$\mathbf{V}\boldsymbol{\mu}(x) := (\mathbf{V}\boldsymbol{\mu})(x) \equiv \boldsymbol{\mu}(\cdot) \underset{\Gamma}{\circ} \mathbf{E}(x-\cdot),\ x \in \Omega^+, \tag{43}$$

and

$$\mathbf{W}\boldsymbol{\lambda}(x) := (\mathbf{W}\boldsymbol{\lambda})(x) \equiv \partial_{\bar\nu(\cdot)}\mathbf{E}(x-\cdot) \underset{\Gamma}{\circ} \boldsymbol{\lambda}(\cdot),\ x \in \Omega^+, \tag{44}$$

are called the single and the double layer potentials of the operator $\mathbf{G}$ on the surface Γ, correspondingly.

Lemma 1. *For arbitrary sequences $\boldsymbol{\lambda} \in \left(H^{1/2}(\Gamma)\right)^\infty$ and $\boldsymbol{\mu} \in \left(H^{-1/2}(\Gamma)\right)^\infty$ the layer potentials $\boldsymbol{u} = \boldsymbol{V\mu}$ and $\boldsymbol{u} = \boldsymbol{W\lambda}$ are the solutions of the homogeneous equation*

$$\boldsymbol{Gu} = \boldsymbol{0} \ \ (in\ \mathbb{R}^3 \setminus \Gamma). \tag{45}$$

Proof. Proof of the lemma regarding the domain Ω can be found in lemma 5.3 [7] and in case of the domain Ω^+ can be done analogously. □

Similarly to the layer potentials $\mathbf{V}$ and $\mathbf{W}$, by means of the q-convolution we can define the volume potential for the domain Ω^+ and use it to obtain a partial solution of the system (1). Since in this case the difference from the interior problems discussed in [9] is minor we will consider only problems for the homogeneous system (45).

Let $\gamma_0^- : H^1(\Omega) \to H^{1/2}(\Gamma)$ be a trace operator, $\gamma_1^- : H^1(\Omega, P) \to H^{-1/2}(\Gamma)$ be a normal derivative operator and $[\gamma_0 u] := \gamma_0^+ u - \gamma_0^- u$, $[\gamma_1 u] := \gamma_1^+ u - \gamma_1^- u$ are their jumps across the boundary Γ.

Theorem 5. *For the sequence $\boldsymbol{u} \in \left(H^1(\mathbb{R}^3 \setminus \Gamma, P)\right)^\infty$ which satisfies the equation* (45) *in $\mathbb{R}^3 \setminus \Gamma$ the following representation takes place*

$$\boldsymbol{u}(x) = \boldsymbol{W\lambda}(x) - \boldsymbol{V\mu}(x),\ x \in \mathbb{R}^3 \setminus \Gamma, \tag{46}$$

where $\boldsymbol{\lambda} := [\gamma_0 \boldsymbol{u}]$ and $\boldsymbol{\mu} := [\gamma_1 \boldsymbol{u}]$.

Proof. As we can see, the layer potentials consist of the components

$$\begin{aligned} (V_j\mu)(x) &:= \langle \mu(\cdot),\ E_j(x-\cdot)\,\rangle_\Gamma,\ \mu \in H^{-1/2}(\Gamma); \\ (W_j\lambda)(x) &:= \langle \partial_{\bar\nu(\cdot)}E_j(x-\cdot),\ \lambda(\cdot)\,\rangle_\Gamma,\ \lambda \in H^{1/2}.(\Gamma),\ \ j \in \mathbb{N}_0. \end{aligned} \tag{47}$$

Let some function $u_0 \in H^1(\mathbb{R}^3 \setminus \Gamma, P)$ satisfy the equation $Pu = 0$ in $\mathbb{R}^3 \setminus \Gamma$. Then the third Green's formula holds

$$u_0(x) = (W_0\lambda_0)(x) - (V_0\mu_0)(x),\ x \in \mathbb{R}^3 \setminus \Gamma, \tag{48}$$

where $\lambda_0 := [\gamma_0 u_0]$ and $\mu_0 := [\gamma_1 u_0]$. Note that this formula can be derived from the first equality in (36) if we take $v_0(\cdot) = E_0(x,\cdot)$. For the explanation of the corresponding procedure and some aspects of usage of this formula see, e.g. [1, 14] and [3, 4] for the case of operator (37).

We can use this approach for the following components of the sequence $\mathbf{u}$ as well. Let us assume we also have a function $u_1 \in H^1(\mathbb{R}^3 \setminus \Gamma, P)$ provided the pair u_0 and u_1 satisfies the second equation in (45). Then from the second equality in (36) we obtain:

$$\begin{aligned} &-(c_{1,0}v_0 + Pv_1, u_0)_{\Omega^+} - (Pv_0, u_1)_{\Omega^+} = \\ &= \langle \gamma_1^+ u_1,\ \gamma_0^+ v_0 \rangle_\Gamma + \langle \gamma_1^+ u_0,\ \gamma_0^+ v_1 \rangle_\Gamma - \langle \gamma_1^+ v_1,\ \gamma_0^+ u_0 \rangle_\Gamma - \\ &- \langle \gamma_1^+ v_0,\ \gamma_0^+ u_1 \rangle_\Gamma. \end{aligned} \tag{49}$$

If we take $v_0(\cdot) = E_0(x, \cdot)$ and $v_1(\cdot) = E_1(x, \cdot)$ and keep in mind the first two equalities of (42) we obtain for $\forall x \in \Omega^+$:

$$\begin{aligned} -u_1(x) = \langle \gamma_1^+ u_1,\ \gamma_0^+ E_0 \rangle_\Gamma + \langle \gamma_1^+ u_0,\ \gamma_0^+ E_1 \rangle_\Gamma - \\ - \langle \gamma_1^+ E_1,\ \gamma_0^+ u_0 \rangle_\Gamma - \langle \gamma_1^+ E_0,\ \gamma_0^+ u_1 \rangle_\Gamma. \end{aligned}$$

If we use the second Green's formula for the interior domain Ω [9] we will have

$$0 = -\langle \gamma_1^- u_1,\ \gamma_0^+ E_0 \rangle_\Gamma - \langle \gamma_1^- u_0,\ \gamma_0^+ E_1 \rangle_\Gamma + \langle \gamma_1^+ E_1,\ \gamma_0^- u_0 \rangle_\Gamma + \langle \gamma_1^+ E_0,\ \gamma_0^- u_1 \rangle_\Gamma.$$

Therefore, by adding the last two formulae we obtain the representation formula for the component u_1 for $\forall x \in \Omega^+$:

$$u_1(x) = (W_0\lambda_1)(x) + (W_1\lambda_0)(x) - (V_0\mu_1)(x) - (V_1\mu_0)(x). \tag{50}$$

It is straightforward to see that there is the same representation formula for $\forall x \in \Omega$.

Now we consider the equality in (36) with index $k > 1$. After the substitution $v_0(\cdot) = E_0(x, \cdot)$, $v_1(\cdot) = E_1(x, \cdot)$, ..., and $v_k(\cdot) = E_k(x, \cdot)$ all components in it's left hand side will disappear except $(Pv_0, u_k)_{\Omega^+}$. As in previous cases from $(Pv_0, u_k)_{\Omega^+}$ we get $u_k(x)$ for $\forall x \in \Omega^+$ and 0 for $\forall x \in \Omega$. The rest of the proof repeats the same operations as for $k = 1$. □

Main properties of the potentials $\mathbf{V}$ and $\mathbf{W}$ have been studied in the aforementioned work [9]. Here we recall some of them. Let us consider the boundary operators

$$\begin{aligned} \mathbf{V} &: (H^{-1/2}(\Gamma))^\infty \to (H^{1/2}(\Gamma))^\infty, \quad &\mathbf{K}' &: (H^{-1/2}(\Gamma))^\infty \to (H^{-1/2}(\Gamma))^\infty, \\ \mathbf{K} &: (H^{1/2}(\Gamma))^\infty \to (H^{1/2}(\Gamma))^\infty, \quad &\mathbf{D} &: (H^{1/2}(\Gamma))^\infty \to (H^{-1/2}(\Gamma))^\infty, \end{aligned}$$

defined by means of q-convolution in the following way:

$$\begin{aligned} (\mathbf{V}\boldsymbol{\mu})_i &:= \sum_{j=0}^{i} V_j \mu_{i-j}, \quad (\mathbf{K}\boldsymbol{\lambda})_i := \sum_{j=0}^{i} K_j \lambda_{i-j}, \\ (\mathbf{K}'\boldsymbol{\mu})_i &:= \sum_{j=0}^{i} K'_j \mu_{i-j}, \quad (\mathbf{D}\boldsymbol{\lambda})_i := \sum_{j=0}^{i} D_j \lambda_{i-j},\ i \in \mathbb{N}_0, \end{aligned}$$

for arbitrary sequences $\boldsymbol{\lambda} \in \left(H^{1/2}(\Gamma)\right)^{\infty}$ and $\boldsymbol{\mu} \in \left(H^{-1/2}(\Gamma)\right)^{\infty}$. Components of these operators are defined as follows:

$$V_j\mu := \gamma_0^+ V_j\mu, \; D_j\lambda := -\gamma_1^+ W_j\lambda, \;\; j \in \mathbb{N}_0,$$
$$K_0'\mu := \gamma_1^+ V_0\mu - \frac{1}{2}\mu, \;\; K_j'\mu := \gamma_1^+ V_j\mu, \;\; j \in \mathbb{N},$$
$$K_0\lambda := \gamma_0^+ W_0\lambda + \frac{1}{2}\lambda, \;\; K_j\lambda := \gamma_0^+ W_j\lambda, \;\; j \in \mathbb{N}.$$

Hence, according to the theorem 5 the generalized solution of the homogeneous system (45) can be given by its trace and the normal derivative on the boundary – the Cauchy data. As it can be seen from the boundary conditions (5) and (7), in each of the boundary problems these data are incomplete. To get the complete Cauchy data we need to consider corresponding BIEs that can be obtained by means of the presentation (46). Note that this is the so-called direct approach [2] to replacement of BVPs by BIEs and in our case it could be implemented taking into account the results obtained in [14, 8]. As a result, the following theorem defines the relation between the Cauchy data of some generalized solution of the homogeneous system and BIEs.

Theorem 6. ***(i)*** *If a pair of sequences* $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \left(H^{1/2}(\Gamma)\right)^{\infty} \times \left(H^{-1/2}(\Gamma)\right)^{\infty}$ *are the Cauchy data of some generalized solution of the equation* (45)*, then they satisfy both equations*

$$\left(\frac{1}{2}\boldsymbol{I} - \boldsymbol{K}\right)\boldsymbol{\lambda} + \boldsymbol{V}\boldsymbol{\mu} = 0 \quad in \;\; (H^{1/2}(\Gamma))^{\infty} \tag{51}$$

and

$$\boldsymbol{D}\boldsymbol{\lambda} + \left(\frac{1}{2}\boldsymbol{I} + \boldsymbol{K}'\right)\boldsymbol{\mu} = 0 \quad in \;\; (H^{-1/2}(\Gamma))^{\infty}. \tag{52}$$

(ii) *If a pair of sequences* $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \left(H^{1/2}(\Gamma)\right)^{\infty} \times \left(H^{-1/2}(\Gamma)\right)^{\infty}$ *satisfy one of the equations* (51) *or* (52)*, then they satisfy the second one and are the Cauchy data of some generalized solution of the equation* (45).

Note that for the integral representation of the solution of the PDEs with variable coefficients it is possible to use a parametrix (Levi function) associated with a fundamental solution of corresponding operator with frozen coefficients [11].

5. Boundary integral equations

Theorem 6 gives us reason for the replacement of boundary value problems with corresponding boundary integral equations in regards to the Cauchy datum that is not given explicitly in the formulation of the problem. Due to the similarity of the boundary integral equations that are obtained for interior and exterior problems we will demonstrate this procedure for the Dirichlet problem (1), (5) only. In this case the boundary condition contains the given sequence $\boldsymbol{\lambda} = \tilde{\mathbf{h}} \in (H^{1/2}(\Gamma))^{\infty}$. Then, taking into account the equation (51), after substitution of the given trace into it we will obtain the following boundary integral equation of the first kind in regards to the sequence $\boldsymbol{\mu}$:

$$\mathbf{V}\boldsymbol{\mu} = \left(-\frac{1}{2}\mathbf{I} + \mathbf{K}\right)\tilde{\mathbf{h}} \quad \text{in} \quad (H^{1/2}(\Gamma))^{\infty}. \tag{53}$$

If we substitute the known trace into the equation (52), we will come to the following boundary integral equation of the second kind

$$\left(\frac{1}{2}\mathbf{I} + \mathbf{K}'\right)\boldsymbol{\mu} = -\mathbf{D}\tilde{\mathbf{h}} \quad \text{in} \quad (H^{-1/2}(\Gamma))^{\infty}. \tag{54}$$

Theorem 7. *The normal derivative of the generalized solution* $\boldsymbol{u} \in (H^1(\Omega, P))^{\infty}$ *of the Dirichlet problem* (1), (5) *satisfies both boundary integral equations* (53) *and* (54). *Conversely, if a sequence* $\boldsymbol{\mu} \in (H^{-1/2}(\Gamma))^{\infty}$ *is a solution of one of the boundary integral equations* (53) *or* (54) *then it will satisfy the other one and the function built by formula* (46) *with* $\boldsymbol{\lambda} = \tilde{\boldsymbol{h}}$ *will be the generalized solution of the Dirichlet problem* (1), (5).

Proof. Since boundary integral equations (53) and (54) are only modifications of the relations (51) and (52), then the validity of the direct and the inverse statements of this theorem is granted by the theorem 6. □

Obtained sequences of BIEs have some important recurrent properties. Consider the BIEs (53). It can be reduced to a sequence of equations

$$V_0\mu_k = -\frac{1}{2}\tilde{h}_k + \sum_{i=0}^{k} K_{k-i}\tilde{h}_i - \sum_{i=0}^{k-1} V_{k-i}\mu_i \quad \text{in} \quad H^{1/2}(\Gamma), \quad k \in \mathbb{N}_0.$$

Applying the same approach for equations (54) we get the following sequences of BIEs of the second kind

$$\frac{1}{2}\mu_k + K_0'\mu_k = -\sum_{i=0}^{k} D_{k-i}\tilde{h}_i - \sum_{i=0}^{k-1} K_{k-i}'\mu_i \quad \text{in} \quad H^{-1/2}(\Gamma), \ k \in \mathbb{N}_0,$$

As we see, after the application of q-convolution to the BVPs in the operator form, all of the obtained sequences of BIEs will have the same important property. It consists in the fact that their boundary operators in the left hand sides remain the same for each $k \in \mathbb{N}_0$. Solvability of such integral equations and numerical methods for their solution are well studied in the literature. At the other point of view, the structure of the obtained BIEs allows us to build efficient algorithms for their numerical solution. The same applies for BIEs that correspond to other BVPs. Such equations are discussed in details in [13].

Thus, variational problems for infinite triangular systems, which consist of elliptic equations with variable coefficients, have been formulated and their well-posedness has been shown. By using the q-convolution of sequences, in the case of constant coefficients the representation of generalized solutions in the form of potentials has been obtained, with which variational problems have been reduced to triangular systems of BIEs. Components of the solution of the system of BIEs can consistently be found from the relevant equations which differ only in the right hand side. In this case the right hand side consists of the components of the solutions, found on previous steps, besides of the given Cauchy data. A numerical method for the solution of such systems, developed on the basis of the boundary elements method in [15], gives us ability to efficiently solve the considered boundary problems.

Acknowledgements. The author expresses his sincere gratitude to Prof. W. Wendland for the attention to this research and valuable comments, and to Prof. R. Chapko for ongoing support.

Bibliography

1. McLean W. Strongly Elliptic Systems and Boundary Integral Equations / William McLean. – Cambridge: Cambridge University Press, 2000. – 372 p.
2. Hsiao G. C. Boundary Integral Equations / George Hsiao, Wolfgang Wendland. – Berlin: Springer-Verlag, 2008. – 640 p.
3. Bamberger A. Formulation variationnelle espace-temps pour le calcul par potentiel retarde de la diffraction d'une onde acoustique (I) / A. Bamberger, T. Ha-Duong // Math. Methods Appl. Sci. – 1986. – No 8 (3). – P. 405-435.
4. Bamberger A. Formulation variationnelle pour le calcul de la diffraction d'une onde acoustique par une surface rigide / A. Bamberger, T. Ha-Duong // Math. Methods Appl. Sci. – 1986. – No 8 (4). – P. 598-608.
5. Lubich C. On the multistep time discretization of linear initial-boundary value problems and their boundary integral equations / Christian Lubich // Numer. Math. – 1994. – No 67. – P. 365-389.
6. Laliena A. R. A distributional version of Kirchhoff's formula / A. R. Laliena, F. J. Sayas // J. Math. Anal. Appl. – 2009. – No 359. – P. 197-208.
7. Litynskyy S. On week solutions of boundary value problems for an infinite system of elliptic equations / Svyatoslav Litynskyy, Yuriy Muzychuk, Anatoliy Muzychuk // Visn. Lviv. Univ. Ser. Prykl. Matem. Inform. – 2009. – No 15. – P. 52-70 (in Ukrainian).
8. Chapko R. On the numerical solution of initial boundary value problems by the Laguerre transformation and boundary integral equations / Roman Chapko, Rainer Kress // Integral and Integrodifferential Equations: Theory, Methods and Applications. Series in Mathematical Analysis and Applications. – 2000. – No 2. – P. 55-69.
9. Muzychuk Yu. A. On variational formulations of inner boundary value problems for infinite systems of elliptic equations of special kind / Yuriy Muzychuk, Roman Chapko // Matematychni Studii. – 2012. – No 38 (1). – P. 12-34.
10. Sybil Yu. Three-dimensional elliptic boundary value problems for an open Lipschitz surface / Yuriy Sybil // Matematychni Studii. – 1997. – No 8. – P. 79-96.
11. Chkadua O. Analysis of segregated boundary-domain integral equations for mixed variable-coefficient BVPs in exterior domains / O. Chkadua, S. Mikhailov, D. Natroshvili. – Eds. C. Constanda, P. Harris // Integral Methods in Science and Engineering: Computational and Analytic Aspects. – Springer (Birkhauser). – Boston. – ISBN 978-0-8176-8237-8. – 2011. – P. 109-128.
12. Dautray R. Mathematical analysis and numerical methods for science and technology. Vol. 2. Functional and Variational Methods / Robert Dautray, Jacques-Louis Lions. – Berlin: Springer, 2000. – 592 p.
13. Muzychuk Yu. A. On the boundary integral equation method for boundary-value problems for a system of elliptic equations of the special type in partially semi-infinite domains / Yuriy Muzychuk, Roman Chapko // Reports of the National Academy of Sciences of Ukraine. – 2012. – No 11. – P. 20-27 (in Ukrainian).
14. Costabel M. Boundary integral operators on Lipschitz domains: elementary results / Martin Costabel // SIAM J.Math.Anal. – 1988. – No 19. – P. 613-626.
15. Litynskyy S. Boundary elements method for some triangular system of boundary integral equations / Svyatoslav Litynskyy, Yuriy Muzychuk // Proceedings of XIV International

Seminar/Workshop on Direct and Inverse problem of Electromagnetic and Acoustic Wave Theory (DIPED-2009). – 2009. – P. 208-211.
16. Funaro D. Polynomial Approximations of Differential Equations / Daniele Funaro. – Springer-Verlag, 1992. – 305 p.

Yuriy Muzychuk,
Ivan Franko National University of Lviv,
1, Universytets'ka Str., Lviv, 79000, Ukraine

Received 12.03.2014