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TWO-STEP COMBINED METHOD FOR SOLVING NONLINEAR OPERATOR EQUATIONS

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Резюме. У статті вивчено напівлокальну збіжність двокрокового комбінованого методу для розв'язування нелінійних операторних рівнянь, побудованого на базі двох методів з порядками збіжності $1+\sqrt{2}$. Аналіз збіжності проведено за узагальнених умов Ліпшиця для перших і других похідних та поділених різниць першого порядку.

Abstract. In this paper we study a semilocal convergence of the two-step combined method for solving nonlinear operator equations. It method is based on two methods of convergence orders $1+\sqrt{2}$. Convergence analysis is provided for generalized Lipschits condition for Frechet derivates of the first and second orders and for divided differences of the first order.

1. Introduction

Consider the equation

$$H(x) \equiv F(x) + G(x) = 0, \tag{1}$$

where F and G are nonlinear operators, defined on a convex subset D of a Banach space X with values in a Banach space Y. F is a Fréchet-differentiable operator, G is a continuous operator, differentiability of which is not required.

The well-known Newton's method cannot be applied, as differentiability of operator H is required. For solving nonlinear equation (1) very often use the two-point iterative process [1]

$$x_{n+1} = x_n - A_n^{-1}(F(x_n) + G(x_n)), \quad n = 0, 1, \ldots, \tag{2}$$

where $A_n = A(x_{n-1}, x_n) \in L(X,Y)$. The convergence analysis of the method (2) in general and for $A_n = F'(x_n)$, $A_n = F'(x_n) + G(x_{n-1}; x_n)$, $A_n = H(x_{n-1}; x_n)$ and its modifications was provided by authors [1, 2, 3, 4, 5, 6, 18]. Here $G(x;y)$ $(H(x;y))$ is a first order divided difference of the operator $G(H)$ at the points x and y [13, 14, 15]. In papers [7, 11] we researched a semilocal convergence of the method (2) for $A_n = F'(x_n) + G(x_{n-1}; x_n)$ and $A_n = F'(x_n) + G(2x_n - x_{n-1}; x_{n-1})$.

In works [10, 12] we proposed a two-step method that is based on the methods with the convergence orders $1+\sqrt{2}$ [9, 17]. Its iterative formula is:

$$\begin{aligned} x_{n+1} &= x_n - \Big[F'\Big(\frac{x_n + y_n}{2}\Big) + G(x_n; y_n)\Big]^{-1} H(x_n), \\ y_{n+1} &= x_{n+1} - \Big[F'\Big(\frac{x_n + y_n}{2}\Big) + G(x_n; y_n)\Big]^{-1} H(x_{n+1}), \quad n = 0, 1, \ldots. \end{aligned} \tag{3}$$

Key words. Generalized Lipschitz condition, nondifferentiable operator, semilocal convergence.

We provided a local and a semilocal convergence analysis for method (3) under classical Lipschitz conditions for the first and second order derivatives and divided differences of the first order and established the convergence order. Also we showed results of the numerical solving of the nonlinear equations and systems of nonlinear equations by this iterative process. In paper [8] we proved the local convergence theorem of the (3) under generalized Lipschitz conditions.

In this paper, we study the semilocal convergence of the method (3) under generalized Lipschitz conditions for the first and second order derivatives and divided differences of the first order. These conditions are more general and include classical Lipschitz conditions. Therefore our results have the theoretical interest.

2. Preliminaries

We will need the following definition and lemmas [8, 16].

Definition 7. Let G be a nonlinear operator defined on a subset D of a linear space X with values in a linear space Y and let $x,\ y$ be two points of D. A linear operator from X into Y, denoted as $G(x;y)$, which satisfies the condition

$$G(x;y)(x-y)=G(x)-G(y)$$

is called a divided difference of the first order of G at the points x and y.

In the study of iterative methods very often use the Lipschitz conditions with constant L. Parameter L under Lipschitz conditions does not necessarily has to be a constant, but may also be a positive integrable function. In work [16] Wang suggested generalized Lipschitz conditions for the derivative operator in which instead of constant there was used a certain positive integrable function. In the work [9] we introduce analogous generalized Lipschitz conditions for the divided difference of the first order operator.

Let us denote as $U_0=\{x:\ \|x-x_0\|\le r_0\}$ a closed ball of radius r_0 with center at the point x_0. If L in Lipschitz conditions is a positive integrable function, we consider the conditions

$$\|F'(x)-F'(y)\|\le\int_0^{\|x-y\|}L(u)du,\ x,y\in U_0 \tag{4}$$

and

$$\|G(x;y)-G(u;v)\|\le\int_0^{\|x-u\|+\|y-v\|}M(z)dz,\ x,y,u,v\in U_0, \tag{5}$$

where L and M are positive integrable functions. Lipschitz conditions (4) and (5) we will call generalized Lipschitz conditions or Lipschitz conditions with the L (or M) average. Note that in the case of constants L and M we obtain from (4) and (5) the classical Lipschitz conditions.

Lemma 1. [16]. *Let* $h(t)=\dfrac{1}{t}\displaystyle\int_0^t L(u)du,\ 0\le t\le r$, *where* $L(u)$ *is a positive integrable function that is nondecreasing monotonically in* $[0,r]$. *Then* $h(t)$ *is nondecreasing monotonically with respect to* t.

Lemma 2. [8]. *Let* $g(t) = \dfrac{1}{t^3}\displaystyle\int_0^t N(u)(t-u)^2 du$, $0 \le t \le r$, *where* $N(u)$ *is a positive integrable function that is nondecreasing monotonically in* $[0, r]$. *Then* $g(t)$ *is a nondecreasing monotonically with respect to* t.

3. Semilocal convergence analysis of the two-step iterative process (3)

We can show the following semilocal convergence theorem for the method (3). Imposed terms guarantee the convergence of the iterative process (3) to the solution x^* and its uniqueness.

Theorem 1. *Let* F *and* G *be nonlinear operators, defined on an open convex subset* D *of a Banach space* X *with values in a Banach space* Y. F *is a Fréchet-differentiable operator,* G *is a continuous operator, differentiability of which is not required. Assume that the linear operator* $A_0 = F'\Big(\dfrac{x_0+y_0}{2}\Big) + G(x_0; y_0)$, *where* $x_0,\ y_0 \in D$, *is invertible and in* $U_0 = \{x : \ \|x - x_0\| \le r_0\} \subset D$ *the Lipschitz conditions are fulfilled*

$$\|A_0^{-1}(F'(x) - F'(y))\| \le \int_0^{\|x-y\|} L(z)dz, \tag{6}$$

$$\|A_0^{-1}(F''(x)h - F''(y)h)\| \le \|h\| \int_0^{\|x-y\|} N(z)dz,\ h \in X, \tag{7}$$

$$\|A_0^{-1}(G(x;y) - G(u;v))\| \le \int_0^{\|x-u\|+\|y-v\|} M(z)dz, \tag{8}$$

where L, M, *and* N *are positive integrable and nondecreasing monotonically functions.*

Let a, c $(c > a)$, r_0 *be nonnegative numbers such that*

$$\|x_0 - y_0\| \le a, \quad \|A_0^{-1}(F(x_0) + G(x_0))\| \le c, \tag{9}$$

$$r_0 \ge \frac{c}{1-\gamma}, \quad \int_0^{(2r_0-a)/2} L(z)dz + \int_0^{2r_0-a} M(z)dz < 1, \tag{10}$$

$$\gamma = \frac{\frac{1}{8}c\int_0^c N(z)\Big(1-\frac{z}{c}\Big)^2 dz + \int_0^{(c-a)/2} L(z)dz + \int_0^{c-a} M(z)dz}{1 - \int_0^{(2r_0-a)/2} L(z)dz - \int_0^{2r_0-a} M(z)dz}, \quad 0 \le \gamma < 1.$$

Then the iterative process (3) is well-defined and sequences $\{x_n\}_{n\ge0}$, $\{y_n\}_{n\ge0}$ *generated by it remain in* U_0 *and converge to the solution* x^* *of equation (1) and, for all* $n \ge 0$, *the following inequalities are satisfied*

$$\|x_n - x_{n+1}\| \le t_n - t_{n+1}, \quad \|y_n - x_{n+1}\| \le s_n - t_{n+1}, \tag{11}$$

$$\|x_n - x^*\| \le t_n - t^*, \quad \|y_n - x^*\| \le s_n - t^*, \tag{12}$$

where sequences $\{t_n\}_{n\ge0}$ *and* $\{s_n\}_{n\ge0}$ *defined by the formulas*

$$t_0 = r_0, \quad s_0 = r_0 - a, \quad t_1 = r_0 - c,$$

$$t_{n+1}-t_{n+2}=$$
$$=\frac{1}{8c^3}\frac{\int_0^c N(z)(c-z)^2dz(t_n-t_{n+1})^3}{1-\int_0^{(t_0-t_{n+1}+s_0-s_{n+1})/2}L(z)dz-\int_0^{t_0-t_{n+1}+s_0-s_{n+1}}M(z)dz}+$$
$$+\frac{1}{c-a}\frac{\Big[\int_0^{(c-a)/2}L(z)dz+\int_0^{c-a}M(z)dz\Big](t_n-t_{n+1})(s_n-t_{n+1})}{1-\int_0^{(t_0-t_{n+1}+s_0-s_{n+1})/2}L(z)dz-\int_0^{t_0-t_{n+1}+s_0-s_{n+1}}M(z)dz},$$
$$n\geq 0, \tag{13}$$

$$t_{n+1}-s_{n+1}=$$
$$=\frac{1}{8c^3}\frac{\int_0^c N(z)(c-z)^2dz(t_n-t_{n+1})^3}{1-\int_0^{(t_0-t_n+s_0-s_n)/2}L(z)dz-\int_0^{t_0-t_n+s_0-s_n}M(z)dz}+$$
$$+\frac{1}{c-a}\frac{\Big[\int_0^{(c-a)/2}L(z)dz+\int_0^{c-a}M(z)dz\Big](t_n-t_{n+1})(s_n-t_{n+1})}{1-\int_0^{(t_0-t_n+s_0-s_n)/2}L(z)dz-\int_0^{t_0-t_n+s_0-s_n}M(z)dz},$$
$$n\geq 0 \tag{14}$$

are nonincreasing nonnegative and converge to certain t^* such that

$$r_0-\frac{c}{1-\gamma}\leq t^*<t_0.$$

Proof. Let us show by the mathematical induction method that, for all $k\geq 0$

$$t_{k+1}\geq s_{k+1}\geq t_{k+2}\geq r_0-\frac{c}{1-\gamma}\geq 0, \tag{15}$$

$$t_{k+1}-t_{k+2}\leq\gamma(t_k-t_{k+1}),\quad t_{k+1}-s_{k+1}\leq\gamma(t_k-t_{k+1}) \tag{16}$$

are satisfied. For $k=0$, from (13) and (14), we get

$$t_1-t_2=\frac{1}{c^3}\frac{\int_0^c N(z)(c-z)^2dz(t_0-t_1)^3}{1-\int_0^{(t_0-t_1+s_0-s_1)/2}L(z)dz-\int_0^{t_0-t_1+s_0-s_1}M(z)dz}+$$
$$+\frac{1}{c-a}\frac{\Big[\int_0^{(c-a)/2}L(z)dz+\int_0^{c-a}M(z)dz\Big](t_0-t_1)(s_0-t_1)}{1-\int_0^{(t_0-t_1+s_0-s_1)/2}L(z)dz-\int_0^{t_0-t_1+s_0-s_1}M(z)dz}$$

and

$$t_2=r_0-c-\Big[\frac{\frac{1}{8}c\int_0^c N(z)(1-\frac{z}{c})^2dz+\int_0^{(c-a)/2}L(z)dz+\int_0^{c-a}M(z)dz}{1-\int_0^{(2r_0-a)/2}L(z)dz-\int_0^{2r_0-a}M(z)dz}\Big]c\geq$$
$$\geq r_0-(1+\gamma)c=r_0-\frac{(1-\gamma^2)c}{1-\gamma}\geq r_0-\frac{c}{1-\gamma}\geq 0.$$

Similarly, we have

$$t_1-s_1=\frac{1}{8c^3}\int_0^c N(z)(c-z)^2dz(t_0-t_1)^3+$$
$$+\frac{1}{c-a}\Big[\int_0^{(c-a)/2}L(z)dz+\int_0^{c-a}M(z)dz\Big](t_0-t_1)(s_0-t_1)$$

and

$$s_1 = r_0 - c - \Big[\frac{1}{8}c\int_0^c N(z)\Big(1-\frac{z}{c}\Big)^2 dz + \int_0^{(c-a)/2} L(z)dz + \int_0^{c-a} M(z)dz\Big]c.$$

From the last equalities it follows that

$$t_1 \geq t_2,\ s_1 \geq t_2,\ t_1 \geq s_1 \geq t_2 \geq r_0 - \frac{c}{1-\gamma} \geq 0.$$

Assume that that inequalities (15) and (16) are satisfied for $k = \overline{0, n-1}$. Then, for $k = n$, we obtain

$$\begin{gathered} t_{n+1} - t_{n+2} = \\ \frac{1}{8c^3}\frac{\int_0^c N(z)(c-z)^2dz(t_n-t_{n+1})^3}{1-\int_0^{(t_0-t_{n+1}+s_0-s_{n+1})/2}L(z)dz - \int_0^{t_0-t_{n+1}+s_0-s_{n+1}}M(z)dz} + \\ +\frac{1}{c-a}\frac{\Big[\int_0^{(c-a)/2}L(z)dz+\int_0^{c-a}M(z)dz\Big](t_n-t_{n+1})(s_n-t_{n+1})}{1-\int_0^{(t_0-t_{n+1}+s_0-s_{n+1})/2}L(z)dz - \int_0^{t_0-t_{n+1}+s_0-s_{n+1}}M(z)dz} \leq \\ \leq \frac{\frac{1}{8}c\int_0^c N(z)(1-\frac{z}{c})^2dz + \int_0^{(c-a)/2}L(z)dz + \int_0^{c-a}M(z)dz}{1-\int_0^{(2r_0-a)/2}L(z)dz - \int_0^{2r_0-a}M(z)dz}(t_n-t_{n+1}) = \\ = \gamma(t_n-t_{n+1}), \end{gathered}$$

$$\begin{gathered} t_{n+1} - s_{n+1} = \frac{1}{8c^3}\frac{\int_0^c N(z)(c-z)^2dz(t_n-t_{n+1})^3}{1-\int_0^{(t_0-t_n+s_0-s_n)/2}L(z)dz - \int_0^{t_0-t_n+s_0-s_n}M(z)dz} + \\ +\frac{1}{c-a}\frac{\Big[\int_0^{(c-a)/2}L(z)dz+\int_0^{c-a}M(z)dz\Big](t_n-t_{n+1})(s_n-t_{n+1})}{1-\int_0^{(t_0-t_n+s_0-s_n)/2}L(z)dz - \int_0^{t_0-t_n+s_0-s_n}M(z)dz} \leq \\ \leq \frac{\frac{1}{8}c\int_0^c N(z)(1-\frac{z}{c})^2dz + \int_0^{(c-a)/2}L(z)dz + \int_0^{c-a}M(z)dz}{1-\int_0^{(2r_0-a)/2}L(z)dz - \int_0^{2r_0-a}M(z)dz}(t_n-t_{n+1}) = \\ = \gamma(t_n-t_{n+1}) \end{gathered}$$

and

$$\begin{gathered} t_{n+1} \geq s_{n+1} \geq t_{n+2} \geq t_{n+1} - \gamma(t_n - t_{n+1}) \geq \\ \geq r_0 - \frac{1-\gamma^{n+2}}{1-\gamma}c \geq r_0 - \frac{c}{1-\gamma} \geq 0. \end{gathered}$$

So, we prove, that $\{t_n\}_{n\geq 0}$ and $\{s_n\}_{n\geq 0}$ are nonincreasing, nonnegative sequences and converge to $t^* \geq 0$.

Let us prove, by mathematical induction, that the iterative process (3) is well-defined and inequalities (11) are satisfied for all $n \geq 0$.

Taking into account (9) and that $t_0 - t_1 = c$, we establish that $x_1 \in U_0$ and (11) are satisfied for $n = 0$.

Denote $A_n = F'\Big(\dfrac{x_n+y_n}{2}\Big) + G(x_n;y_n)$. Using the Lipschitz conditions (6) and (8), we have

$$\|I - A_0^{-1}A_{n+1}\| = \|A_0^{-1}[A_0 - A_{n+1}]\| \le$$
$$\le \Big\|A_0^{-1}[F'\Big(\frac{x_0+y_0}{2}\Big) - F'\Big(\frac{x_{n+1}+y_{n+1}}{2}\Big) + G(x_0;y_0) - G(x_{n+1};y_{n+1})]\Big\| \le$$
$$\le \int_0^{(\|x_0-x_{n+1}\|+\|y_0-y_{n+1}\|)/2} L(z)dz + \int_0^{\|x_0-x_{n+1}\|+\|y_0-y_{n+1}\|} M(z)dz \le$$
$$\le \int_0^{(t_0-t_{n+1}+s_0-s_{n+1})/2} L(z)dz + \int_0^{t_0-t_{n+1}+s_0-s_{n+1}} M(z)dz \le$$
$$\le \int_0^{(t_0+s_0)/2} L(z)dz + \int_0^{t_0+s_0} M(z)dz < 1.$$

According to the Banach lemma on the invertible operator, A_{n+1} is invertible and

$$\|A_{n+1}^{-1}A_0\| \le$$
$$\le \Big(1 - \int_0^{(\|x_0-x_{n+1}\|+\|y_0-y_{n+1}\|)/2} L(z)dz - \int_0^{\|x_0-x_{n+1}\|+\|y_0-y_{n+1}\|} M(z)dz\Big)^{-1}.$$

Let us prove that iterative process (3) is well-defined for $k = n+1$. Taking into account the definition of the first order divided difference, conditions (6), (8) and identity [17]

$$F(x) - F(y) - F'\Big(\frac{x+y}{2}\Big)(x-y) = \frac{1}{4}\int_0^1 (1-t)\Big[F''\Big(\frac{x+y}{2} + \frac{t}{2}(x-y)\Big) -$$
$$-F''\Big(\frac{x+y}{2} + \frac{t}{2}(y-x)\Big)\Big]dt(x-y)(x-y),$$

we obtain

$$\|A_0^{-1}H(x_{n+1})\| =$$
$$= \|A_0^{-1}\Big[F(x_{n+1}) - F(x_n) - F'\Big(\frac{x_n+x_{n+1}}{2}\Big)(x_{n+1}-x_n) +$$
$$+ F'\Big(\frac{x_n+x_{n+1}}{2}\Big)(x_{n+1}-x_n) - F'\Big(\frac{x_n+y_n}{2}\Big)(x_{n+1}-x_n) +$$
$$+ G(x_{n+1}) - G(x_n) - G(x_n;y_n)(x_{n+1}-x_n)]\,\| \le$$
$$\le \frac{1}{8}\int_0^{\|x_n-x_{n+1}\|} N(z)(\|x_n-x_{n+1}\|-z)^2 dz +$$
$$+ \int_0^{\|y_n-x_{n+1}\|/2} L(z)dz\|x_n-x_{n+1}\| +$$
$$+ \int_0^{\|y_n-x_{n+1}\|} M(z)dz\|x_n-x_{n+1}\|.$$

Denote

$$A_n = \frac{1}{8}\int_0^{\|x_n - x_{n+1}\|} N(z)(\|x_n - x_{n+1}\| - z)^2 dz,$$

$$B_n = \int_0^{\|y_n - x_{n+1}\|/2} L(z)dz,\ C_n = \int_0^{\|y_n - x_{n+1}\|} M(z)dz,$$

$$Q_{n+1} = 1 - \int_0^{(\|x_0 - x_{n+1}\| + \|y_0 - y_{n+1}\|)/2} L(z)dz - \int_0^{\|x_0 - x_{n+1}\| + \|y_0 - y_{n+1}\|} M(z)dz.$$

Hence, taking into account lemmas 1, 2 and inequalities (11), we have

$$\|x_{n+1} - x_{n+2}\| = \|A_{n+1}^{-1} H(x_{n+1})\| \le \|A_{n+1}^{-1} A_0\| \|A_0^{-1} H(x_{n+1})\| \le$$
$$\le \frac{A_n + [B_n + C_n]\|x_n - x_{n+1}\|}{Q_{n+1}} =$$
$$= \frac{A_n \|x_n - x_{n+1}\|^3}{Q_{n+1}\|x_n - x_{n+1}\|^3} + \frac{[B_n + C_n]\|x_n - x_{n+1}\|\|y_n - x_{n+1}\|}{Q_{n+1}\|y_n - x_{n+1}\|} \le$$
$$\le \frac{A_0 \|x_n - x_{n+1}\|^3}{Q_{n+1}\|x_0 - x_1\|^3} + \frac{[B_0 + C_0]\|x_n - x_{n+1}\|\|y_n - x_{n+1}\|}{Q_{n+1}\|y_0 - x_1\|} \le$$
$$\le \frac{1}{8(t_0 - t_1)^3} \frac{\int_0^{t_0 - t_1} N(z)(t_0 - t_1 - z)^2 dz (t_n - t_{n+1})^3}{1 - \int_0^{(t_0 - t_{n+1} + s_0 - s_{n+1})/2} L(z)dz - \int_0^{t_0 - t_{n+1} + s_0 - s_{n+1}} M(z)dz} +$$
$$+ \frac{1}{s_0 - t_1} \frac{\left[\int_0^{(s_0 - t_1)/2} L(z)dz + \int_0^{s_0 - t_1} L(z)dz\right](t_n - t_{n+1})(s_n - t_{n+1})}{1 - \int_0^{(t_0 - t_{n+1} + s_0 - s_{n+1})/2} L(z)dz - \int_0^{t_0 - t_{n+1} + s_0 - s_{n+1}} M(z)dz} =$$
$$= \frac{1}{8c^3} \frac{\int_0^{c} N(z)(c - z)^2 dz (t_n - t_{n+1})^3}{1 - \int_0^{(t_0 - t_{n+1} + s_0 - s_{n+1})/2} L(z)dz - \int_0^{t_0 - t_{n+1} + s_0 - s_{n+1}} M(z)dz} +$$
$$+ \frac{1}{c - a} \frac{\left[\int_0^{(c-a)/2} L(z)dz + \int_0^{c-a} M(z)dz\right](t_n - t_{n+1})(s_n - t_{n+1})}{1 - \int_0^{(t_0 - t_{n+1} + s_0 - s_{n+1})/2} L(z)dz - \int_0^{t_0 - t_{n+1} + s_0 - s_{n+1}} M(z)dz} =$$
$$= t_{n+1} - t_{n+2}$$

and

$$\|x_{n+2} - y_{n+2}\| = \|A_{n+1}^{-1} H(x_{n+2})\| \le \|A_{n+1}^{-1} A_0\| \|A_0^{-1} H(x_{n+2})\| \le$$
$$\le \frac{A_{n+1} + [B_{n+1} + C_{n+1}]\|x_n - x_{n+1}\|}{Q_{n+1}} =$$
$$= \frac{A_{n+1} \|x_{n+1} - x_{n+2}\|^3}{Q_{n+1}\|x_n - x_{n+1}\|^3} + \frac{[B_{n+1} + C_{n+1}]\|x_{n+1} - x_{n+2}\|\|y_{n+1} - x_{n+2}\|}{Q_{n+1}\|y_{n+1} - x_{n+2}\|} \le$$
$$\le \frac{A_0 \|x_{n+1} - x_{n+2}\|^3}{Q_{n+1}\|x_0 - x_1\|^3} + \frac{[B_0 + C_0]\|x_{n+1} - x_{n+2}\|\|y_{n+1} - x_{n+2}\|}{Q_{n+1}\|y_0 - x_1\|} \le$$
$$\le \frac{1}{8(t_0 - t_1)^3} \frac{\int_0^{t_0 - t_1} N(z)((t_0 - t_1) - z)^2 dz (t_{n+1} - t_{n+2})^3}{1 - \int_0^{(t_0 - t_{n+1} + s_0 - s_{n+1})/2} L(z)dz - \int_0^{t_0 - t_{n+1} + s_0 - s_{n+1}} M(z)dz} +$$

$$+\frac{1}{s_0-t_1}\frac{\left[\int_0^{(s_0-t_1)/2}L(z)dz+\int_0^{s_0-t_1}M(z)dz\right](t_{n+1}-t_{n+2})(t_{n+2}-s_{n+1})}{1-\int_0^{(t_0-t_{n+1}+s_0-s_{n+1})/2}L(z)dz-\int_0^{t_0-t_{n+1}+s_0-s_{n+1}}M(z)dz}=$$

$$=\frac{1}{8c^3}\frac{\int_0^c N(z)(c-z)^2dz(t_{n+1}-t_{n+2})^3}{1-\int_0^{(t_0-t_{n+1}+s_0-s_{n+1})/2}L(z)dz-\int_0^{t_0-t_{n+1}+s_0-s_{n+1}}M(z)dz}+$$

$$+\frac{1}{c-a}\frac{\left[\int_0^{(c-a)/2}L(z)dz+\int_0^{c-a}M(z)dz\right](t_{n+1}-t_{n+2})(s_{n+1}-t_{n+2})}{1-\int_0^{(t_0-t_{n+1}+s_0-s_{n+1})/2}L(z)dz-\int_0^{t_0-t_{n+1}+s_0-s_{n+1}}M(z)dz}=$$

$$=t_{n+2}-s_{n+2}.$$

Thus, the iterative process (3) is well-defined for all $n\geq 0$. Hence it follows that

$$\|x_n-x_k\|\leq t_n-t_k,\ \|y_n-x_k\|\leq s_n-t_k,\|y_n-y_k\|\leq s_n-s_k,\quad 0\leq n\leq k,\ (17)$$

i.e., the sequence $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ are fundamental in a Banach space X and convergence to x^*. From (17) for $k\to\infty$ it follows inequalities (12). Let us show that x^* is the solution of the equation (1). Indeed,

$$\|A_0^{-1}H(x_{n+1})\|\leq\frac{1}{8}\int_0^{\|x_n-x_{n+1}\|}N(z)(\|x_n-x_{n+1}\|-z)^2dz+$$

$$+\int_0^{\|y_n-x_{n+1}\|/2}L(z)dz\|x_n-x_{n+1}\|+\int_0^{\|y_n-x_{n+1}\|}M(z)dz\|x_n-x_{n+1}\|\leq$$

$$\leq\frac{1}{24}N(\|x_n-x_{n+1}\|)\|x_n-x_{n+1}\|^3+\int_0^{\|y_n-x_{n+1}\|/2}L(z)dz\|x_n-x_{n+1}\|+$$

$$+\int_0^{\|y_n-x_{n+1}\|}M(z)dz\|x_n-x_{n+1}\|\to 0,\quad\text{when } n\to\infty.$$

Thus, $H(x^*)=0$. The theorem is proven. □

Theorem 2. *Let F and G be nonlinear operators, defined on an open convex subset D of a Banach space X with values in a Banach space Y. F is a Fréchet-differentiable operator, G is a continuous operator, differentiability of which is not required. Assume that:*
1) conditions of Theorem 1 are satisfied;
2) r_0 from Theorem 1 additionally satisfies condition

$$\gamma_1=\frac{\frac{1}{8}r_0\int_0^{r_0}N(z)\Big(1-\frac{z}{r_0}\Big)^2dz+\int_0^{(r_0-a)/2}L(z)dz+\int_0^{r_0-a}M(z)dz}{1-\int_0^{(2r_0-a)/2}L(z)dz-\int_0^{2r_0-a}M(z)dz}<1. \tag{18}$$

Then the iterative process (3) is well-defined and generated by it $\{x_n\}_{n\geq 0}$ belongs to U_0 and converges to the unique solution x^* of the equation $F(x)=0$ in U_0.

Proof. To show the uniqueness, we assume that there exists a second solution x^{**}.

Using the approximation

$$x_{n+1}-x^{**}=x_n-x^{**}-A_n^{-1}[H(x_n)-H(x^{**})]=$$
$$=A_n^{-1}\big[F'\Big(\frac{x_n+y_n}{2}\Big)(x_n-x^{**})-F(x_n)+F(x^{**})\big]+$$
$$+A_n^{-1}\big[G(x_n;y_n)-G(x_n;x^{**})\big](x_n-x^{**}),$$

we obtain

$$\|x_{n+1}-x^{**}\|\le\Big\|A_n^{-1}\Big[F'\Big(\frac{x_n+y_n}{2}\Big)(x_n-x^{**})-F(x_n)+F(x^{**})\Big]\Big\|+$$
$$+\|A_n^{-1}\big[G(x_n;y_n)-G(x_n;x^{**})\big](x_n-x^{**})\|\le$$
$$\le\Big\|A_n^{-1}\Big[F'\Big(\frac{x_n+x^{**}}{2}\Big)(x_n-x^{**})-F(x_n)+F(x^{**})\Big]\Big\|+$$
$$+\Big\|A_n^{-1}\Big[F'\Big(\frac{x_n+y_n}{2}\Big)-F'\Big(\frac{x_n+x^{**}}{2}\Big)\Big](x_n-x^{**})\Big\|+$$
$$+\|A_n^{-1}\big[G(x_n;y_n)-G(x_n;x^{**})\big](x_n-x^{**})\|\le$$
$$\le\|A_n^{-1}A_0\|\Big\|A_0^{-1}\Big[F(x_n)-F(x^{**})-F'\Big(\frac{x_n+x^{**}}{2}\Big)(x_n-x^{**})\Big]\Big\|+$$
$$+\|A_n^{-1}A_0\|\Big\|A_0^{-1}\Big[F'\Big(\frac{x_n+y_n}{2}\Big)-F'\Big(\frac{x_n+x^{**}}{2}\Big)\Big]\Big\|\|x_n-x^{**}\|+$$
$$+\|A_n^{-1}A_0\|\|A_0^{-1}\big[G(x_n;y_n)-G(x_n;x^{**})\big]\|\|x_n-x^{**}\|\le$$
$$\le\frac{1}{4}\frac{\int_0^1(1-t)\int_0^{t\|x_n-x^{**}\|}N(z)dzdt}{Q_n}\|x_n-x^{**}\|^2+$$
$$+\frac{\int_0^{\|y_n-x^{**}\|/2}L(z)dz}{Q_n}\|x_n-x^{**}\|+\frac{\int_0^{\|y_n-x^{**}\|}M(z)dz}{Q_n}\|x_n-x^{**}\|=$$
$$=\frac{\frac{1}{4}\int_0^{\|x_n-x^{**}\|}N(z)\int_{z/\|x_n-x^{**}\|}^1(1-t)dzdt\|x_n-x^{**}\|^2}{Q_n}+$$
$$+\frac{\int_0^{\|y_n-x^{**}\|/2}L(z)dz+\int_0^{\|y_n-x^{**}\|}M(z)dz}{Q_n}\|x_n-x^{**}\|\le$$
$$=\frac{\frac{1}{8}\int_0^{\|x_n-x^{**}\|}N(z)\big(1-\frac{z}{\|x_n-x^{**}\|}\big)^2dz\|x_n-x^{**}\|^2}{Q_n}+$$
$$+\frac{\int_0^{\|y_n-x^{**}\|/2}L(z)dz+\int_0^{\|y_n-x^{**}\|}M(z)dz}{Q_n}\|x_n-x^{**}\|\le$$
$$\le\gamma_1\|x_n-x^{**}\|\le\ldots\le\gamma_1^{n+1}\|x_0-x^{**}\|,$$

which implies $x^{**}=\lim\limits_{n\to\infty}x_n=x^*$. The theorem is proven. □

Let $L(z)=L=\mathrm{const}$, $N(z)=N=\mathrm{const}$ and $M(z)=M=\mathrm{const}$. Then we get the following result.

Theorem 3. *Let F and G be nonlinear operators, defined on an open convex subset D of a Banach space X with values in a Banach space Y. F is a Fréchet-differentiable operator, G is a continuous operator, differentiability of which is* *not required. Assume that the linear operator* $A_0 = F'\Big(\dfrac{x_0+y_0}{2}\Big) + G(x_0;y_0)$, *where* $x_0,\ y_0 \in D$, *is invertible and in* $U_0 = \{x:\ \|x-x_0\| \le r_0\} \subset D$ *the Lipschitz conditions are fulfilled*

$$\|A_0^{-1}(F'(x)-F'(y))\| \le L\|x-y\|,$$
$$\|A_0^{-1}(F''(x)h-F''(y)h)\| \le N\|x-y\|\|h\|,\ h \in X,$$
$$\|A_0^{-1}(G(x;y)-G(u;v))\| \le M(\|x-u\|+\|y-v\|),$$

where L, M *and* N *are positive numbers.*

Let a, c $(c>a)$, r_0 *be nonnegative numbers such that*

$$\|x_0-y_0\| \le a, \quad \|A_0^{-1}(F(x_0)+G(x_0))\| \le c,$$
$$r_0 \ge \frac{c}{1-\gamma}, \quad (L/2+M)(2r_0-a)<1,$$
$$\gamma = \frac{c^2N/24+(L/2+M)(c-a)}{1-(L/2+M)(2r_0-a)}, \qquad 0 \le \gamma < 1.$$

Then the iterative process (3) is well-defined and sequences $\{x_n\}_{n\ge0}$, $\{y_n\}_{n\ge0}$ *generated by it remain in* U_0 *and converge to the solution* x^* *of equation (1) and, for all* $n \ge 0$, *the following inequalities are satisfied*

$$\|x_n-x_{n+1}\| \le t_n-t_{n+1}, \quad \|y_n-x_{n+1}\| \le s_n-t_{n+1},$$
$$\|x_n-x^*\| \le t_n-t^*, \quad \|y_n-x^*\| \le s_n-t^*,$$

where sequences $\{t_n\}_{n\ge0}$ *and* $\{s_n\}_{n\ge0}$ *defined by the formulas*

$$t_0=r_0, \quad s_0=r_0-a, \quad t_1=r_0-c,$$
$$t_{n+1}-t_{n+2}=$$
$$\frac{N(t_n-t_{n+1})^3/24+(L/2+M)(t_n-t_{n+1})(s_n-t_{n+1})}{1-(L/2+M)(t_0-t_{n+1}+s_0-s_{n+1})}, \quad n\ge0, \tag{19}$$
$$t_{n+1}-s_{n+1}=$$
$$\frac{N(t_n-t_{n+1})^3/24+(L/2+M)(t_n-t_{n+1})(s_n-t_{n+1})}{1-(L/2+M)(t_0-t_n+s_0-s_n)}, \quad n\ge0$$

are nonincreasing nonnegative and converge to certain t^* *such that* $r_0 - \dfrac{c}{1-\gamma} \le$ $\le t^* < t_0$.

Remark 1. *If* $F(x)=0$, $L=0$ *and* $N=0$ *then the sequences* $\{t_n\}_{n\ge0}$ *and* $\{s_n\}_{n\ge0}$, *defined by the formulas (19), reduce to similar ones in* [9] *for the case* $\alpha = 1$.

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