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NUMERICAL ANALYSIS OF THE GIRKMANN PROBLEM WITH FEM/BEM COUPLING USING DOMAIN DECOMPOSITION

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Резюме. Ми розглядаємо поєднану модель для задачі Гіркмана. Ця задача полягає в обчисленні плоского деформованого стану для тіла, що складається з основної частини та тонкої частини, що прикріплена до основної частини. Для побудови наближеного розв'язку цієї задачі ми використовуємо метод граничних елементів (МГЕ) та метод скінченних елементів (МСЕ), поєднані за допомогою алгоритму декомпозиції областей. Наведено результати числових експериментів. Порівняно напружено-деформований стан конструкцій для різних форм оболонок.

Abstract. We consider a coupled model for the Girkmann problem. The problem involves computation of the plane strain state for the body which consists of a massive part and a thin part, which is attached to the massive part. For the numerical solution of this problem we use boundary element method (BEM) and finite element method (FEM) for different parts of the body, which are coupled using domain decomposition. We provide the results of some numerical simulations. The stress-strain state for the structures having shells of different shapes are compared.

1. Introduction

A lot of structures, that occur in engineering, are inhomogeneous and contain thin parts and massive parts. Therefore, it is important to develop both analytical methods and numerical algorithms for the analysis of the stress-strain state of such structures. Different aspects of such problems were discussed in [3, 6, 8, 2] (in [8] the case of the bodies with thin inclusions is considered; in [2] the bodies with thin covers are considered). Papers [3] and [6] are devoted to the numerical solution of the Girkmann problem.

In this article, we solve numerically the Girkmann problem which involves computation of a plane strain state for the body consisting of a massive part and a thin part, which is attached to the massive part. The thin part is modeled using Timoshenko shell theory equations and its stress-strain state is numerically computed using FEM with bubble shape functions. The massive part is modeled using the theory of linear elasticity and the numerical solution is obtained using boundary element method (BEM). The approximate solutions in both parts are connected using domain decomposition algorithm.

The application of domain decomposition method allows us to decouple problems in both parts and solve the problems independently in each part. As a

Key words. Girkmann problem, elasticity theory, Timoshenko shell theory, finite element method, boundary element method, domain decomposition.

result, it is possible to compute the stress-strain state accurately even for small shell thicknesses without having problems with stability issues of the coupled problem.

We compare the stress-strain state for different shapes of the middle line of the shells: circular, parabolic and of the form of chain curve. Although the curves lie close to each other, the stress-strain states in these cases are very different from each other.

2. Problem statement

Let us consider a problem of plane strain of an elastic body which consists of a massive part Ω_1 with the thin part in Ω_2 attached to Ω_1 by its end face (Fig. 1). Let us denote by Γ_i the outer boundary of the bodies in Ω_i, $i = 1, 2$ and by Γ_I the common boundary between bodies in Ω_1 and Ω_2.

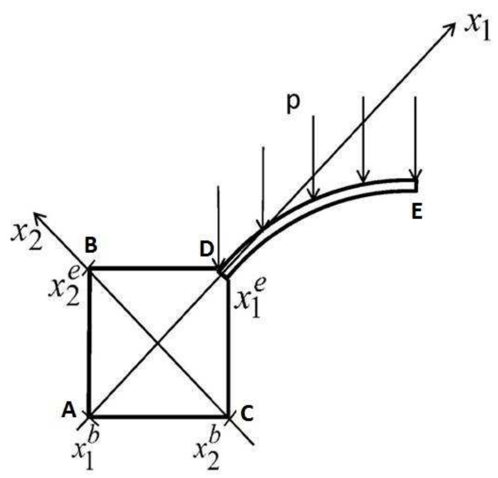

Fig. 1. Elastic Body

The plane strain stress of the body in Ω_1 can be described by

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} = f_1$$
$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} = f_2 \tag{1}$$

that holds for $x \in \Omega_1$, $x = (x_1, x_2)$.

Here $f = (f_1, f_2)$ denotes the volume forces that act on the body in Ω_1.

From the Hook's law it follows that the components of the stress tensor can be written as

$$\sigma_{ij} = \frac{1}{2} E_1 \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \qquad i, j = 1, 2,$$

where $u(x) = (u_1(x), u_2(x))$ is the displacement vector with u_i being the displacements in the directions x_i for $i = 1, 2$; E_1 is the Young's modulus of the body in Ω_1. In the following we assume that no volume forces act on the body in Ω_1.

Let us denote by n the outer normal vector to Ω_1, and by τ – the tangent vector.

Equations (1) are considered together with the boundary conditions

$$u_n = 0, u_\tau = 0, x \in \Gamma_D$$

and

$$\sigma_{nn} = 0, \sigma_{n\tau} = 0, x \in \Gamma_N,$$

where $\Gamma_1 = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$; u_n and u_τ are the components of the displacement vector in the coordinate system n, τ. Similarly, σ_{nn} and $\sigma_{n\tau}$ are the components of the stress tensor in the n, τ coordinate system.

For the description of the thin part in Ω_2 we use the equations of Timoshenko shell theory of the form [4]

$$-\frac{1}{A_1}\frac{dT_{11}}{d\xi_1} - k_1 T_{13} = p_1,$$

$$-\frac{1}{A_1}\frac{dT_{13}}{d\xi_1} + k_1 T_{11} = p_3, \qquad (2)$$

$$-\frac{1}{A_1}\frac{dM_{11}}{d\xi_1} + T_{13} = m_1, 0 \le \xi_1 \le 1,$$

where v_1, w, γ_1 are the displacements and angle of revolution in the shell; T_{11}, T_{13}, M_{11} are the forces and momentum in the shell; $A_1 = A_1(\xi_1)$, $k_1 = k_1(\xi_1)$ correspond to Lame parameter and middle line curvature parameter; p_1, p_3, m_1 are given functions; it holds

$$T_{11} = \frac{E_2 h}{1 - v_2^2}\varepsilon_{11},\ T_{13} = k' G' h \varepsilon_{13},\ M_{11} = \frac{E_2 h^3}{12\left(1 - v_2^2\right)}\chi_{11},$$

$$\varepsilon_{11} = \frac{1}{A_1}\frac{d\mathrm{v}_1}{d\xi_1} + k_1 w,\ \varepsilon_{13} = \frac{1}{A_1}\frac{dw}{d\xi_1} + \gamma_1 - k_1 \mathrm{v}_1,\ \chi_{11} = \frac{1}{A_1}\frac{d\gamma_1}{d\xi_1},$$

$$p_1 = \left(1 + k_1 \tfrac{h}{2}\right)\sigma_{13}^{+} - \left(1 - k_1 \tfrac{h}{2}\right)\sigma_{13}^{-},$$

$$p_3 = \left(1 + k_1 \tfrac{h}{2}\right)\sigma_{33}^{+} - \left(1 - k_1 \tfrac{h}{2}\right)\sigma_{33}^{-},$$

$$m_1 = \tfrac{h}{2}\left(\left(1 + k_1 \tfrac{h}{2}\right)\sigma_{13}^{+} - \left(1 - k_1 \tfrac{h}{2}\right)\sigma_{13}^{-}\right).$$

Here E_2 is the Young's modulus for the shell, v_2 is the Poisson's ratio; $\sigma_{ij}^{+}, \sigma_{ij}^{-}$, $i, j = 1, 3$ are the components of the stress tensor on the outer ($\xi_3 = \frac{h}{2}$) and inner ($\xi_3 = -\frac{h}{2}$) boundaries of the shell. It is known, that in the case of isotropic bodies we have $k' = \frac{5}{6}$, $G' = \frac{E_2}{2(1+v_2)}$.

At the free end of the thin part we impose boundary conditions either on the displacements v_1, w and γ_1 or on the forces T_{11}, T_{13} and momentum M_{11} in the shell (if the end is subjected to load or free). At the top and bottom outer boundaries of the shell we prescribe to σ_{13}^{+} and σ_{33}^{+} some given stresses.

Remark. The choice of 2D curvilinear coordinate system for the shell as ξ_1, ξ_3 (instead of ξ_1, ξ_2) is based on the fact, that 2D problem is obtained from the 3D case by assuming the body being infinite in the direction of ξ_2.

On the boundary Γ_I, common to both Ω_1 and Ω_2 we prescribe the following coupling conditions:

$$u_n = \mathrm{v}_1 + \xi_3\gamma_1, u_\tau = w,$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{nn} d\xi_3 = T_{11}, \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{n\tau} d\xi_3 = T_{13}, \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{nn}\xi_3 d\xi_3 = M_{11}. \tag{3}$$

3. Numerical Approximation of the Model

For the numerical solution of the model domain decomposition algorithm is used. Inside the main part we construct the approximate solution using boundary element method (BEM) applied to the integral equations based on the Green's representation formula for the solution of the following form [1]

$$\frac{1}{2}u_j(x_0)\int_\Gamma (t_i(x)G_{ij}(x,x_0) - F_{ij}(x,x_0)u_i(x))d\Gamma(x), \tag{4}$$

where $\Gamma = \Gamma_1 \cup \Gamma_I$, $x_0 \in \Gamma$;

$G_{ij}(x,\zeta) = C_1(C_2\delta_{ij}\log r - \frac{y_i y_j}{r^2})$ is the matrix Green's function;

$F_{ij}(x,\zeta) = \frac{C_3}{r^2}(C_4(\delta_{ik}y_j + \delta_{jk}y_i - \delta_{ij}y_k) + 2\frac{y_i y_j y_k}{r^2})$ is a co-normal derivative of the matrix Green's function;

$r^2 = y_i y_i$;

$y_i = x_i - \zeta_i$;

$\mu_1 = \frac{E_1}{2(1+\nu_1)}$ is a shear modulus of the body in Ω_1;

$C_1 = -\frac{1}{8\pi\mu(1-\nu_1)}$,

$C_2 = 3 - 4\nu_1$,

$C_3 = -\frac{1}{4\pi(1-\nu_1)}$,

$C_4 = 1 - 2\nu_1$,

In order to apply BEM we divide the boundary $\Gamma_1 \cup \Gamma_I$ of Ω_1 into the elements and then choose the appropriate shape functions $\phi_j(\xi)$, $j = 1, 2, ..., m$, to construct the approximation.

The approximate solution can be written in the form

$$u_i(\xi) = \sum_{j=1}^m u_{ij}\phi_j(\xi), \quad i = 1, 2,$$

$$t_i(\xi) = \sum_{j=1}^m t_{ij}\phi_j(\xi), \quad i = 1, 2, \xi \in \Gamma_1 \cup \Gamma_I,$$

where u_{ij} and t_{ij} are the unknown coefficients that are found by applying Galerkin method to the integral equation (4) (see [1]).

The approximate solution of the boundary value problem inside Ω_2 is found using finite element method with bubble shape functions. On each element the shape functions are given by

$$\Phi_0(\xi) = \frac{1-\xi}{2}, \quad \Phi_1(\xi) = \frac{1+\xi}{2},$$

$$\Phi_j(\xi) = \sqrt{\frac{2j-1}{2}} \int_{-1}^{\xi} P_{j-1}(t)dt, \quad j = 2, 3, ...,$$

where $\xi \in [-1, 1]$ is the local coordinate, obtained by mapping each element onto the inverval [-1,1]; $P_j(t)$ are the Legendre polynomials.

In order to find the approximate solution of the boundary-value problem (2), we apply to the system (2) Galerkin approach.

The approximate solutions in both domains are connected using domain decomposition algorithm (Dirichlet-Neumann scheme) [5]. The domain decomposition algorithm has the following form:

1) set an initial guess λ^0 for the unknown displacements on the interface Γ_I, set $\varepsilon > 0$;

2) for k=0,1,... solve the boundary value problem in Ω_2 with the displacements equal to λ^k to obtain the apporimation for the loads in Ω_1 using (3);

3) solve the corresponding integral equations in Ω_1 to find the displacements u_n^1 and u_τ^1 on Γ_I;

4) update the displacements λ^k on Γ_I:

$$\lambda_1^{k+1} = \lambda_1^k + \theta u_n^1,$$

$$\lambda_2^{k+1} = \lambda_2^k + \theta u_\tau^1,$$

where $\theta > 0$ is a relaxation parameter;

5) if $\|\lambda^{k+1} - \lambda^k\| \geq \varepsilon$ then go to step 2, otherwise the algorithm ends.

It is known, that the Steklov-Poincare equation that corresponds to our problem, possesses a unique solution [7]. Moreover, domain decomposition algorithm converges for appropriately chosen (empirically) relaxation parameter θ $(0 \leq \theta \leq \theta_{max})$ [7].

4. Numerical experiments

Let Ω_1 be a polygon with $x_1^b = -1$, $x_2^b = -1$, $x_1^e = 1$, $x_2^e = 1$. To the main part in Ω_1 a thin body in Ω_2 is attached on its edge. The thickness of the body in Ω_2 is $h = 0.01$ (Fig. 1).

On the boundaries AC and AB the structure is fixed (the displacements are equal to zero); we prescribe a load of $p = 1Pa/m$ on the outer boundary of the body in Ω_2 (Fig. 1); on the edge with the point E the symmetry conditions are set; all the other parts of the outer boundary are traction-free.

We consider the following physical parameters of the bodies: Young's modulus of the main part in Ω_1 is equal to $E_1 = 25000$ MPa, which corresponds to concrete; the Young's modulus of the thin part in Ω_2 is equal to $E_2 = 20580$

MPa, which corresponds to cork. Poisson's ratio of the body in Ω_1 is equal to $\nu_1 = 0.33$, in Ω_2 – $\nu_2 = 0$.

For the numerical solution we use FEM in the shell with bubble shape functions. For the main part we use boundary element method with quadratic shape functions. Problems in both parts are connected using domain decomposition algorithm (Dirichlet-Neumann scheme) [5].

In all the cases under consideration the convergence is obtained in around 5 iterations. The results correspond to a case of 202 boundary elements, 32 finite elements of the fourth order. We find, that the mesh refinement or the change of the order of the shape functions don't change the solution significantly.

Let us consider different cases of the curve shapes, that describe middle line of the body in Ω_2: circle arc, parabola and chain curve. The unknown coefficients of the parametric representation of the curves are chosen in such a way, that all the curves have the same endpoints D and E. Moreover, all the curves are symmetric with respect to the axis, which passes through the point E and is colinear to AB.

In the case of the circle arc the parametric representation has the form

$$x_1(\alpha) = R\sin\alpha,$$

$$x_2(\alpha) = R\cos\alpha, \quad \tfrac{\pi}{4} \le \alpha \le \tfrac{\pi}{2}.$$

Let us choose $R = 5.005$.

In the case of parabola parametric representation has the form

$$x_1(\alpha) = -\tfrac{2-\sqrt{2}}{R}x_2^2 + R,$$

$$x_2(\alpha) = R\cos\alpha, \quad \tfrac{\pi}{4} \le \alpha \le \tfrac{\pi}{2}.$$

In the case of chain curve parametric representation has the form

$$x_1(\alpha) = -\tfrac{4.497}{2}(e^{\frac{x_2}{4.497}} + e^{-\frac{x_2}{4.497}}) + 9.502,$$

$$x_2(\alpha) = R\cos\alpha, \quad \tfrac{\pi}{4} \le \alpha \le \tfrac{\pi}{2}.$$

The graphs of three curves are shown on Fig. 2

We can conclude from Fig. 2, that the graphs of the curves lie close to each other.

Formulae for the calculation of Lame parameter A_1 and curvatures k_1 of the middle line of the shells have the form

$$A_1 = \sqrt{x_1'^2 + x_2'^2},$$

$$k_1 = \tfrac{x_1'' x_2' - x_1' x_2''}{A_1^3}.$$

Let us calculate the stress-strain state for the body depicted on the Fig. 1.

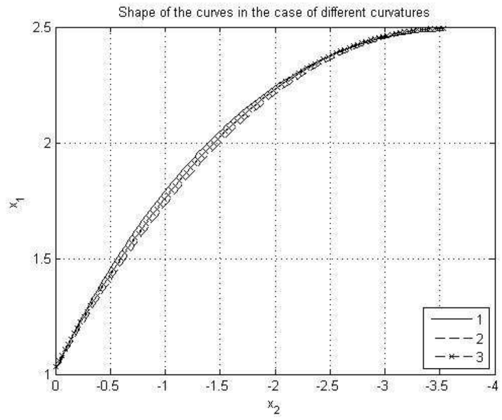

FIG. 2. Middle Line of Different Curves

Fig. 3, 4 show the displacements in the case of different shapes of middle lines, Fig. 5-7 show the momenta that arise on the middle line of Ω_2 in the case of different shapes of middle lines.

Curve 1 on Fig. 3 corresponds to the case of the middle line having the shape of part of the parabola, curve 2 – middle line being the chain curve.

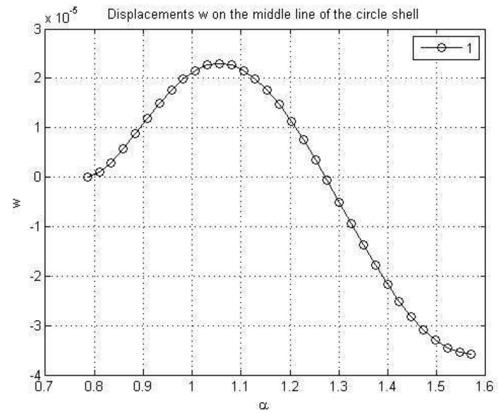

FIG. 3. Displacements w on the middle line of the shell in the case of the circle-shaped shell

On the interface $0 \leq x_2 \leq h$, $x_1 = x_1^e$ we have to set the Neumann condition for the problem in main part, and Dirichlet condition for the problem in the shell. The displacements on the interface for the shell are found using the conditions

$$u_n = v_1 + \xi_3 \gamma_1,$$

$$u_\tau = w.$$

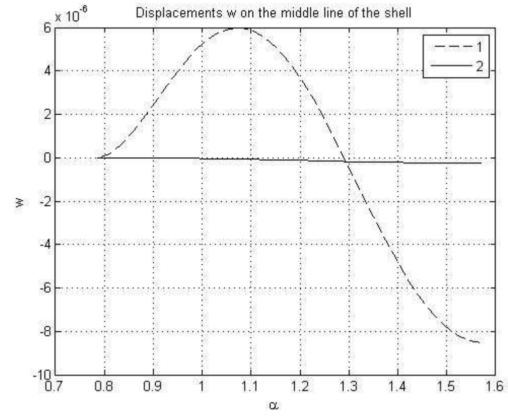

FIG. 4. Displacements w on the middle line of the shell in the case of parabola and chain curve

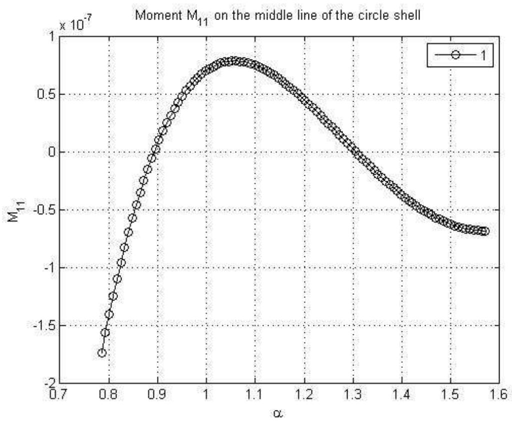

FIG. 5. Momentum m_{11} on the middle line of the shell in the case of the circle-shaped shell

Applying the first condition at the points $\xi_3 = 0$ and $\xi_3 = h/2$, we find that

$$v_1|_{\xi_1=0} = -u_n|_{\xi_3=0},$$

$$\gamma_1|_{\xi_1=0} = \frac{2}{h}(u_n|_{\xi_3=\frac{h}{2}} - u_n|_{\xi_3=0}).$$

Applying the second condition at the point $\xi_3 = 0$, we find that

$$w|_{\xi_1=0} = u_\tau|_{\xi_3=0}.$$

Let us consider the conditions on the loads, that need to be imposed on the interface for the problem in the main part. In order to express $\sigma_{n\tau}$ we use conditions

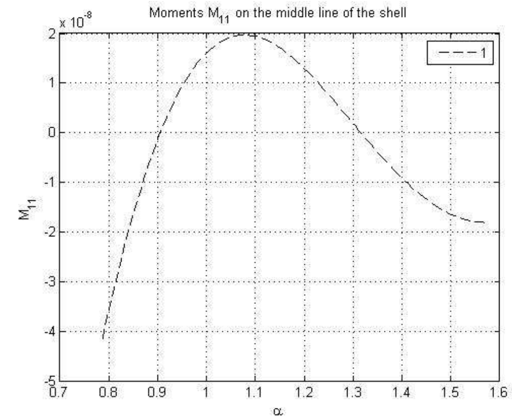

FIG. 6. Momentum m_{11} on the middle line of the shell in the case of parabola

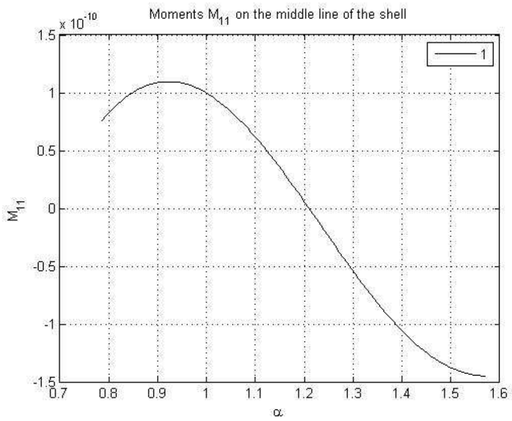

FIG. 7. Momentum m_{11} on the middle line of the shell in the case of chain curve

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{n\tau} d\xi_3 = T_{13}, \quad \sigma_{n\tau}(\xi_3) = \sigma_{13}^{-}|_{\xi_1=0}, \quad \sigma_{n\tau}(\xi_3) = -\sigma_{13}^{+}|_{\xi_1=0}.$$

In order to express σ_{nn} we use conditions

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{nn} d\xi_3 = T_{11}, \quad \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{nn} \xi_3 d\xi_3 = M_{11}.$$

Let us assume that on the interface $\sigma_{n\tau} = a\xi_3^2 + b\xi_3 + c$, $\sigma_{nn} = e\xi_3 + f$, where a, b, c, e, f are the unknown coefficients. These assumptions are based on the fact, that we have three conditions for $\sigma_{n\tau}$ and two conditions on σ_{nn}.

The computations yield

$$\sigma_{nn}(\xi_3) = M_{11}\frac{12}{h^3}\xi_3 + \frac{T_{11}}{h},$$

$$\sigma_{n\tau} = (\frac{3}{h^2}(\sigma_{13}^-|_{\xi_1=0} - \sigma_{13}^+|_{\xi_1=0}) - \frac{6}{h^3}T_{13})\xi_3^2 -$$

$$-\frac{1}{h}(\sigma_{13}^-|_{\xi_1=0} + \sigma_{13}^+|_{\xi_1=0})\xi_3 + \frac{1}{h}(T_{13} - \frac{1}{4}(h(\sigma_{13}^-|_{\xi_1=0} - \sigma_{13}^+|_{\xi_1=0}) - 2T_{13}).$$

From Fig. 3-4 we can conclude, that the smallest displacement in the normal direction is achieved when the middle line of the thin part of the body is a chain curve. The largest displacement in the normal direction arises when the middle line of the thin part is a circle segment.

Fig. 5-7 show, that the smallest momentum is achieved when the middle line of the thin part of the body is a chain curve. The largest momentum arises when the middle line of the thin part is a circle segment.

Therefore, the stress-strain state of the bodies inside the thin part in the case of the Girkmann problem heavily depends on the geometrical parameters of the middle line of the shell (shape, curvature).

5. Conclusions

We conclude, that the stress-strain state of the bodies inside the shell in the case of the Girkmann problem heavily depends on the geometrical parameters of the middle line of the shell (shape, curvature). The elastic body where the shell has the shape of the chain curve, is the best since almost no momentum arises in this case.

The convergence of our algorithm is obtained in around 5 iterations. Therefore, the proposed algorithm can be efficiently applied for the numerical solution of the Girkmann problem.

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