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NUMERICAL MODELLING OF SHALLOW-WATER FLOW IN HYDRODYNAMIC APPROXIMATIONS

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Резюме. Сформульовано двовимірну початково-крайову задачу руху води на території водозбору. Для виводу рівнянь руху проводилися усереднення доданків за глибиною потоку та враховувалися умови мілкості потоків. Побудовано відповідну варіаційну задачу, для якої при дискретизації за просторовими змінними використано метод скінченних елементів і за часом – однокрокову рекурентну схему. Для великих чисел Рейнольдса побудовано стабілізаційну схему, що базується на функціях-бульбашках із використанням методу найменших квадратів.Числові результати апробовано на тестових прикладах для різних початкових та крайових умов, у різні моменти часу і при виборі великих значень чисел Рейнольдса.

Abstract. Formulated a two-dimensional initial-boundary value problem of movement of water in the watershed. To derive the equations of motion were carried averaging summands in the depth flow and conditions of shallow flows were taken into account. The variational problem was built for it in discretization for spatial variables used finite element method and time - one-step recurrent scheme. For large Reynolds numbers built stabilization scheme based on functional bubbles by the method of least squares. Numerical results tested on test examples for different initial and boundary conditions, at different times and in selecting high values of Reynolds numbers.

1. Introduction

One of the most important processes of a hydrological cycle concerns to a shallow water flows to which belong rain and channels flows, water flow from a watershed surface, motion of water in ocean, etc. Processes which underlie of this model have wave nature, with wave length is much greater then the vertical dimensions. To describe these processes is possible outgoing from general equations of Navier-Stokes or from equations of Reynolds. From supposition, that the horizontal scales of fluid motion are much more vertical, the average on vertical component of a flow is realized. The detailed derivation of average equations of shallow water from equations of the Reynolds can be found in works [3],[6]. Equations looks like following:

† *Key words.* Variational problem, initial-boundary value problem, Galerkin approximations, shallow-water flow, Navier-Stokes equations, hydrodynamic approximations.

$$\left\{\begin{array}{l} \frac{\partial q_i}{\partial t} + \sum\limits_{j=1}^{2} \frac{\partial}{\partial x_j}\left(\frac{q_i q_j}{h}\right) = \sum\limits_{j=1}^{2} \frac{\partial N_{ij}}{\partial x_j} - \frac{\partial N_p}{\partial x_i} + B_i, \\ N_{ij} \approx \varepsilon_{ij}\left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i}\right), \\ \frac{\partial(\rho h)}{\partial t} + \sum\limits_{j=1}^{2} \frac{\partial q_j}{\partial x_j} = 0, \quad i,j = 1,2, \end{array}\right. \tag{1}$$

where ε_{ij} - vortex viscosity coefficient, $q_i = hu_i$ – unknowns of value flows, $B_i = \tau_i|_\xi - \tau_i|_\eta + p_a \frac{\partial h}{\partial x_i} + \rho g h \frac{\partial \eta}{\partial x_i}$ (i = 1,2), $N_p = \rho g \frac{h^2}{2} + hp_a, p_a$ - atmospheric pressure, ξ - free surface of flow, - bottom contour, h - flow depth, $\tau_i|_\xi$ та $\tau_i|_\eta$ - stresses on free surface and bottom contour accordingly.

Average equations of shallow water deduced from general Navier-Stokes equations in the works [1,2],[4],[7]. It looks like

$$\left\{\begin{array}{l} \frac{\partial u_i}{\partial t} + \sum\limits_{j=1}^{2} u_j \frac{\partial u_i}{\partial x_j} + g\frac{\partial h}{\partial x_i} + \frac{\left(u_i - u_i^0\right)R - u_i I}{h} = -g\frac{\partial \eta}{\partial x} - \frac{F_i}{h} - \frac{\partial(R\Lambda)}{\partial x_i}, \\ \frac{\partial h}{\partial t} + \frac{\partial(hu_j)}{\partial x_j} = R - I, \quad i = 1,2, \end{array}\right. \tag{2}$$

where u_i - unknowns of speed value, h - unknown flow depth, u_{i_0} - velocity on a free surface, g - acceleration of gravity, I - speed of fluid infiltration into the ground, R - rain inflow velocity, η - bottom contour, Λ - speed of falling of rain drops, F_i - items which allow for tangential stresses on the bottom and on the free surface of a flow.

In motion equations from viscous terms there are only tangential stresses on a free surface and at the bottom, others are rejected in conditions of shallow water. In a result of averaging system of equations set by depth of a flow and allowing conditions of shallow water, the third equation of motion will be converted to the hydrostatic law of pressure, which is characteristic for shallow water equations

$$p(z) = p(\xi) - \rho f_3 (\xi - z).$$

For completion of problem formulation equation of shallow-water supplement by an initial and boundary conditions. The boundary conditions in the literature partition on two kinds: those which are set on hard boundary of flow and on opened boundary. On each of boundaries it is necessary to set two conditions: normal and tangent components of stresses or speeds. For model (2) are set only normal components [2]:

on hard boundary

$$q_n = 0 \; or \; q_n = \overline{q}_n;$$

on opened boundary

$$N_{nn} = \overline{N}_{nn}.$$

It is explained to those that in model (1) the terms that take into account votex viscosity are discarded, therefore tangent components of stresses or flows are not set.

Let's consider one more version of assigning of a boundary conditions. Let Ω - projection of a fluid flow on a two-dimension plane. The boundary of area Ω is partitioned on following parts: Γ_B - fixed boundary of a watershed, Γ_R-boundary of a channel (the fluid inflows), Γ_S - opened sea border (the fluid can both inflow and outflow, see Fig. 1).

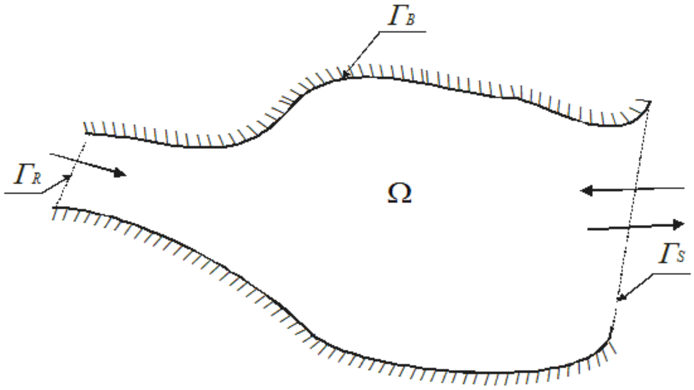

FIG. 1. Projection of a fluid flow on a two-dimension plane

More often boundary conditions for two-dimension problem of shallow water write down [2,4,9-13]:

- on fixed boundary Γ_B of a flow set

$$U \cdot \nu = 0, \quad \nabla U_\tau \cdot \nu = 0,$$

  where ν and τ - units normal and tangent to bound of domain,U_τ - tangential components of velocity;
- on boundary of fluid inflow:

$$U \cdot \nu = \hat{U} \cdot \nu, \quad \mu \frac{\partial U}{\partial \nu} \cdot \tau = 0,$$

  where μ - coefficient of viscosity;
- on opened sea border the boundary conditions it is possible to set as

$$\frac{\partial U}{\partial \nu} = 0.$$

In considered above shallow water models all items which contain component of stresses are skipped. Component of stresses are saved only on a free surface and on the bottom of flow. Scientific approach, which is submitted in this work saves all components of stresses in motion equations. For solving of shallow water problem the finite element method was selected.

2. Formulation of initial-boundary problem

Suppose that flow of viscous incompressible fluid in each point of time $t \in [0, T], 0 < T < +\infty$, forms on an immovable surface $x_3 = \eta(x_1, x_2)$ of watershed some fluid layer $D = D(t)$ (Fig.2).

Let's designate through $\xi(x, t)$ a free surface of this flow, which contacts to atmosphere, where $x = (x_1, x_2, x_3) \in R^3, \nu$ - unit outward normal of domain

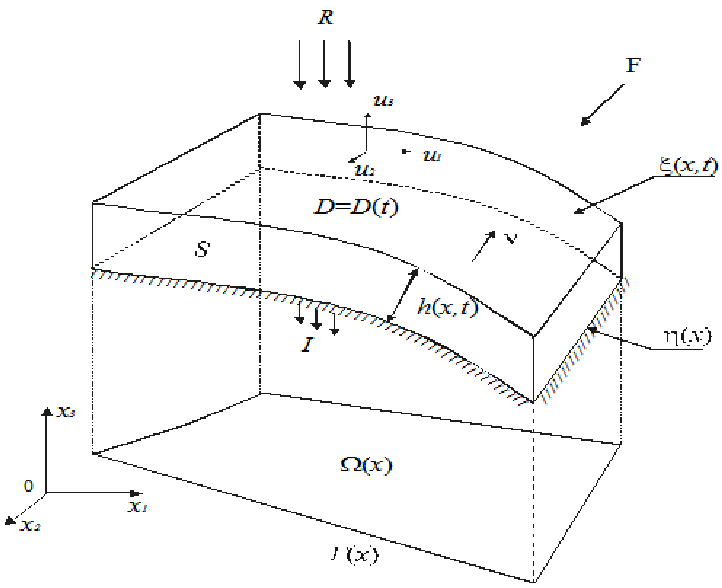

FIG. 2. Model of shallow water flow

$D = D(t)$. Lateral (vertical) surface of this flow, if such exists we shall designate through S. Let's mark, that the part of a surface S can be degenerated in boundary Γ of watershed river. So $\partial D(t) = \eta \cup \xi(t) \cup S$.

Projection of a fluid layer $D(t)$ on a horizontal plane we will denote as Ω. Assume, that boundary γ of domain continuous by Lipschitz.

Let's guess, that a fluid state under the influence of mass forces $F = \{f_i(x)_{i=1}^3$ in each point of time $t \in [0, T], 0 < t < +\infty$ is described by of the Navier-Stokes equations

$$\begin{cases} \rho\left(\frac{\partial u_i}{\partial t} + \sum\limits_{k=1}^{3} \frac{\partial}{\partial x_k}(u_i u_k) - f_i\right) - \sum\limits_{k=1}^{3} \frac{\partial \sigma_{ik}}{\partial x_k} = 0, \\ \sigma_{ij} = -p\delta_{ij} + \tau_{ij}, \\ \tau_{ij} = 2\mu e_{ij}, \\ e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \\ div\ \vec{u} = 0, \quad i, j = 1, 2, 3, \end{cases} \tag{3}$$

where $div\ \vec{u} = \sum\limits_{k=1}^{3} \frac{\partial u_k}{\partial x_k}, \vec{u} = \{u_i(x,t)\}_{i=1}^3$ and $p = p(x,t)$ - velocity vector and hydrostatic pressure accordingly, $F = fi(x,t)_{i=1}^3$ - vector of mass forces, $\rho = const > 0$ and $\mu = const > 0$- density and viscosity, $\{e_{ij}\}_{i,j=1}^3$, $\{\sigma_{ij}\}_{i,j=1}^3$ - velocity and stresses tensors, δ_{ij} - Kroneker symbol.

Let in an initial time water flow described by conditions

$$u_i|_{t=0} = u_i^0\ in\ D(0), i = 1, 2, 3. \tag{4}$$

Except of initial conditions, the equations are necessary supplement by the applicable boundary conditions, which determine interaction of flow water with atmosphere, surface of ground, groundwater etc.The main factors which influence on a fluid state:

– intensive rain precipitations, evaporation of water;
– replenishing of water from channel;
– infiltration of water in soil(groundwater replenishment);
– atmospheric wind, etc.

Attempts to describe characteristic modes of shallow-water flows result in simplification of equations (3) and respective to them boundary conditions and will be reviewed later. At the given stage we will limit by a typical boundary conditions for this equations [2,5,7,9-13]:

$$u = \widehat{u} \; on \; B_u \times (0,T], mes(B_u) > 0, B_u \subset \partial D(t), \tag{5}$$

$$\tau_{ij}\nu_j = \widehat{\tau}_i \; on \; B_\tau \times (0,T], B_\tau \subset \partial D(t) \backslash B_u, \; i,j = 1,2,3, \tag{6}$$

where $u = \{u_i\}_{j=1}^3, \nu = \{\nu_i\}_{j=1}^3$ - unit outward normal of bound $\partial D(t), \nu_i = cos(\nu, x_i)$.

Generally free surface of a flow $\xi(x,t)$ is unknown, therefore it is necessary to set conditions for definition of its position in space in each point time. For finding of a free surface $x_3 = \xi(x_1, x_2, t)$ we shall use a kinematic condition [16]:

$$u_3 + R = \frac{\partial \xi}{\partial t} + u_1^0 \frac{\partial \xi}{\partial x_1} + u_2^0 \frac{\partial \xi}{\partial x_2}, \tag{7}$$

where R – rain velocity, u_1^0, u_2^0 - horizontal components of velocity on a free surface and initial condition

$$\xi|_{t=0} = \xi^0 \; in \, \Omega. \tag{8}$$

On the bottom of flow the fluid can flow in a soil in a direction of an axis x_3

$$u_3 = -I \; on \; [0,T], \tag{9}$$

where I - velocity of seepage water in soil. If I $= 0$ does it mean that surface is impermeable ; I > 0 - fluid particles seepage in a soil with a preset speed; I $<$ 0 - the groundwaters rise on a back surface of ground.

On a base surface for velocity we shall allow for a condition of adhesion

$$u_1 = u_2 = 0. \tag{10}$$

The initial-boundary problem (3)-(10) is difficult to applying for a nature watersheds and requires simplifications. At the first stage (3) we will reduce equation to a undimensional kind. Such form will give a chance to receive numbers, which characterize motion of water (Reynold's number), and also the parameters of equations are such normalized that their values will change in definite limits. At the second stage, allowing conditions of shallow water, neglect terms order of smallness $\varepsilon = \delta/L$ (the maximum thickness of a flow does not exceed the size δ, and characteristic horizontal dimensions value $L, and(\delta/L << 1)$).

All components of stresses in two first equations of motion remain saved after simplification. The following step of simplifications is reduction of a problem dimension at the expense of a depth averaging of equations. After an average is received a two-dimension problem of a water flow in hydrodynamic approximation concerning three unknowns - two components of flow and depth:

$$\begin{cases} \frac{\partial q_i}{\partial t} + \sum\limits_{j=1}^{2} \frac{\partial}{\partial x_j}\left(q_i \frac{q_j}{h}\right) + Gh\left(\frac{\partial h}{\partial x_i} + \frac{\partial \eta}{\partial x_i}\right) - \frac{1}{\rho Re}\sum\limits_{j=1}^{2} \frac{\partial(\tau_{ij} h)}{\partial x_j} - \frac{g|q|q_i}{ReC^2h^2} = 0, \\ \frac{\partial h}{\partial t} + \sum\limits_{j=1}^{2} \frac{\partial q_j}{\partial x_j} = R - I, \\ \tau_{ij} = \mu\left(\frac{\partial(q_i/h)}{\partial x_j} + \frac{\partial(q_j/h)}{\partial x_i}\right), \quad i, j = 1,\ 2, \end{cases} \tag{11}$$

where h - unknown depth, $q = (q_1, q_2)$ - unknown vector of flow, η - bottom contour,ρ - density of water, Re - Reynolds number, τ_{ij} - stresses tensor, μ - viscosity of water, C - Shezi factor,g - gravitational acceleration, $G = \frac{gL}{V_\infty^2}$, L - typical spatial size, V_∞ - typical velocity, R - rain inflow, I - water seepage in a soil.

The first two equations of system are averaged equations of motion, which are parabolic type. Their novelty consists in preservation of addend with internal stresses of a flow, which are essential on surfaces with considerably change gradients. In the literature the hyperbolic equations of a shallow water flow are considered where the stresses only on the bottom and on a free surface of a flow are taking into consideration. In this case it is supposed that the wind stresses are negligible. The third equation of a system is an averaged equation of continuity, which describes a free surface of a flow.

Let's consider a water flow from a surface watershed in a projection on a horizontal plane. Here Ω - two-dimension domain which restricted by curve Γ_B (watershed line) and Γ_P (outflow line), n, ζ - normal and tangent to boundary of area accordingly.

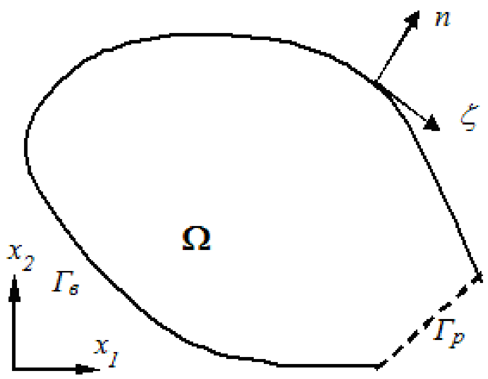

FIG. 3. Water flow projection on a horizontal plane

Equations of system (11) are added by boundary conditions

$$\tau_\zeta|_{\Gamma_B} = 0, q \cdot n|_{\Gamma_B} = 0, \quad q \cdot \zeta|_{\Gamma_P} = 0, q \cdot n|_{\Gamma_P} = \hat{q} \tag{12}$$

and initial conditions

$$h|_{t=0} = h_0, \quad q|_{t=0} = q_{0\text{B}} \ in\, \Omega, \tag{13}$$

where $\hat{q}$ – known water outflow.

In outcome we received a system of three equations (11) for searching components of vector of a flow and depth with boundary (12) and initial (13) conditions. We will decide a problem (11)-(13) by a finite element method (FEM)[8,12-13].

3. Applying of a FEM to the problem solution

According to a procedure of a FEM it is necessary to make a variational formulation. For formulation of variational initial-boundary target setting (11) – (13) we will enter set of allowed functions for flows

$$Q\left(\hat{q}\right) := \left\{ q = \{q_i\}_{i=1}^{2} \in H^1\left(\Omega\right)^2 \,\middle|\, q\cdot n|_{\Gamma_P} = \hat{q},\, q\cdot n|_{\Gamma_B} = 0,\, q\cdot\zeta|_{\Gamma_P} = 0 \right\}$$

and space $Q_0 = Q(0)$. Space of allowed(permissible) functions for depth - $\Phi := L^2(\Omega)$. Let's search a flow as $q = q_* + \bar{q}$ with unknown $q_* \in Q_0, \bar{q}.n = \hat{q}\; on\, \Gamma_p$. Further, for simplicity of identifications we will use instead of q_* identification q.

Let's enter the following forms

$$\begin{cases} a(q,p) = \int\limits_{\Omega} q\cdot p dx, \quad b(w;q,p) = \int\limits_{\Omega} \sum\limits_{i,j=1}^{2} p_i \frac{\partial}{\partial x_j}\left(q_i w_j\right) dx, \\ c\left(h;w,p\right) = \frac{1}{\rho}\int\limits_{\Omega} h \sum\limits_{i,j=1}^{2} \tau_{ij}\left(w\right) \frac{\partial p_i}{\partial x_j} dx, \quad d\left(z;h,p\right) = \frac{1}{2}\int\limits_{\Omega} Gzh\left(\nabla\cdot p\right) dx, \\ l\left(\eta;h,p\right) = \int\limits_{\Omega} G\eta\nabla\cdot\left(hp\right) dx, \quad \bar{R}\left(h,q,p\right) = \int\limits_{\Omega} \frac{g|q|(q\cdot p)}{C^2h^2} dx, \\ \qquad \forall p,\ q,\ w \in Q_0, \\ m\left(q,\theta\right) = \int\limits_{\Omega} \left(\nabla\cdot q\right)\theta dx, \quad \langle s,\theta\rangle = \int\limits_{\Omega}\left(R-I\right)\theta dx, \quad \forall\theta,\ z\in\Phi. \end{cases} \tag{14}$$

Then, take into 3, the variational initial-boundary target setting to become

$$\begin{cases} Given\; q_0 \in Q_0, h_0 \in \Phi; \\ Find\; q \in Q_0, h \in \Phi\; such\; that \\ a\left(q'\left(t\right),p\right) + b\left(q\left(t\right)/h(t); q\left(t\right),p\right) - d\left(h\left(t\right);h\left(t\right),p\right) - \\ -l\left(\eta;h\left(t\right),p\right) + \frac{1}{Re}\left[c\left(h\left(t\right);q/h\left(t\right),p\right) - \bar{R}\left(h\left(t\right);q\left(t\right),p\right)\right] + \\ \qquad +a\left(F(\hat{q}),p\right) = 0, \\ a\left(h'\left(t\right),\theta\right) + m\left(q\left(t\right),\theta\right) + a(V(\hat{q}),\theta) = \langle s\left(t\right),\theta\rangle \forall t\in\left[0,T\right], \\ a(q\left(0\right) - q_0, p) = 0,\ a\left(h\left(0\right) - h_0,\theta\right) = 0 \quad \forall p\in Q_0, \forall\theta\in\Phi, \end{cases} \tag{15}$$

where $F(\hat{q})\; and\; V(\hat{q})$ – items accordingly of first and second equations of a system, which are formed by a flow components $\hat{q}$.

We will decide the variational problem with usage of a projective-net scheme of FEM. Let's conduct a discretization of a problem in time. Interval of time [0,T] we will divide into $N_T + 1$ identical parts $[t_k, t_{k+1}]$ by length Δt and we will select approximations for depth and flows as

$$h(x,t) \approx h_{\Delta t}(x,t) = h^k\left(x\right) + H^{k+\frac{1}{2}}\left(x\right)\Delta t\omega\left(t\right), \tag{16}$$

$$q(x,t) \approx q_{\Delta t}(x,t) = q^k\left(x\right) + U^{k+\frac{1}{2}}\left(x\right)\Delta t\omega\left(t\right), \tag{17}$$

where

$$H^{k+\frac{1}{2}} = \frac{h^{k+1} - h^k}{\Delta t}, H^{k+\frac{1}{2}} \in \Phi, \quad \forall x \in \Omega, \ \forall t \in [t_k, t_{k+1}], \quad k = 0, \quad \dots, N_T.$$

It is known, that if we approximate a function by an interpolation polynomial of the first order, the precision greater than Δt^2 cannot be obtained. Therefore at the given stage (phase) we can conduct a linearization of a problem by throwing off terms of the order. By substituting (16) – (17) in a variational problem (15) and ignore terms of the order Δt^2, we receive a linearized problem as the one-step recurrent scheme of integrating in time

$$\begin{cases} Given \quad q^0 \in Q_0, \ h^0 \in such \ that \ \lambda \in (0, \ 1] \ ; \\ Find \quad U^{k+\frac{1}{2}} \in Q_0, \ H^{k+\frac{1}{2}} \in Phi \ , \ such \ that \\ a(U^{k+\frac{1}{2}}, p) + \\ +\lambda \Delta t \Big[b(q^k/h^k; U^{k+\frac{1}{2}}, p) + b(U^{k+\frac{1}{2}}; q^k/h^k, p) - 2d(H^{k+\frac{1}{2}}; h^k, p) - \\ -l(\eta; H^{k+\frac{1}{2}}, p) + \frac{1}{Re} \Big(c(H^{k+\frac{1}{2}}; q^k/h^k, p) + c(h^k; U^{k+\frac{1}{2}}/h^k, p) \Big) \Big] = \\ = d\left(h^k; h^k, p\right) + l\left(\eta; h^k, p\right) - b\left(q^k/h^k; q^k, p\right) - \\ -\frac{1}{Re}\left[c\left(h^k; q^k/h^k, p\right) - \bar{R}\left(h^k; q^k, p\right)\right] - a(F_{k+1/2}, p), \\ a(H^{k+\frac{1}{2}}, \theta) + \lambda \Delta t m(U^{k+\frac{1}{2}}, \theta) = \\ = < s_{k+1/2}, \theta > -m(q^k, \theta) - a(V_{k+1/2}, \theta), \\ q^{k+1} = q^k + \Delta t U^{k+\frac{1}{2}}, \ h^{k+1} = h^k + \Delta t H^{k+\frac{1}{2}}, \ k = 0, ..., N_T, \end{cases} \tag{18}$$

where $F_{k+1/2} = F(t_k + \Delta t/2), V_{k+1/2} = V(t_k + \Delta t/2), s_{k+1/2} = s(t_k + \Delta t/2)$.

At a discretization of a problem (18) according to space variables are utilised piecewise linear approximatings on triangular elements for flows and piecewise constant approximatings of depthes. Such selection of approximatings allows to eliminate depth of a flow and to receive a system of simple equations only concerning vector of a flow.

For a discretization of a problem according space variables the domain Ω is divided into triangular finite elements. Let's enter the spaces for flows $Q_0^h \subset Q_0, \dim\ Q_0^h = N_p < \infty$ and for depthes $\Phi^h \subset \Phi, dim\ \Phi_0^h = N_e < \infty$. Let's select piecewise linear approximatings for flows

$$\varphi_i\left(x_1, x_2\right) = \begin{cases} L_i\left(x_1, x_2\right), P_i \in \Omega_e, \\ 0, P_i \notin \Omega_e \end{cases}$$

and piecewise constant for depthes

$$\psi_e\left(x_1, x_2\right) = \begin{cases} 1, P \in \Omega_e, \\ 0, P \notin \Omega_e. \end{cases}$$

Further using a procedure of a Galorkin method, we will obtain a system of simple equations concerning unknowns of vector of a flow W in nodal values of a grid and vector of depths S in center of gravity of triangles:

$$\begin{cases} Given\ q^0 \in Q_0, \quad h^0 \in \Phi \quad and \quad \lambda \in (0,1]\ ; \\ Find\ U_h^{k+\frac{1}{2}} = \sum\limits_{i=1}^{N_p} W_i^{k+\frac{1}{2}} \varphi_i \in Q_h^0, \quad H_h^{k+\frac{1}{2}} = \sum\limits_{e=1}^{N_e} S_e^{k+\frac{1}{2}} \psi_e \in \Phi^h \\ such,\ \ that\ \ a(U_h^{k+\frac{1}{2}}, p) + \\ +\lambda\Delta t \Big[b(q^k/h^k; U_h^{k+\frac{1}{2}}, p) + b(U_h^{k+\frac{1}{2}}; q^k/h^k, p) - 2d(H_h^{k+\frac{1}{2}}, h^k, p) - \\ -l(\eta, H_h^{k+\frac{1}{2}}, p) + \frac{1}{Re}\Big(c(H_h^{k+\frac{1}{2}}; q^k/h^k, p) + c(h^k; U_h^{k+\frac{1}{2}}/h^k, p)\Big)\Big] = \\ = d\left(h^k, h^k, p\right) + l\left(\eta, h^k, p\right) - b\left(q^k/h^k; q^k, p\right) - \\ -\frac{1}{Re}\left[c\left(h^k; q^k/h^k, p\right) - \bar{R}\left(h^k; q^k, p\right)\right] - a(F_{k+\frac{1}{2}}, p) \quad \forall p \in Q_0, \\ a(H_h^{k+\frac{1}{2}}, \theta) + \lambda\Delta t m(U_h^{k+\frac{1}{2}}, \theta) = \\ =< s_{k+\frac{1}{2}}, \theta > -m(q^k, \theta) - a(V_{k+\frac{1}{2}}, \theta) \quad \forall \theta \in \Phi, \\ q^{k+1} = q^k + \Delta t U_h^{k+\frac{1}{2}}, \qquad h^{k+1} = h^k + \Delta t H_h^{k+\frac{1}{2}}, \qquad k = 0, ..., N_T. \end{cases} \tag{19}$$

On a Fig. 4 completely sampled equations are sketched on one finite element

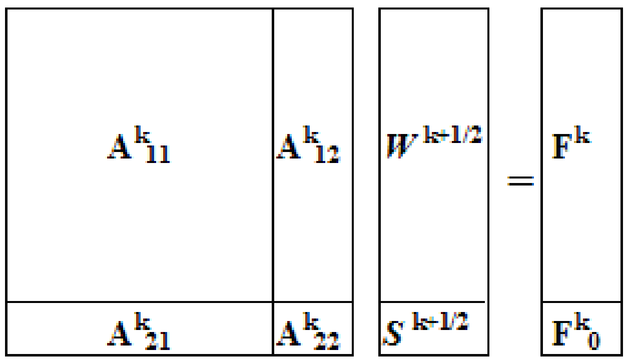

FIG. 4. Diagrammatic representation of a system of simple equations

A_{22} – diagonal matrix. At the expense of condensation of internal parameters we can eliminate depth on one finite element by using a ratio

$$S_e^{k+\frac{1}{2}} = A_{22}^{k^{-1}}(F_0^k - A_{21}^k W_e^{k+\frac{1}{2}}). \tag{20}$$

In outcome we will obtain a system of simple equations concerning two unknowns – flow components

$$(A_{11}^k - A_{12}^k A_{22}^{k^{-1}} A_{21}^k) W_e^{k+\frac{1}{2}} = F^k - A_{12}^k A_{22}^{k^{-1}} F_0^k.$$

4. Stabilization scheme FEM

At large values of Reynold's numbers (Re>100) flows and their gradients change sharply. As outcome the obtained solution of a shallow water problem loses the stability and appears oscillations. On this case, stabilization scheme is obtained, which is based on bubble functions with usage of a least-squares method. As the depth of a fluid is considered as a constant on one finite element, it does not influence behaviour of the solution. In a system (18) the stabilization addend is added to equations of flows in the next view

$$\begin{aligned} S(U^{k+\frac{1}{2}},H^{k+\frac{1}{2}},p) = M_e \Bigg[\int\limits_{\Omega_e} U^{k+\frac{1}{2}} \cdot p dx + \\ + \Delta t\lambda \Bigg[\int\limits_{\Omega_e} \sum_{j=1}^{2} \left((\frac{\partial}{\partial x_j}((q_i^k U_j^{k+\frac{1}{2}}) + (q_j^k U_i^{k+\frac{1}{2}}))/h^k \right) p dx + \\ + \int\limits_{\Omega_e} \left(U_i^{k+\frac{1}{2}} + \sum_{j=1}^{2} \frac{\partial}{\partial x_j}(q_i^k q_j^k)/h^k + Gh^k \frac{\partial \eta}{\partial x_i} - \frac{g|q^k|q_i^k}{ReC^2\,(h^k)^2} \right) \times \\ \times \left(\sum_{j=1}^{2} \frac{\partial}{\partial x_j}((q_i^k p_j) + (q_j^k p_i))/h^k \right) dx + \int\limits_{\Omega_e} GH^{k+\frac{1}{2}} \frac{\partial \eta}{\partial x_i} p dx \Bigg] - \\ - \int\limits_{\Omega_e} \frac{g|q^k|q_i^k p_i}{ReC^2(h^k)^2} dx + \int\limits_{\Omega_e} \sum_{j=1}^{2} \frac{\partial}{\partial x_j}((q_i^k q_j^k)/h^k) p dx + \int\limits_{\Omega_e} Gh^k \frac{\partial \eta}{\partial x_i} p dx \Bigg], \end{aligned} \quad (21)$$

where M_e -– stabilization factor on each finite element.

For stabilization factor M_e using the upper-bound estimate μ_0 obtained in the work [6] for approximating scheme of Navier-Stokes equations

$$\mu_0 = \frac{7}{5}\left(\frac{1}{7kd^2/\Delta^2 - e}\right), \quad (22)$$

where Δ - square of finite triangle element, $d^2 = l_1^2 + l_2^2 + l_3^2$, l_i - length of triangle side (i=1, 2, 3), $e = div\, w$, w - know velocity from previous step, k - kinematic viscosity of a fluid.

5. Test examples

Example 1. Let's consider a problem of shallow water flow from a surface some watershed. All parameters of a problem are set in a dimensionless view. Let's select a test surface watershed $\eta(x, y)$ as Fig. 5, where x, y change from 0 to 2. In an initial time we will enable that h_0 =0.01, q_i=0 (i=1,2). Concerning boundary conditions, we enable, that the water does not outflow and normal component of flow velocities on boundary of domain is equal zero $q\dot{n}$=0. We enable, that constant rain influx R=1, infiltration of a fluid in a ground I=0, coefficient factor Shezi C=60, Reynold's number Re=0.1. Quantity of splitting points of domain 60x60. For the solution of a problem we apply the numeric scheme (19), in which parameters $\lambda = 0.5$, $\Delta t = 0.005$. Let's consider result in a point of time t = 0.195 (quantity of steps in time tt=40).

In a Fig. 5 the depth of a flow H (quantity of water is figured, which collects at the bottom surface with constant rain influx). As the water does not outflow, cavities are filled by the water. From results apparently, that the maximum value of depth is reached in the middle of a bottom surface, where there is a greatest cavity. In the highest points of surface watershed values of depth are approaches to zero, as the water flows down.

The law of conservation of mass for the given example is tested. The conducted calculations have shown, that the volume of the fall out precipitations approximately coincides with water volume on a given surface in the given point of time 0.78105.

In Fig.7 and Fig.8 are figured values components of a fluid flow accordingly on axes x and y. In a Fig.9 the module of a flow is figured. From results it is possible to see, that the flow has zero values in those points of a bottom surface, where the fluid collects and whence the water flows off, in these extreme points water is not gone. The maximum values of a flow are reached in currents, where there is a maximum slope of a bottom surface to horizont.

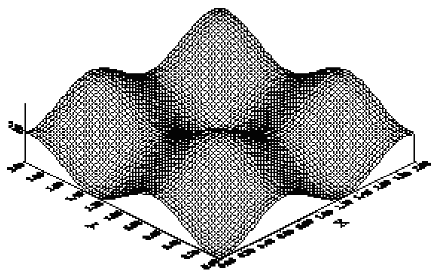

FIG. 5. Bottom surface $\eta(x, y)$

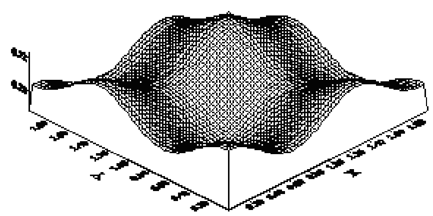

FIG. 6. Flow depth H

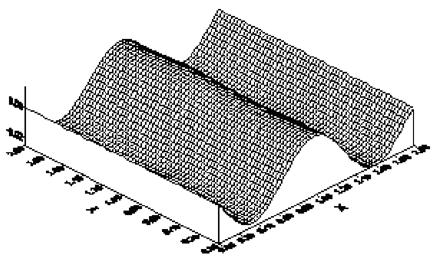

FIG. 7. Flow component Qx

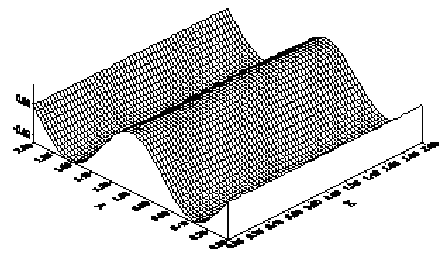

FIG. 8. Flow component Qy

Example 2. By important point at problem solving of shallow water is selection of a Reynold's number values. When parameter receives large values (Re>100), solution obtained with the help of the numeric scheme (19), loses the stability, values of flows and their gradients are very large, as a result of it there are oscillations. In the Fig. 10 the values of depthes of a problem with parameters by given in an example 1 and Reynold's number Re = 150 are figured. On Fig. 11 the values of component flows accordingly on axis x are figured. The results are displayed in a point of time t = 0.073 (quantity of steps in time tt = 15, $\Delta t = 0.005$).

For the solution of this problem the stabilization scheme of a finite element method with stabilization factor (21) was obtained. We apply the stabilization

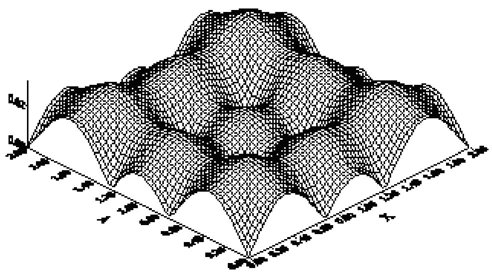

FIG. 9. Module of flow

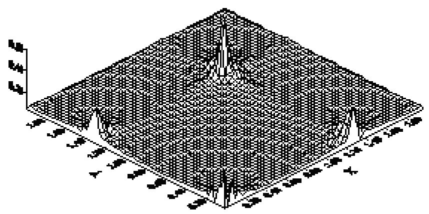

FIG. 10. Flow depth H

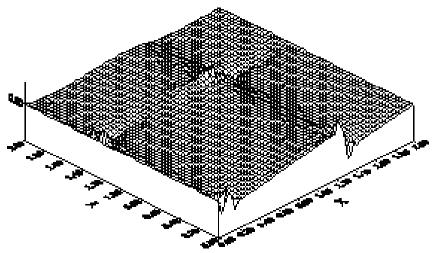

FIG. 11. Flow component Qx

scheme to the solution of our problem with a Reynold's number Re=150 and stabilization factor $M_e = -0.5$. Let's consider computing results in a point of time t = 0.586 (quantity of steps in time tt=60), quantity of splitting points of domain 30x30. In a Fig. 12 the values of depth are figured, the Fig. 13, Fig. 14, Fig. 15 - represent values components and module of a flow accordingly.

From results it is possible to see, that the problem, which has arisen, at applying the numeric scheme (19) to the solution of a problem, is decided positively

The results are smoothed at the expense of the introducing of a stabilization factor. The computing results have shown, that the problems of a shallow water flow can be decided with any values of Reynold's numbers, applying the stabilization scheme of a finite element method.

The law of conservation of mass for the obtained outcomes is executed. The volume of the fall out precipitations coincides with a volume of a fluid on a surface watershed 2.34314.

Example 3. Let's consider a water flow from a surface watershed Fig. 16 (part of Perespil countryside in the Lvov area). Boundary and initial conditions we will select similarly to the previous example, quantity of splitting points of domain 60x60, stabilization a factor M_e = -0.5. Let's consider the results in a point of time t = 0.146 (quantity of steps in time tt=30) with a Reynold's number Re=150. In a Fig. 17 the depth H of a water flow is displayed. For the greater visualization we compare isolines of a watershed surface (Fig. 18)

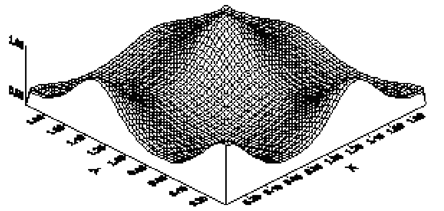

FIG. 12. Flow depth H

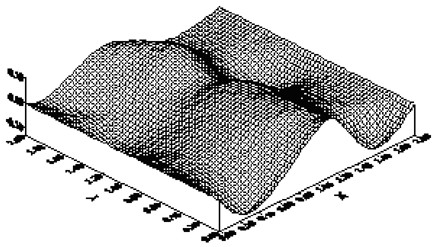

FIG. 13. Flow component Qx

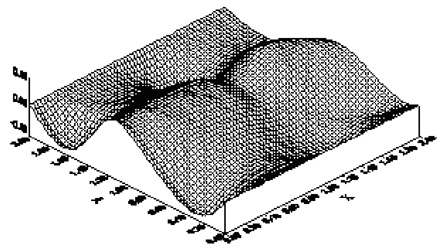

FIG. 14. Flow component Qy

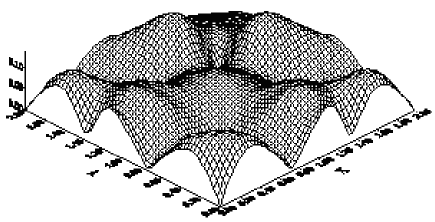

FIG. 15. Module of flow

and depth (Fig. 19). As the water does not outflow, cavities are filled by water. From results we can see, that the filling of a watershed surface by water implements according to isolines.In Fig. 20 is displayed module of flow.

7. CONCLUSIONS

For a selected example with stabilization factor is the laws of conservation of mass and flow of fluids are fair. The obtained model enables to conduct calculations of values of depth and speeds of fluid flows on columbines with rain and lateral influxes for different initial and boundary conditions in different point of time with large values of a Reynold's number.

The above examples indicate that significant influence on the solution of the problem of shallow water on the surface of a watershed has a choice of Reynolds number. For small values of this number of problem can be solved by using numerical scheme (19). Choosing Re> 100, the solution loses its stability (Fig. 10 – Fig. 13). This is because for large values of the Reynolds number solutions of problems may have internal and boundary layers - a very narrow area where most solutions and their gradients change sharply. As a result, numerical solutions, built on the Galerkin scheme, where the parameter discretization is too large to consider all these layers can ostcillate throughout the domain.

Considering it was built stabilization scheme FEM. Applying this scheme to solving problems of shallow water on the surface of a watershed above mentioned problem disappears (Fig.14 - Fig.20).

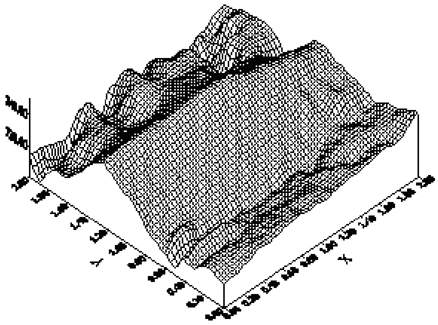

FIG. 16. Bottom surface (x, y)

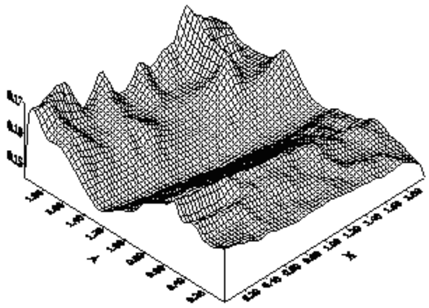

FIG. 17. Flow depth H

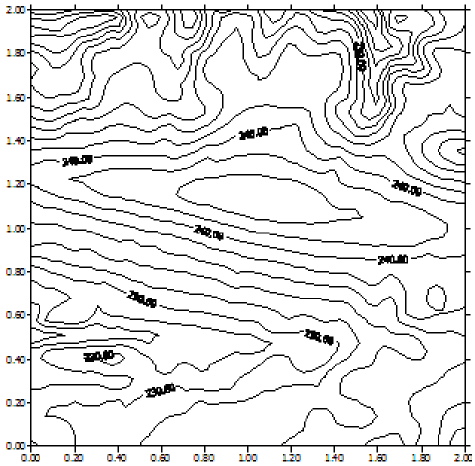

FIG. 18. Isolines of bottom surface η(x,y)

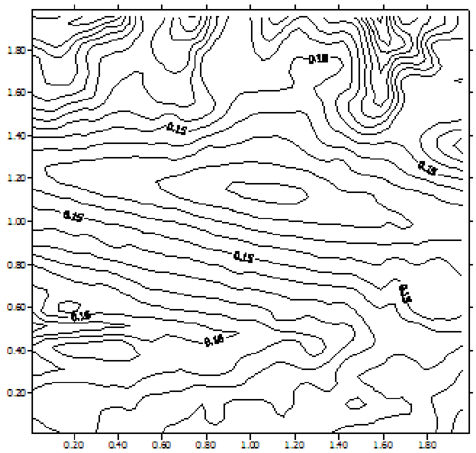

FIG. 19. Isolines of depth surface H(x,y)

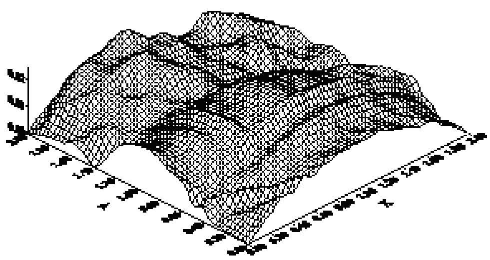

FIG. 20. Module of flow

Thus, based stabilization scheme FEM can be effective in solving the problem of shallow water from any surface water catchment for large Reynolds numbers.

Bibliography

1. Venherskyi P. Mathematical modeling of rain water runoff in the relief area / P. Venherskyi, N. Smushak // Intern. scientific-practical. conf. "GIS technology today": Abstracts ext. – Lviv, 1999. — P. 35.
2. Venherskyi P. Construction of mathematical models of rain water flow on surface watershed / P. Venherskyi, N. Smushak, G. Shinkarenko // Visnyk of Lviv University. A series of mech-mat. – Lviv, 1998. – No 50. – P. 41-45.
3. Venherskyi P. Numerical study of regularized problems of shallow water in kinematic approximation / P. Venherskyi, V. Trushevskyi // Visnyk of Lviv University. A series of App. Math. Inform. – 2003. – No 6. – P. 116-125.
4. Venherskyi P. Numerical modeling of shallow overland flow in the kinematic approximation / P. Venherskyi, V. Trushevskyi // Visnyk of Lviv University. A series of App. Math. Inform. – 1999. – No 1. – P. 44-49.
5. Venherskyi P. Estimation of the depth of recursion of a class of bilateral methods / P. Venherskyi, V. Trushevskyi, P. Senyo // Visnyk of Lviv. University. A series of mechanics and Math. – 1998. – No 50. – P. 45-48.
6. Venherskyi P. Stabilization of the numerical solution of a variational problem of shallow water / P.Venherskyi,V. Trushevskyi, G.Shinkarenko // Visnyk of Lviv University. A series of App. Math. Inform. – 2002. – No 4. – P. 102-109.
7. Venherskyi P. Numerical solution of variational problems of surface flow / P. Venherskyi, V.Trushevskyi, G.Shinkarenko // Visnyk of Kyiv. University. A series of kibernet. – 2002. – No 3. – P. 26-30.
8. Venherskyi P. Applying object-oriented approach for describing algorithms for solving boundary value problems by finite element method / P. Venherskyi, O.Yefremov, B. Stryhaliuk // Visnyk of nat. Univ "Lviv politehnyka." Electronics and telecommunications. — 2002. — No 443. — P. 190-192.
9. Weiyan T. Shallow Water Hydrodynamics: Mathem. Theory Numer. Solution for a Two-dimens. System Shallow Water Equations / T. Weiyan. — Amsterdam: Elsevier, 1992. -– 434 p.
10. Westerink J. J. Tide and Storm Surge Predictions Using a Finite Element Model / J. J. Westerink, R. A. Luettich, A. M. Baptista, N. W. Schefiner, P. Farrar // J. Hydraulic Eng. — 1992. — No 118. — P. 1373-1390.
11. Wunsch O. Numerical simulation of 3d viscous fluid flow and convective mixing in a static mixer / O. Wunsch, G. Bohme // Archive of Applied Mechanics. — 2000. — No 70. — P. 91-102.
12. Yang C.-T. An analysis of flow over a backward-facing step by an assumed stress mixed finite element method. / C.-T. Yang, S. N. Alturi // Numer. Meth. Laminar and Turbulent Flow Proc. 3 Int. Conf. Seattle. — Swansea. — 1983. — P. 302-316.
13. Yang C.-T. An assumed deviatoric stress-pressure-velocity mixed finite element method for unsteady, convective, incompressible viscose flow: Part 2: Computational studies / C.-T. Yang, S. N. Alturi // Int. J. Num. Meth. Fluids. — 1984. -– No 4. — P. 43-69.

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